Specifying & Refining a Bridge Controller

MEB: Chapter 2

EECS3342 Z: System Specification and Refinement
Winter 2022

CHEN-WEI WANG

Learning Outcomes

This module is designed to help you understand:

- What a ***Requirement Document*** (***RD***) is
- What a ***refinement*** is
- Writing ***formal specifications***
  - (Static) contexts: constants, axioms, theorems
  - (Dynamic) machines: variables, invariants, events, guards, actions
- ***Proof Obligations*** (***POs***) associated with proving:
  - ***refinements***
  - system ***properties***
- Applying ***inference rules*** of the ***sequent calculus***

Recall: Correct by Construction

- Directly reasoning about **source code** (written in a programming language) is too complicated to be feasible.
- Instead, given a ***requirements document***, prior to **implementation**, we develop ***models*** through a series of ***refinement*** steps:
  - Each model formalizes an ***external observer***'s perception of the system.
  - Models are "sorted" with ***increasing levels of accuracy*** w.r.t. the system.
  - The ***first model***, though the most ***abstract***, can already be proved satisfying some ***requirements***.
  - Starting from the ***second model***, each model is analyzed and proved ***correct*** relative to two criteria:
    1. Some ***requirements*** (i.e., R-descriptions)
    2. ***Proof Obligations*** (***POs***) related to the ***preceding model*** being ***refined by*** the ***current model*** (via "extra" ***state*** variables and ***events***).
  - The ***last model*** (which is *correct by construction*) should be **sufficiently close** to be transformed into a **working program** (e.g., in C).

State Space of a Model

- A model's *state space* is the set of **all** configurations:
  - Each ***configuration*** assigns values to **constants** & **variables**, subject to:
    - ***axiom*** (e.g., typing constraints, assumptions)
    - ***invariant*** properties/theorems
  - Say an initial model of a bank system with two constants and a variable:

    $c \in \mathbb{N}1 \wedge L \in \mathbb{N}1 \wedge accounts \in String \mathrel{+\!\!\!\!\rightarrow} \mathbb{Z}$ /* typing constraint */

    $\forall id \bullet id \in \mathrm{dom}(accounts) \Rightarrow -c \leq accounts(id) \leq L$ /* desired property */

    Q. What is the ***state space*** of this initial model?

    A. All valid combinations of *c*, *L*, and *accounts*.

    - Configuration 1: $(c = 1{,}000, L = 500{,}000, b = \varnothing)$
    - Configuration 2: $(c = 2{,}375, L = 700{,}000, b = \{(\text{"}id1\text{"}, 500), (\text{"}id2\text{"}, 1{,}250)\})$

      $\dots$ [Challenge: ***Combinatorial Explosion***]
  - Model Concreteness ↑ ⇒ (State Space ↑ ∧ Verification Difficulty ↑)
- A model's ***complexity*** should be guided by those properties intended to be verified against that model.

  ⇒ ***Infeasible*** to prove **all** desired properties on a model.

  ⇒ ***Feasible*** to distribute desired properties over a list of ***refinements***.

Roadmap of this Module

- We will walk through the *development process* of constructing ***models*** of a control system regulating cars on a bridge.

  Such controllers exemplify a ***reactive system***.

  (with **sensors** and **actuators**)

- Always stay on top of the following roadmap:

  1. A ***Requirements Document*** (***RD***) of the bridge controller
  2. A brief overview of the ***refinement strategy***
  3. An initial, the most ***abstract*** model
  4. A subsequent ***model*** representing the ***1st refinement***
  5. A subsequent ***model*** representing the ***2nd refinement***
  6. A subsequent ***model*** representing the ***3rd refinement***

Requirements Document: Mainland, Island

Imagine you are asked to build a bridge (as an alternative to ferry) connecting the downtown and Toronto Island.

Page Source: `https://soldbyshane.com/area/toronto-islands/`

Requirements Document: E-Descriptions

Each ***E-Description*** is an **atomic** ***specification*** of a ***constraint*** or an ***assumption*** of the system's working environment.

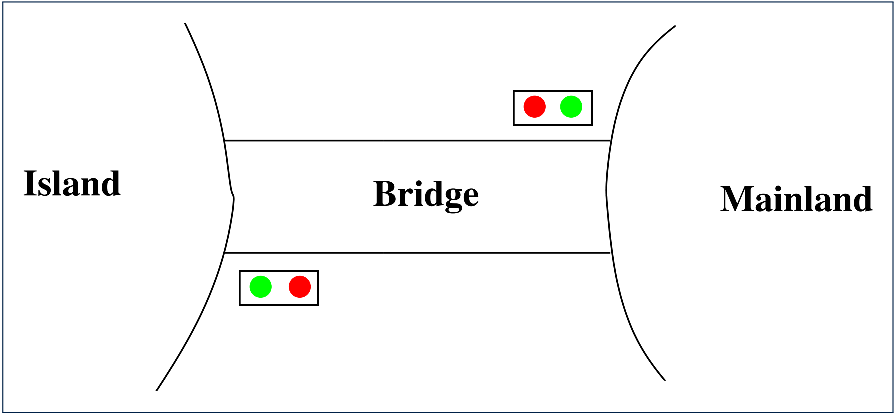
ENV1	The system is equipped with two traffic lights with two colors: green and red.
ENV2	The traffic lights control the entrance to the bridge at both ends of it.
ENV3	Cars are not supposed to pass on a red traffic light, only on a green one.
ENV4	The system is equipped with four sensors with two states: on or off.
ENV5	The sensors are used to detect the presence of a car entering or leaving the bridge: "on" means that a car is willing to enter the bridge or to leave it.

Requirements Document: R-Descriptions

Each ***R-Description*** is an **atomic** ***specification*** of an intended ***functionality*** or a desired ***property*** of the working system.

REQ1	The system is controlling cars on a bridge connecting the mainland to an island.
REQ2	The number of cars on bridge and island is limited.
REQ3	The bridge is one-way or the other, not both at the same time.

Requirements Document:
Visual Summary of Equipment Pieces



Refinement Strategy

- Before diving into details of the ***models***, we first clarify the adopted ***design strategy of progressive refinements***.

  0. The ***initial model*** (m_0) will address the intended functionality of a limited number of cars on the island and bridge.

  [**REQ2**]

  1. A ***1st refinement*** (m_1 which ***refines*** m_0) will address the intended functionality of the *bridge being one-way*.

  [**REQ1**, **REQ3**]

  2. A ***2nd refinement*** (m_2 which ***refines*** m_1) will address the environment constraints imposed by *traffic lights*.

  [**ENV1, ENV2, ENV3**]

  3. A ***final, 3rd refinement*** (m_3 which ***refines*** m_2) will address the environment constraints imposed by *sensors* and the *architecture*: controller, environment, communication channels.

  [**ENV4, ENV5**]

- Recall *Correct by Construction* :

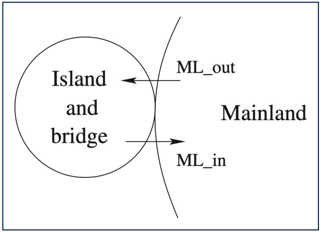
  From each ***model*** to its ***refinement***, only a manageable amount of details are added, making it ***feasible*** to conduct **analysis** and **proofs**.

Model m_0: Abstraction

- In this most ***abstract*** perception of the bridge controller, we do **not** even consider the bridge, traffic lights, and sensors!
- Instead, we focus on this single ***requirement***:

REQ2	The number of cars on bridge and island is limited.

- **Analogies**:
  - Observe the system from the sky: island and bridge appear only as a compound.



  - "***Zoom in***" on the system as ***refinements*** are introduced.

Model m_0: State Space

1. The ***static*** part is fixed and may be seen/imported.

   A ***constant*** d denotes the maximum number of cars allowed to be on the ***island-bridge compound*** at any time.

   (whereas cars on the mainland is unbounded)

   constants: d

   axioms:
   axm0_1 : $d \in \mathbb{N}$

   Remark. ***Axioms*** are assumed true and may be used to prove theorems.

2. The ***dynamic*** part changes as the system ***evolves***.

   A ***variable*** n denotes the actual number of cars, at a given moment, in the ***island-bridge compound***.

   variables: n

   invariants:
   inv0_1 : $n \in \mathbb{N}$
   inv0_2 : $n \leq d$

   Remark. ***Invariants*** should be (subject to **proofs**):

   - ***Established*** when the system is first initialized
   - ***Preserved***/***Maintained*** after any ***enabled event***'s actions take effect

Model m_0: State Transitions via Events

- The system acts as an **ABSTRACT STATE MACHINE (ASM)**: it ***evolves*** as ***actions of enabled events*** change values of variables, subject to ***invariants***.
- At any given ***state*** (a valid ***configuration*** of constants/variables):
  - An event is said to be ***enabled*** if its guard evaluates to ***true***.
  - An event is said to be ***disabled*** if its guard evaluates to ***false***.
  - An ***enabled*** event makes a ***state transition*** if it occurs and its ***actions*** take effect.
- *1st* ***event***: A car exits mainland (and enters the island-bridge compound).

```
ML_out
  begin
    $n := n + 1$
  end
```

Correct Specification? Say $d = 2$.
Witness: ***Event Trace*** $\langle init, ML\_in \rangle$

- *2nd* ***event***: A car enters mainland (and exits the island-bridge compound).

```
ML_in
  begin
    $n := n - 1$
  end
```

Correct Specification? Say $d = 2$.
Witness: ***Event Trace*** $\langle init, ML\_out, ML\_out, ML\_out \rangle$

Model m_0: Actions vs. Before-After Predicates

- When an enabled event *e* occurs there are two notions of ***state***:
  - ***Before-/Pre-State***: Configuration just ***before*** *e*'s actions take effect
  - ***After-/Post-State***: Configuration just ***after*** *e*'s actions take effect

  Remark. When an enabled event occurs, its ***action(s)*** cause a **transition** from the ***pre-state*** to the ***post-state***.
- As examples, consider ***actions*** of m_0's two events:

Events	ML_out $n := n + 1$	ML_in $n := n - 1$
before–after predicates	$n' = n + 1$	$n' = n - 1$

  - An event ***action*** "$n := n + 1$" is not a variable assignment; instead, it is a ***specification***: "*n becomes n + 1 (when the state transition completes)*".
  - The ***before-after predicate*** (***BAP***) "$n' = n + 1$" expresses that n' (the ***post-state*** value of *n*) is one more than *n* (the ***pre-state*** value of *n*).
- When we express ***proof obligations*** (***POs***) associated with ***events***, we use ***BAP***.

Design of Events: Invariant Preservation

- Our design of the two events

```
ML_out
  begin
    $n := n + 1$
  end
```

```
ML_in
  begin
    $n := n - 1$
  end
```

  only specifies how the ***variable*** *n* should be updated.

- Remember, ***invariants*** are conditions that should never be ***violated***!

```
invariants:
  inv0_1 : $n \in \mathbb{N}$
  inv0_2 : $n \leq d$
```

- By simulating the system as an ***ASM***, we discover ***witnesses*** (i.e., event traces) of the ***invariants*** not being preserved all the time.

$$\exists s \bullet s \in \textbf{\textsc{State Space}} \Rightarrow \neg \textbf{\textit{invariants}}(s)$$

- We formulate such a commitment to preserving ***invariants*** as a ***proof obligation (PO)*** rule (a.k.a. a ***verification condition (VC)*** rule).

Sequents: Syntax and Semantics

- We formulate each *PO/VC* rule as a (horizontal or vertical) ***sequent***:

$$\boxed{H \vdash G} \qquad \boxed{\begin{array}{c} H \\ \vdash \\ G \end{array}}$$

  - The symbol $\vdash$ is called the ***turnstile***.
  - H is a set of predicates forming the ***hypotheses/assumptions***.
    [assumed as ***true***]
  - G is a set of predicates forming the ***goal/conclusion***.
    [claimed to be ***provable*** from H]
- Informally:
  - $H \vdash G$ is ***true*** if G can be proved by assuming H.
    [i.e., We say "H ***entails*** G" or "H ***yields*** G"]
  - $H \vdash G$ is ***false*** if G cannot be proved by assuming H.
- Formally: $H \vdash G \iff (H \Rightarrow G)$

  Q. What does it mean when H is empty (i.e., no hypotheses)?

  A. $\boxed{\vdash G} \equiv \boxed{\textit{true} \vdash G}$ [Why not $\boxed{\vdash G} \equiv \boxed{\textit{false} \vdash G}$?]

PO of Invariant Preservation: Sketch

- Here is a sketch of the PO/VC rule for *invariant preservation*:

Axioms
Invariants Satisfied at ***Pre-State***
Guards of the Event
⊢
Invariants Satisfied at ***Post-State***

INV

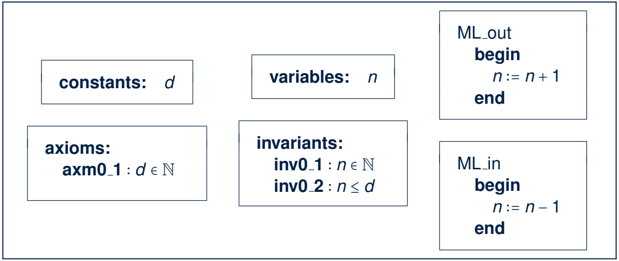
- Informally, this is what the above PO/VC *requires to prove*:

Assuming **all** axioms, invariants, and the event's guards hold at the ***pre-state***,

after the ***state transition*** is made by the event,

all invariants hold at the ***post-state***.

PO of Invariant Preservation: Components



constants: d

variables: n

ML_out
begin
$n := n + 1$
end

axioms:
axm0_1 : $d \in \mathbb{N}$

invariants:
inv0_1 : $n \in \mathbb{N}$
inv0_2 : $n \leq d$

ML_in
begin
$n := n - 1$
end

- c: list of ***constants*** $\langle d \rangle$
- $A(c)$: list of ***axioms*** $\langle \textbf{axm0\_1} \rangle$
- ***v*** and ***v'***: list of ***variables*** in ***pre***- and ***post***-states $\boldsymbol{v} \mathrel{\widehat{=}} \langle n \rangle$, $\boldsymbol{v'} \mathrel{\widehat{=}} \langle n' \rangle$
- $I(c, v)$: list of ***invariants*** $\langle \textbf{inv0\_1}, \textbf{inv0\_2} \rangle$
- $G(c, v)$: the ***event***'s list of guards

  $G(\langle d \rangle, \langle n \rangle)$ of $ML\_out \mathrel{\widehat{=}} \langle \textbf{\textit{true}} \rangle$, $G(\langle d \rangle, \langle n \rangle)$ of $ML\_in \mathrel{\widehat{=}} \langle \textbf{\textit{true}} \rangle$
- $E(c, v)$: effect of the ***event***'s actions i.t.o. what variable values ***become***

  $E(\langle d \rangle, \langle n \rangle)$ of $ML\_out \mathrel{\widehat{=}} \langle \boldsymbol{n} + 1 \rangle$, $E(\langle d \rangle, \langle n \rangle)$ of $ML\_out \mathrel{\widehat{=}} \langle \boldsymbol{n} - 1 \rangle$
- $v' = E(c, v)$: ***before-after predicate*** formalizing E's actions

  BAP of ML_out: $\langle \boldsymbol{n'} \rangle = \langle \boldsymbol{n} + 1 \rangle$, BAP of ML_in: $\langle \boldsymbol{n'} \rangle = \langle \boldsymbol{n} - 1 \rangle$

Rule of Invariant Preservation: Sequents

- Based on the components (c, $A(c)$, v, $I(c,v)$, $E(c,v)$), we are able to formally state the ***PO/VC Rule of Invariant Preservation***:

$$\boxed{\begin{array}{l} A(c) \\ I(c, \mathbf{v}) \\ G(c, \mathbf{v}) \\ \vdash \\ I_i(c, \mathbf{E(c, v)}) \end{array}} \quad \underline{\text{INV}} \qquad \text{where } I_i \text{ denotes a single invariant condition}$$

  - Accordingly, how many ***sequents*** to be proved? [# events × # invariants]
  - We have <u>two</u> ***sequents*** generated for ***event*** *ML_out* of model m_0:

$$\boxed{\begin{array}{l} d \in \mathbb{N} \\ n \in \mathbb{N} \\ n \leq d \\ \vdash \\ n + 1 \in \mathbb{N} \end{array}} \quad \underline{\text{ML\_out}/\textbf{inv0\_1}/\text{INV}} \qquad \boxed{\begin{array}{l} d \in \mathbb{N} \\ n \in \mathbb{N} \\ n \leq d \\ \vdash \\ n + 1 \leq d \end{array}} \quad \underline{\text{ML\_out}/\textbf{inv0\_2}/\text{INV}}$$

  <u>Exercise</u>. Write the ***POs of invariant preservation*** for event *ML_in*.

- Before claiming that a ***model*** is *correct*, outstanding ***sequents*** associated with <u>all</u> ***POs*** must be <u>proved</u>/<u>discharged</u>.

Inference Rules: Syntax and Semantics

- An *inference rule (IR)* has the following form:

$$\frac{A}{C}\ \mathbf{L}$$

Formally: $A \Rightarrow C$ is an axiom.

Informally: To prove C, it is sufficient to prove A instead.

Informally: C is the case, assuming that A is the case.

  - L is a name label for referencing the ***inference rule*** in proofs.
  - ***A*** is a **set** of sequents known as ***antecedents*** of rule L.
  - ***C*** is a **single** sequent known as ***consequent*** of rule L.

- Let's consider ***inference rules (IRs)*** with two different flavours:

$$\frac{H1 \ \vdash \ G}{H1, H2 \ \vdash \ G}\ \mathbf{MON} \qquad \frac{}{n \in \mathbb{N} \ \vdash \ n+1 \in \mathbb{N}}\ \mathbf{P2}$$

  - IR **MON**: To prove $H1, H2 \ \vdash \ G$, it suffices to prove $H1 \ \vdash \ G$ instead.
  - IR **P2**: $n \in \mathbb{N} \ \vdash \ n+1 \in \mathbb{N}$ is an ***axiom***.

[proved automatically without further justifications]

Proof of Sequent: Steps and Structure

- To prove the following sequent (related to ***invariant preservation***):

$$\boxed{\begin{array}{l} d \in \mathbb{N} \\ n \in \mathbb{N} \\ n \leq d \\ \vdash \\ n + 1 \in \mathbb{N} \end{array}} \quad \underline{\text{ML\_out/}\textbf{inv0\_1}\text{/INV}}$$

1. Apply a ***inference rule***, which ***transforms*** some “outstanding” **sequent** to <u>one</u> or <u>more</u> other **sequents** to be proved instead.
2. Keep applying ***inference rules*** until <u>**all**</u> ***transformed*** **sequents** are ***axioms*** that do <u>**not**</u> require any further justifications.

- Here is a *formal proof* of ML_out/**inv0_1**/INV, by applying IRs **MON** and **P2**:

$$\boxed{\begin{array}{l} d \in \mathbb{N} \\ n \in \mathbb{N} \\ n \leq d \\ \vdash \\ n + 1 \in \mathbb{N} \end{array}} \quad \textbf{MON} \quad \boxed{\begin{array}{l} n \in \mathbb{N} \\ \vdash \\ n + 1 \in \mathbb{N} \end{array}} \quad \textbf{P2}$$

Example Inference Rules (1)

$$\frac{}{\vdash\ 0 \in \mathbb{N}}\ \textbf{P1}$$

1st Peano axiom: 0 is a natural number.

$$\frac{}{n \in \mathbb{N}\ \vdash\ n+1 \in \mathbb{N}}\ \textbf{P2}$$

2nd Peano axiom: $n+1$ is a natural number, assuming that n is a natural number.

$$\frac{}{0 < n\ \vdash\ n-1 \in \mathbb{N}}\ \textbf{P2'}$$

$n-1$ is a natural number, assuming that n is positive.

$$\frac{}{n \in \mathbb{N}\ \vdash\ 0 \leq n}\ \textbf{P3}$$

3rd Peano axiom: n is non-negative, assuming that n is a natural number.

Example Inference Rules (2)

$$\frac{}{n < m \;\vdash\; n + 1 \leq m}\ \textbf{INC}$$

$n + 1$ is less than or equal to m,
assuming that n is strictly less than m.

$$\frac{}{n \leq m \;\vdash\; n - 1 < m}\ \textbf{DEC}$$

$n - 1$ is strictly less than m,
assuming that n is less than or equal to m.

Example Inference Rules (3)

$$\frac{H1 \vdash G}{H1, H2 \vdash G}\ \textbf{MON}$$

To prove a goal under certain hypotheses,
it suffices to prove it under less hypotheses.

$$\frac{H, P \vdash R \qquad H, Q \vdash R}{H, P \lor Q \vdash R}\ \textbf{OR\_L}$$

Proof by Cases:
To prove a goal under a disjunctive assumption,
it suffices to prove **independently**
the same goal, twice, under each disjunct.

$$\frac{H \vdash P}{H \vdash P \lor Q}\ \textbf{OR\_R1}$$

To prove a disjunction,
it suffices to prove the left disjunct.

$$\frac{H \vdash Q}{H \vdash P \lor Q}\ \textbf{OR\_R2}$$

To prove a disjunction,
it suffices to prove the right disjunct.

Revisiting Design of Events: *ML_out*

- Recall that we already proved ***PO*** $\boxed{\text{ML\_out/}\textbf{inv0\_1}\text{/INV}}$:

$$\boxed{\begin{array}{l} d \in \mathbb{N} \\ n \in \mathbb{N} \\ n \leq d \\ \vdash \\ n + 1 \in \mathbb{N} \end{array}} \quad \textbf{MON} \quad \boxed{\begin{array}{l} n \in \mathbb{N} \\ \vdash \\ n + 1 \in \mathbb{N} \end{array}} \quad \textbf{P2}$$

∴ ***ML_out/inv0_1/INV*** succeeds in being discharged.

- How about the other ***PO*** $\boxed{\text{ML\_out/}\textbf{inv0\_2}\text{/INV}}$ for the same event?

$$\boxed{\begin{array}{l} d \in \mathbb{N} \\ n \in \mathbb{N} \\ n \leq d \\ \vdash \\ n + 1 \leq d \end{array}} \quad \textbf{MON} \quad \boxed{\begin{array}{l} n \leq d \\ \vdash \\ n + 1 \leq d \end{array}} \quad \textbf{?}$$

∴ ***ML_out/inv0_2/INV*** fails to be discharged.

Revisiting Design of Events: *ML_in*

- How about the ***PO*** ML_in/**inv0_1**/INV for *ML_in*:

$$\boxed{\begin{array}{l} d \in \mathbb{N} \\ n \in \mathbb{N} \\ n \leq d \\ \vdash \\ n - 1 \in \mathbb{N} \end{array}} \quad \textbf{MON} \quad \boxed{\begin{array}{l} n \in \mathbb{N} \\ \vdash \\ n - 1 \in \mathbb{N} \end{array}} \quad \textbf{?}$$

∴ ***ML_in/inv0_1/INV*** <u>fails</u> to be discharged.

- How about the other ***PO*** ML_in/**inv0_2**/INV for the same event?

$$\boxed{\begin{array}{l} d \in \mathbb{N} \\ n \in \mathbb{N} \\ n \leq d \\ \vdash \\ n - 1 \leq d \end{array}} \quad \textbf{MON} \quad \boxed{\begin{array}{l} n \leq d \\ \vdash \\ n - 1 < d \lor n - 1 = d \end{array}} \quad \textbf{OR\_1} \quad \boxed{\begin{array}{l} n \leq d \\ \vdash \\ n - 1 < d \end{array}} \quad \textbf{DEC}$$

∴ ***ML_in/inv0_2/INV*** <u>succeeds</u> in being discharged.

Fixing the Design of Events

- Proofs of ***ML_out/inv0_2/INV*** and ***ML_in/inv0_1/INV*** fail due to the two events being *enabled when they should not* .
- Having this feedback, we add proper ***guards*** to *ML_out* and *ML_in*:

```
ML_out
  when
    $n < d$
  then
    $n := n + 1$
  end
```

```
ML_in
  when
    $n > 0$
  then
    $n := n - 1$
  end
```

- Having changed both events, updated ***sequents*** will be generated for the PO/VC rule of ***invariant preservation***.
- All ***sequents*** $(\{ML\_out,\ ML\_in\} \times \{\textbf{inv0\_1},\ \textbf{inv0\_2}\})$ now ***provable***?

Revisiting Fixed Design of Events: *ML_out*

- How about the ***PO*** ML_out/**inv0_1**/INV for *ML_out*:

$$\boxed{\begin{array}{l} d \in \mathbb{N} \\ n \in \mathbb{N} \\ n \leq d \\ n < d \\ \vdash \\ n + 1 \in \mathbb{N} \end{array}} \quad \textbf{MON} \quad \boxed{\begin{array}{l} n \in \mathbb{N} \\ \vdash \\ n + 1 \in \mathbb{N} \end{array}} \quad \textbf{P2}$$

∴ ***ML_out/inv0_1/INV*** still <u>succeeds</u> in being discharged!

- How about the other ***PO*** ML_out/**inv0_2**/INV for the same event?

$$\boxed{\begin{array}{l} d \in \mathbb{N} \\ n \in \mathbb{N} \\ n \leq d \\ n < d \\ \vdash \\ n + 1 \leq d \end{array}} \quad \textbf{MON} \quad \boxed{\begin{array}{l} n < d \\ \vdash \\ n + 1 \leq d \end{array}} \quad \textbf{INC}$$

∴ ***ML_out/inv0_2/INV*** now <u>succeeds</u> in being discharged!

Revisiting Fixed Design of Events: *ML_in*

- How about the ***PO*** $\boxed{\text{ML\_in/}\textbf{inv0\_1}\text{/INV}}$ for *ML_in*:

$$\boxed{\begin{array}{l} d \in \mathbb{N} \\ n \in \mathbb{N} \\ n \le d \\ n > 0 \\ \vdash \\ n - 1 \in \mathbb{N} \end{array}} \quad \textbf{MON} \quad \boxed{\begin{array}{l} n > 0 \\ \vdash \\ n - 1 \in \mathbb{N} \end{array}} \quad \textbf{P2'}$$

∴ ***ML_in/inv0_1/INV*** now <u>succeeds</u> in being discharged!

- How about the other ***PO*** $\boxed{\text{ML\_in/}\textbf{inv0\_2}\text{/INV}}$ for the same event?

$$\boxed{\begin{array}{l} d \in \mathbb{N} \\ n \in \mathbb{N} \\ n \le d \\ n > 0 \\ \vdash \\ n - 1 \le d \end{array}} \quad \textbf{MON} \quad \boxed{\begin{array}{l} n \le d \\ \vdash \\ n - 1 < d \lor n - 1 = d \end{array}} \quad \textbf{OR\_1} \quad \boxed{\begin{array}{l} n \le d \\ \vdash \\ n - 1 < d \end{array}} \quad \textbf{DEC}$$

∴ ***ML_in/inv0_2/INV*** still <u>succeeds</u> in being discharged!

Initializing the Abstract System m_0

- Discharging the four ***sequents*** proved that both ***invariant*** conditions are ***preserved*** between occurrences/interleavings of ***events*** `ML_out` and `ML_in`.
- But how are the ***invariants established*** in the first place?

  Analogy. Proving P via ***mathematical induction***, two cases to prove:
  - $P(1), P(2), \ldots$ [***base*** cases ≈ ***establishing*** inv.]
  - $P(n) \Rightarrow P(n+1)$ [***inductive*** cases ≈ ***preserving*** inv.]
- Therefore, we specify how the ***ASM*** 's ***initial state*** looks like:

```
init
   begin
      $n := 0$
   end
```

✓ The IB compound, once ***initialized***, has no cars.

✓ Initialization always possible: guard is ***true***.

✓ There is no ***pre-state*** for *init*.

∴ The RHS of := must not involve variables.

∴ The RHS of := may only involve constants.

✓ There is only the ***post-state*** for *init*.

∴ Before-***After Predicate***: $n' = 0$

PO of Invariant Establishment

```
init
  begin
    $n := 0$
  end
```

✓ An ***reactive system***, once ***initialized***, should never terminate.

✓ Event *init* cannot "preserve" the ***invariants***.

$\because$ State before its occurrence (***pre-state***) does not exist.

✓ Event *init* only required to ***establish*** invariants for the first time

- A new formal component is needed:
  - $K(c)$: effect of ***init***'s actions i.t.o. what variable values ***become***

    e.g., $K(\langle d \rangle)$ of *init* $\widehat{=}$ $\langle 0 \rangle$
  - $v' = K(c)$: ***before-after predicate*** formalizing *init*'s actions

    e.g., BAP of *init*: $\langle \boldsymbol{n'} \rangle = \langle 0 \rangle$
- Accordingly, PO of ***invariant establishment*** is formulated as a ***sequent***:

$$\boxed{\begin{array}{l} \text{Axioms} \\ \vdash \\ \textit{Invariants}\ \text{Satisfied at}\ \textit{Post-State} \end{array}}\ \ \underline{\text{INV}} \qquad \boxed{\begin{array}{l} A(c) \\ \vdash \\ I_i(c, \mathbf{K(c)}) \end{array}}\ \ \underline{\text{INV}}$$

Discharging PO of Invariant Establishment

- How many ***sequents*** to be proved? [# invariants]
- We have <u>two</u> ***sequents*** generated for ***event*** *init* of model m_0:

$$\boxed{\begin{array}{l} d \in \mathbb{N} \\ \vdash \\ 0 \in \mathbb{N} \end{array}} \quad \underline{\text{init/}\textbf{inv0\_1}\text{/INV}} \qquad \boxed{\begin{array}{l} d \in \mathbb{N} \\ \vdash \\ 0 \leq d \end{array}} \quad \underline{\text{init/}\textbf{inv0\_2}\text{/INV}}$$

- Can we discharge the ***PO*** $\boxed{\text{init/}\textbf{inv0\_1}\text{/INV}}$?

$$\boxed{\begin{array}{l} d \in \mathbb{N} \\ \vdash \\ 0 \in \mathbb{N} \end{array}} \quad \textbf{MON} \quad \boxed{\begin{array}{l} \\ \vdash \\ 0 \in \mathbb{N} \end{array}} \quad \textbf{P1}$$

∴ ***init/inv0_1/INV***
<u>succeeds</u> in being discharged.

- Can we discharge the ***PO*** $\boxed{\text{init/}\textbf{inv0\_2}\text{/INV}}$?

$$\boxed{\begin{array}{l} d \in \mathbb{N} \\ \vdash \\ 0 \leq d \end{array}} \quad \textbf{P3}$$

∴ ***init/inv0_2/INV***
<u>succeeds</u> in being discharged.

System Property: Deadlock Freedom

- So far we have proved that our initial model m_0 is s.t. all ***invariant conditions*** are:
  - Established when system is first initialized via *init*
  - Preserved whenevner there is a ***state transition***

    (via an enabled event: *ML_out* or *ML_in*)
- However, whenever ***event occurrences*** are conditional (i.e., ***guards*** stronger than ***true***), there is a possibility of ***deadlock*** :
  - A state where ***guards*** of all events evaluate to ***false***
  - When a ***deadlock*** happens, none of the ***events*** is ***enabled***.

    ⇒ The system is blocked and not reactive anymore!
- We express this ***non-blocking*** property as a new requirement:

REQ4	Once started, the system should work for ever.

PO of Deadlock Freedom (1)

- Recall some of the formal components we discussed:
  - c: list of ***constants*** $\langle d \rangle$
  - $A(c)$: list of ***axioms*** $\langle \textbf{axm0\_1} \rangle$
  - ***v*** and ***v'***: list of ***variables*** in ***pre***- and ***post***-states $\boldsymbol{v} \mathrel{\widehat{=}} \langle n \rangle$, $\boldsymbol{v'} \mathrel{\widehat{=}} \langle n' \rangle$
  - $I(c, v)$: list of ***invariants*** $\langle \textbf{inv0\_1}, \textbf{inv0\_2} \rangle$
  - $G(c, v)$: the event's list of ***guards***

$$G(\langle d \rangle, \langle n \rangle) \text{ of } \mathit{ML\_out} \mathrel{\widehat{=}} \langle n < d \rangle,\ G(\langle d \rangle, \langle n \rangle) \text{ of } \mathit{ML\_in} \mathrel{\widehat{=}} \langle n > 0 \rangle$$

- A system is ***deadlock-free*** if <u>at least one</u> of its ***events*** is ***enabled***:

Axioms
Invariants Satisfied at ***Pre-State***
⊢
Disjunction of the guards satisfied at ***Pre-State***

<u>DLF</u>

$$\boxed{\begin{array}{l} A(c) \\ I(c, \boldsymbol{v}) \\ \vdash \\ G_1(c, \boldsymbol{v}) \lor \cdots \lor G_m(c, \boldsymbol{v}) \end{array}} \quad \underline{\text{DLF}}$$

To prove about deadlock freedom

- An event's effect of state transition is <u>**not**</u> relevant.
- Instead, the evaluation of <u>all</u> events' ***guards*** at the ***pre-state*** is relevant.

PO of Deadlock Freedom (2)

- ***Deadlock freedom*** is not necessarily a desired property.

  ⇒ When it is (like m_0), then the generated ***sequents*** must be discharged.
- Applying the PO of ***deadlock freedom*** to the initial model m_0:

$$\boxed{\begin{array}{l} A(c) \\ I(c, \mathbf{v}) \\ \vdash \\ G_1(c, \mathbf{v}) \lor \cdots \lor G_m(c, \mathbf{v}) \end{array}} \quad \underline{\text{DLF}} \quad \boxed{\begin{array}{l} d \in \mathbb{N} \\ n \in \mathbb{N} \\ n \le d \\ \vdash \\ n < d \lor n > 0 \end{array}} \quad \underline{\text{DLF}}$$

  Our bridge controller being ***deadlock-free*** means that cars can ***always*** enter (via *ML_out*) or leave (via *ML_in*) the island-bridge compound.

- Can we formally discharge this ***PO*** for our ***initial model*** m_0?

Example Inference Rules (4)

$$\frac{}{H, P \vdash P}\ \textbf{HYP}$$

A goal is proved if it can be assumed.

$$\frac{}{\perp \vdash P}\ \textbf{FALSE\_L}$$

Assuming ***false*** ($\perp$),
anything can be proved.

$$\frac{}{P \vdash \top}\ \textbf{TRUE\_R}$$

true ($\top$) is proved,
regardless of the assumption.

$$\frac{}{P \vdash E = E}\ \textbf{EQ}$$

An expression being equal to itself is proved,
regardless of the assumption.

Example Inference Rules (5)

$$\frac{H(\boldsymbol{F}), \boldsymbol{E} = \boldsymbol{F} \vdash P(\boldsymbol{F})}{H(\boldsymbol{E}), \boldsymbol{E} = \boldsymbol{F} \vdash P(\boldsymbol{E})} \quad \textbf{EQ\_LR}$$

To prove a goal $P(\boldsymbol{E})$ assuming $H(\boldsymbol{E})$,
where both P and H depend on expression $\boldsymbol{E}$,
it suffices to prove $P(\boldsymbol{F})$ assuming $H(\boldsymbol{F})$,
where both P and H depend on expresion $\boldsymbol{F}$,
given that $\boldsymbol{E}$ is equal to $\boldsymbol{F}$.

$$\frac{H(\boldsymbol{E}), \boldsymbol{E} = \boldsymbol{F} \vdash P(\boldsymbol{E})}{H(\boldsymbol{F}), \boldsymbol{E} = \boldsymbol{F} \vdash P(\boldsymbol{F})} \quad \textbf{EQ\_RL}$$

To prove a goal $P(\boldsymbol{F})$ assuming $H(\boldsymbol{F})$,
where both P and H depend on expression $\boldsymbol{F}$,
it suffices to prove $P(\boldsymbol{E})$ assuming $H(\boldsymbol{E})$,
where both P and H depend on expresion $\boldsymbol{E}$,
given that $\boldsymbol{E}$ is equal to $\boldsymbol{F}$.

Discharging PO of DLF: Exercise

$$\boxed{\begin{array}{l} A(c) \\ I(c, \boldsymbol{v}) \\ \vdash \\ G_1(c, \boldsymbol{v}) \lor \cdots \lor G_m(c, \boldsymbol{v}) \end{array}} \quad \underline{\text{DLF}} \quad \boxed{\begin{array}{l} d \in \mathbb{N} \\ n \in \mathbb{N} \\ n \leq d \\ \vdash \\ n < d \lor n > 0 \end{array}} \quad ??$$

Discharging PO of DLF: First Attempt

$$\boxed{\begin{array}{l} d \in \mathbb{N} \\ n \in \mathbb{N} \\ n \le d \\ \vdash \\ n < d \lor n > 0 \end{array}}$$

$\equiv$

$$\boxed{\begin{array}{l} d\in\mathbb{N}\\ n\in\mathbb{N}\\ n<d\lor n=d\\ \vdash\\ n<d\lor n>0\end{array}} \textbf{ MON } \boxed{\begin{array}{l} n<d\lor n=d\\ \vdash\\ n<d\lor n>0\end{array}} \textbf{ OR\_L } \left\{\begin{array}{l} \boxed{\begin{array}{l}n<d\\ \vdash\\ n<d\lor n>0\end{array}} \textbf{ OR\_R1 } \boxed{\begin{array}{l}n<d\\ \vdash\\ n<d\end{array}} \textbf{ HYP} \\ \boxed{\begin{array}{l}n=d\\ \vdash\\ n<d\lor n>0\end{array}} \textbf{ EQ\_LR, MON } \boxed{\begin{array}{l} \\ \vdash\\ d<d\lor d>0\end{array}} \textbf{ OR\_R2 } \boxed{\begin{array}{l} \\ \vdash\\ d>0\end{array}} \textbf{ ?}\end{array}\right.$$

Why Did the DLF PO Fail to Discharge?

- In our first attempt, proof of the 2nd case failed: $\vdash \ d > 0$
- This ***unprovable*** sequent gave us a good hint:
  - For the model under consideration (m_0) to be ***deadlock-free***, it is required that $d > 0$. [$\geq$ 1 car allowed in the IB compound]
  - But current ***specification*** of m_0 ***not*** strong enough to entail this:
    - $\neg(d > 0) \equiv d \leq 0$ is possible for the current model
    - Given **axm0_1** : $d \in \mathbb{N}$

    $\Rightarrow$ $d = 0$ is allowed by m_0 which causes a ***deadlock***.
- Recall the *init* event and the two ***guarded*** events:

```
init
  begin
    $n := 0$
  end
```

```
ML_out
  when
    $n < d$
  then
    $n := n + 1$
  end
```

```
ML_in
  when
    $n > 0$
  then
    $n := n - 1$
  end
```

When $d = 0$, the disjunction of guards evaluates to ***false***: $0 < 0 \lor 0 > 0$
$\Rightarrow$ As soon as the system is initialized, it ***deadlocks immediately***

as no car can either enter or leave the IR compound!!

Fixing the Context of Initial Model

- Having understood the failed proof, we add a proper ***axiom*** to m_0:

axioms:
  axm0_2 : $d > 0$

- We have effectively elaborated on **REQ2**:

REQ2	The number of cars on bridge and island is limited but positive.

- Having changed the context, an updated ***sequent*** will be generated for the PO/VC rule of ***deadlock freedom***.
- Is this new sequent now ***provable***?

Discharging PO of DLF: Second Attempt

$$\boxed{\begin{array}{l} d \in \mathbb{N} \\ d > 0 \\ n \in \mathbb{N} \\ n \leq d \\ \vdash \\ n < d \lor n > 0 \end{array}}$$

$$\equiv$$

$$\boxed{\begin{array}{l} d \in \mathbb{N} \\ d > 0 \\ n \in \mathbb{N} \\ n < d \lor n = d \\ \vdash \\ n < d \lor n > 0 \end{array}} \ \textbf{MON} \ \boxed{\begin{array}{l} d > 0 \\ n < d \lor n = d \\ \vdash \\ n < d \lor n > 0 \end{array}} \ \textbf{OR\_L} \left\{ \begin{array}{l} \boxed{\begin{array}{l} d > 0 \\ n < d \\ \vdash \\ n < d \lor n > 0 \end{array}} \ \textbf{OR\_R1} \ \boxed{\begin{array}{l} d > 0 \\ n < d \\ \vdash \\ n < d \end{array}} \ \textbf{HYP} \\ \boxed{\begin{array}{l} d > 0 \\ n = d \\ \vdash \\ n < d \lor n > 0 \end{array}} \ \textbf{EQ\_LR, MON} \ \boxed{\begin{array}{l} d > 0 \\ \vdash \\ d < d \lor d > 0 \end{array}} \ \textbf{OR\_R2} \ \boxed{\begin{array}{l} d > 0 \\ \vdash \\ d > 0 \end{array}} \ \textbf{HYP} \end{array} \right.$$

Initial Model: Summary

- The final version of our ***initial model*** m_0 is ***provably correct*** w.r.t.:
  - Establishment of ***Invariants***
  - Preservation of ***Invariants***
  - ***Deadlock*** Freedom
- Here is the final ***specification*** of m_0:

constants: d

variables: n

```
axioms:
    axm0_1 : $d \in \mathbb{N}$
    axm0_2 : $d > 0$
```

```
invariants:
    inv0_1 : $n \in \mathbb{N}$
    inv0_2 : $n \leq d$
```

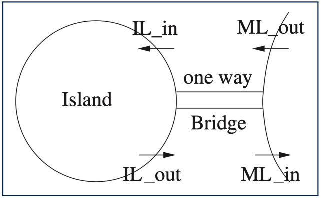
```
init
    begin
        $n := 0$
    end
```

```
ML_out
    when
        $n < d$
    then
        $n := n + 1$
    end
```

```
ML_in
    when
        $n > 0$
    then
        $n := n - 1$
    end
```

Model m_1: "More Concrete" Abstraction

- First ***refinement*** has a more ***concrete*** perception of the bridge controller:
  - We "**zoom in**" by observing the system from **closer to the ground**, so that the island-bridge compound is split into:
    - the island
    - the (one-way) bridge



  - Nonetheless, traffic lights and sensors remain ***abstracted*** away!
- That is, we focus on these two ***requirement***:

REQ1	The system is controlling cars on a bridge connecting the mainland to an island.
REQ3	The bridge is one-way or the other, not both at the same time.

- We are ***obliged to prove*** this ***added concreteness*** is ***consistent*** with m_0.

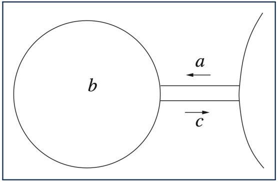
Model m_1: Refined State Space

1. The **static** part is the same as m_0's:

constants: d

axioms:
axm0_1 : $d \in \mathbb{N}$
axm0_2 : $d > 0$

2. The **dynamic** part of the ***concrete state*** consists of three ***variables***:



- ***a***: number of cars on the bridge, heading to the island
- ***b***: number of cars on the island
- ***c***: number of cars on the bridge, heading to the mainland

variables: a, b, c

invariants:
inv1_1 : $a \in \mathbb{N}$
inv1_2 : $b \in \mathbb{N}$
inv1_3 : $c \in \mathbb{N}$
inv1_4 : *??*
inv1_5 : *??*

✓ **inv1_1**, **inv1_2**, **inv1_3** are ***typing*** constraints.

✓ **inv1_4** ***links***/***glues*** the ***abstract*** and ***concrete*** states.

✓ **inv1_5** specifies that the bridge is one-way.

Model m_1: State Transitions via Events

- The system acts as an **ABSTRACT STATE MACHINE (ASM)**: it ***evolves*** as ***actions of enabled events*** change values of variables, subject to ***invariants***.
- We first consider the "old" ***events*** already existing in m_0.
- ***Concrete/Refined*** version of ***event*** *ML_out*:

```
ML_out
  when
    ??
  then
    $a := a + 1$
  end
```

  - Meaning of *ML_out* is ***refined***:
    a car exits mainland (getting on the bridge).
  - *ML_out* ***enabled*** only when:
    - the bridge's current traffic flows to the island
    - number of cars on both the bridge and the island is limited
- ***Concrete/Refined*** version of ***event*** *ML_in*:

```
ML_in
  when
    ??
  then
    $c := c - 1$
  end
```

  - Meaning of *ML_in* is ***refined***:
    a car enters mainland (getting off the bridge).
  - *ML_in* ***enabled*** only when:

    there is some car on the bridge heading to the mainland.

Model m_1: Actions vs. Before-After Predicates

- Consider the ***concrete***/***refined*** version of ***actions*** of m_0's two events:

Events

```
ML_in
  when
    $0 < c$
  then
    $c := c - 1$
  end
```

```
ML_out
  when
    $a + b < d$
    $c = 0$
  then
    $a := a + 1$
  end
```

Before–after predicates

$$a' = a \ \wedge \ b' = b \wedge c' = c - 1$$

$$a' = a + 1 \ \wedge \ b' = b \wedge c' = c$$

  - An event's ***actions*** are a **specification**: "c becomes c - 1 after the transition".
  - The ***before-after predicate*** (***BAP***) "$c' = c - 1$" expresses that c' (the ***post-state*** value of c) is one less than c (the ***pre-state*** value of c).
  - Given that the ***concrete state*** consists of three variables:
    - An event's ***actions*** only specify those changing from ***pre***-state to ***post***-state. [e.g., $c' = c - 1$]
    - Other unmentioned variables have their ***post***-state values remain unchanged. [e.g., $a' = a \wedge b' = b$]

- When we express ***proof obligations*** (***POs***) associated with ***events***, we use ***BAP***.

States & Invariants: Abstract vs. Concrete

- m_0 refines m_1 by introducing more ***variables***:
  - ***Abstract*** State (of m_0 being refined):
  - ***Concrete*** State (of the refinement model m_1):

variables: n

variables: a, b, c

- Accordingly, ***invariants*** may involve different ***states***:
  - ***Abstract*** Invariants (involving the ***abstract*** state only):
  - ***Concrete*** Invariants (involving at least the ***concrete*** state):

invariants:
inv0_1 : $n \in \mathbb{N}$
inv0_2 : $n \leq d$

invariants:
inv1_1 : $a \in \mathbb{N}$
inv1_2 : $b \in \mathbb{N}$
inv1_3 : $c \in \mathbb{N}$
inv1_4 : $a + b + c = n$
inv1_5 : $a = 0 \lor c = 0$

Events: Abstract vs. Concrete

- When an ***event*** exists in both models m_0 and m_1, there are two versions of it:
  - The ***abstract*** version modifies the ***abstract*** state.

```
(abstract_)ML_out
  when
    $n < d$
  then
    $a := n := n + 1$
  end
```

```
(abstract_)ML_in
  when
    $n > 0$
  then
    $n := n - 1$
  end
```

  - The ***concrete*** version modifies the ***concrete*** state.

```
(concrete_)ML_out
  when
    $a + b < d$
    $c = 0$
  then
    $a := a + 1$
  end
```

```
(concrete_)ML_in
  when
    $c > 0$
  then
    $c := c - 1$
  end
```

- A ***new event*** may **only** exist in m_1 (the ***concrete*** model): we will deal with this kind of events later, separately from "redefined/overridden" events.

PO of Refinement: Components (1)

constants: d

axioms:
- **axm0_1** : $d \in \mathbb{N}$
- **axm0_2** : $d > 0$

variables: a, b, c

invariants:
- **inv1_1** : $a \in \mathbb{N}$
- **inv1_2** : $b \in \mathbb{N}$
- **inv1_3** : $c \in \mathbb{N}$
- **inv1_4** : $a + b + c = n$
- **inv1_5** : $a = 0 \lor c = 0$

```
ML_out
  when
    $a + b < d$
    $c = 0$
  then
    $a := a + 1$
  end
```

```
ML_in
  when
    $c > 0$
  then
    $c := c - 1$
  end
```

- c: list of ***constants*** $\langle d \rangle$
- $A(c)$: list of ***axioms*** $\langle$**axm0_1**$\rangle$
- v and v': ***abstract variables*** in pre- & post-states $v \;\widehat{=}\; \langle n \rangle,\ v' \;\widehat{=}\; \langle n \rangle$
- w and w': ***concrete variables*** in pre- & post-states $w \;\widehat{=}\; \langle a, b, c \rangle,\ w' \;\widehat{=}\; \langle a', b', c' \rangle$
- $I(c, v)$: list of ***abstract invariants*** $\langle$**inv0_1**, **inv0_2**$\rangle$
- $J(c, v, w)$: list of ***concrete invariants*** $\langle$**inv1_1**, **inv1_2**, **inv1_3**, **inv1_4**, **inv1_5**$\rangle$

PO of Refinement: Components (2)

```
constants: $d$

axioms:
    axm0_1 : $d \in \mathbb{N}$
    axm0_2 : $d > 0$

variables: $a, b, c$

invariants:
    inv1_1 : $a \in \mathbb{N}$
    inv1_2 : $b \in \mathbb{N}$
    inv1_3 : $c \in \mathbb{N}$
    inv1_4 : $a + b + c = n$
    inv1_5 : $a = 0 \lor c = 0$

ML_out
  when
    $a + b < d$
    $c = 0$
  then
    $a := a + 1$
  end

ML_in
  when
    $c > 0$
  then
    $c := c - 1$
  end
```

- $G(c, v)$: list of guards of the ***abstract event***

  $$G(\langle d \rangle, \langle n \rangle) \text{ of } ML\_out \mathrel{\widehat{=}} \langle n < d \rangle,\ G(c, v) \text{ of } ML\_in \mathrel{\widehat{=}} \langle n > 0 \rangle$$

- $H(c, w)$: list of guards of the ***concrete event***

  $$H(\langle d \rangle, \langle a, b, c \rangle) \text{ of } ML\_out \mathrel{\widehat{=}} \langle a + b < d, c = 0 \rangle,\ H(c, w) \text{ of } ML\_in \mathrel{\widehat{=}} \langle c > 0 \rangle$$

PO of Refinement: Components (3)

constants: d

axioms:
- **axm0_1** : $d \in \mathbb{N}$
- **axm0_2** : $d > 0$

variables: a, b, c

invariants:
- **inv1_1** : $a \in \mathbb{N}$
- **inv1_2** : $b \in \mathbb{N}$
- **inv1_3** : $c \in \mathbb{N}$
- **inv1_4** : $a + b + c = n$
- **inv1_5** : $a = 0 \vee c = 0$

```
ML_out
  when
    $a + b < d$
    $c = 0$
  then
    $a := a + 1$
  end
```

```
ML_in
  when
    $c > 0$
  then
    $c := c - 1$
  end
```

- $E(c, v)$: effect of the ***abstract event***'s actions i.t.o. what variable values **become**

$$E(\langle d \rangle, \langle n \rangle) \text{ of } \mathit{ML\_out} \mathrel{\widehat{=}} \langle \mathbf{n} + 1 \rangle,\ E(\langle d \rangle, \langle n \rangle) \text{ of } \mathit{ML\_out} \mathrel{\widehat{=}} \langle \mathbf{n} - 1 \rangle$$

- $F(c, w)$: effect of the ***concrete event***'s actions i.t.o. what variable values **become**

$$F(c, v) \text{ of } \mathit{ML\_out} \mathrel{\widehat{=}} \langle \mathbf{a} + 1, b, c \rangle,\ F(c, w) \text{ of } \mathit{ML\_out} \mathrel{\widehat{=}} \langle a, b, \mathbf{c} - 1 \rangle$$

Sketching PO of Refinement

The PO/VC rule for a *proper refinement* consists of two parts:

1. ***Guard Strengthening***

Axioms
Abstract Invariants Satisfied at Pre-State
Concrete Invariants Satisfied at Pre-State
Guards of the *Concrete Event*
⊢
Guards of the *Abstract Event*

GRD

- A ***concrete*** event is enabled if its ***abstract*** counterpart is enabled.
- A ***concrete*** transition always has an ***abstract*** counterpart.

2. ***Invariant Preservation***

Axioms
Abstract Invariants Satisfied at Pre-State
Concrete Invariants Satisfied at Pre-State
Guards of the *Concrete Event*
⊢
Concrete Invariants Satisfied at **Post**-State

INV

- A ***concrete*** event performs a ***transition*** on ***concrete*** states.
- This ***concrete*** state ***transition*** must be consistent with how its ***abstract*** counterpart performs a corresponding ***abstract transition***.

Note. ***Guard strengthening*** and ***invariant preservation*** are only applicable to events that might be ***enabled*** after the system is launched.
The special, non-guarded `init` event will be discussed separately later.

Refinement Rule: Guard Strengthening

- Based on the components, we are able to formally state the ***PO/VC Rule of Guard Strengthening for Refinement***:

$$\boxed{\begin{array}{l} A(c)\\ I(c, \mathbf{v})\\ J(c, \mathbf{v}, \mathbf{w})\\ H(c, \mathbf{w})\\ \vdash\\ G_i(c, \mathbf{v}) \end{array}} \quad \underline{\text{GRD}}$$

where G_i denotes a single ***guard*** condition of the ***abstract*** event

  - How many ***sequents*** to be proved? [# *abstract* guards]
  - For *ML_out*, only one ***abstract*** guard, so one ***sequent*** is generated :

$$\boxed{\begin{array}{lllll} d \in \mathbb{N} & d > 0 & & & \\ n \in \mathbb{N} & n \leq d & & & \\ a \in \mathbb{N} & b \in \mathbb{N} & c \in \mathbb{N} & a + b + c = n & a = 0 \lor c = 0 \\ a + b < d & c = 0 & & & \\ \vdash & & & & \\ n < d & & & & \end{array}} \quad \underline{\text{ML\_out/GRD}}$$

- **Exercise**. Write *ML_in*'s ***PO of Guard Strengthening for Refinement***.

PO Rule: Guard Strengthening of *ML_out*

$$\begin{array}{rl}
\textbf{axm0\_1} & \{\ d \in \mathbb{N} \\
\textbf{axm0\_2} & \{\ d > 0 \\
\textbf{inv0\_1} & \{\ n \in \mathbb{N} \\
\textbf{inv0\_2} & \{\ n \leq d \\
\textbf{inv1\_1} & \{\ a \in \mathbb{N} \\
\textbf{inv1\_2} & \{\ b \in \mathbb{N} \\
\textbf{inv1\_3} & \{\ c \in \mathbb{N} \\
\textbf{inv1\_4} & \{\ a + b + c = n \\
\textbf{inv1\_5} & \{\ a = 0 \lor c = 0 \\
\textit{Concrete}\text{ guards of } ML\_out & \left\{ \begin{array}{l} a + b < d \\ c = 0 \end{array} \right. \\
 & \vdash \\
\textit{Abstract}\text{ guards of } ML\_out & \{\ n < d
\end{array}$$

ML_out/GRD

PO Rule: Guard Strengthening of *ML_in*

$$
\begin{array}{rl}
\textbf{axm0\_1} & \{\ d \in \mathbb{N} \\
\textbf{axm0\_2} & \{\ d > 0 \\
\textbf{inv0\_1} & \{\ n \in \mathbb{N} \\
\textbf{inv0\_2} & \{\ n \leq d \\
\textbf{inv1\_1} & \{\ a \in \mathbb{N} \\
\textbf{inv1\_2} & \{\ b \in \mathbb{N} \\
\textbf{inv1\_3} & \{\ c \in \mathbb{N} \\
\textbf{inv1\_4} & \{\ a + b + c = n \\
\textbf{inv1\_5} & \{\ a = 0 \lor c = 0 \\
\textit{Concrete}\text{ guards of } \mathit{ML\_in} & \{\ c > 0 \\
 & \vdash \\
\textit{Abstract}\text{ guards of } \mathit{ML\_in} & \{\ n > 0
\end{array}
$$

ML_in/GRD

Proving Refinement: ML_out/GRD

$$\boxed{\begin{array}{l} d \in \mathbb{N} \\ d > 0 \\ n \in \mathbb{N} \\ n \leq d \\ a \in \mathbb{N} \\ b \in \mathbb{N} \\ c \in \mathbb{N} \\ a + b + c = n \\ a = 0 \lor c = 0 \\ a + b < d \\ c = 0 \\ \vdash \\ n < d \end{array}} \;\textbf{MON}\; \boxed{\begin{array}{l} a + b + c = n \\ a + b < d \\ c = 0 \\ \vdash \\ n < d \end{array}} \;\textbf{EQ\_LR, MON}\; \boxed{\begin{array}{l} a + b + 0 = n \\ a + b < d \\ \vdash \\ n < d \end{array}} \;\textbf{ARI}\; \boxed{\begin{array}{l} a + b = n \\ a + b < d \\ \vdash \\ n < d \end{array}} \;\textbf{EQ\_LR, MON}\; \boxed{\begin{array}{l} n < d \\ \vdash \\ n < d \end{array}} \;\textbf{HYP}$$

Proving Refinement: ML_in/GRD

$$
\boxed{\begin{array}{l} d \in \mathbb{N} \\ d > 0 \\ n \in \mathbb{N} \\ n \leq d \\ a \in \mathbb{N} \\ b \in \mathbb{N} \\ c \in \mathbb{N} \\ a + b + c = n \\ a = 0 \lor c = 0 \\ c > 0 \\ \vdash \\ n > 0 \end{array}}
\;\textbf{MON}\;
\boxed{\begin{array}{l} b \in \mathbb{N} \\ a + b + c = n \\ a = 0 \lor c = 0 \\ c > 0 \\ \vdash \\ n > 0 \end{array}}
\;\textbf{OR\_L}\;
\left\{
\begin{array}{l}
\boxed{\begin{array}{l} b \in \mathbb{N} \\ a + b + c = n \\ a = 0 \\ c > 0 \\ \vdash \\ n > 0 \end{array}}
\;\textbf{EQ\_LR, MON}\;
\boxed{\begin{array}{l} b \in \mathbb{N} \\ 0 + b + c = n \\ c > 0 \\ \vdash \\ n > 0 \end{array}}
\;\textbf{ARI}\;
\boxed{\begin{array}{l} b \in \mathbb{N} \\ b + c = n \\ c > 0 \\ \vdash \\ n > 0 \end{array}}
\;\textbf{ARI}\;
\boxed{\begin{array}{l} c \leq n \\ c > 0 \\ \vdash \\ n > 0 \end{array}}
\;\textbf{ARI}\;
\boxed{\begin{array}{l} n > 0 \\ \vdash \\ n > 0 \end{array}}
\;\textbf{HYP}
\\[2ex]
\boxed{\begin{array}{l} b \in \mathbb{N} \\ a + b + c = n \\ c = 0 \\ c > 0 \\ \vdash \\ n > 0 \end{array}}
\;\textbf{EQ\_LR}\;
\boxed{\begin{array}{l} b \in \mathbb{N} \\ a + b + 0 = n \\ c = 0 \\ 0 > 0 \\ \vdash \\ n > 0 \end{array}}
\;\textbf{EQ\_LR, MON}\;
\boxed{\begin{array}{l} 0 > 0 \\ \vdash \\ n > 0 \end{array}}
\;\textbf{ARI}\;
\boxed{\begin{array}{l} \bot \\ \vdash \\ n > 0 \end{array}}
\;\textbf{FALSE\_L}
\end{array}
\right.
$$

Refinement Rule: Invariant Preservation

- Based on the components, we are able to formally state the ***PO/VC Rule of Invariant Preservation for Refinement***:

$$\boxed{\begin{array}{l} A(c) \\ I(c, \mathbf{v}) \\ J(c, \mathbf{v}, \mathbf{w}) \\ H(c, \mathbf{w}) \\ \vdash \\ J_i(c, E(c, \mathbf{v}), F(c, \mathbf{w})) \end{array}} \quad \underline{\text{INV}}$$

where J_i denotes a single ***concrete invariant***

  - How many ***sequents*** to be proved? [# *concrete* evts × # *concrete* invariants]
  - Here are two (of the ten) ***sequents*** generated:

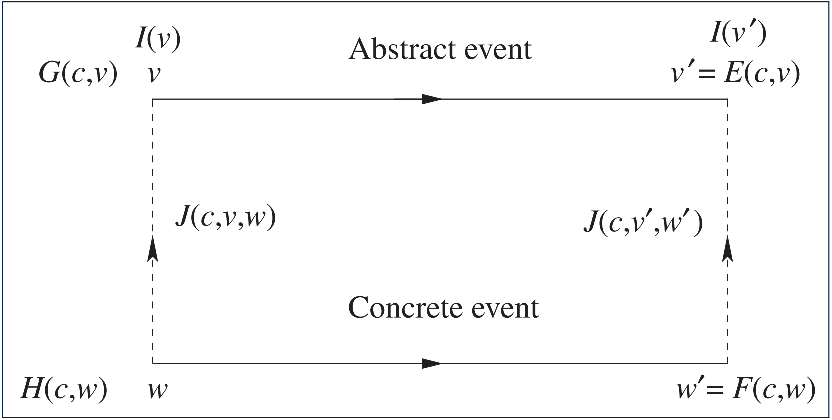
$$\boxed{\begin{array}{l} d \in \mathbb{N} \\ d > 0 \\ n \in \mathbb{N} \\ n \leq d \\ a \in \mathbb{N} \\ b \in \mathbb{N} \\ c \in \mathbb{N} \\ a + b + c = n \\ a = 0 \lor c = 0 \\ a + b < d \\ c = 0 \\ \vdash \\ (a + 1) + b + c = (n + 1) \end{array}} \quad \underline{\text{ML\_out/inv1\_4/INV}} \qquad \boxed{\begin{array}{l} d \in \mathbb{N} \\ d > 0 \\ n \in \mathbb{N} \\ n \leq d \\ a \in \mathbb{N} \\ b \in \mathbb{N} \\ c \in \mathbb{N} \\ a + b + c = n \\ a = 0 \lor c = 0 \\ c > 0 \\ \vdash \\ a = 0 \lor (c - 1) = 0 \end{array}} \quad \underline{\text{ML\_in/inv1\_5/INV}}$$

- **Exercises**. Specify and prove other **eight** ***POs of Invariant Preservation***.

Visualizing Inv. Preservation in Refinement

Each ***concrete*** event (w to w') is ***simulated by*** an ***abstract*** event (v to v'):

- ***abstract*** & ***concrete*** pre-states related by ***concrete*** invariants $J(c, v, w)$
- ***abstract*** & ***concrete*** post-states related by ***concrete*** invariants $J(c, v', w')$



INV PO of m_1: ML_out/inv1_4/INV

axm0_1 $\{ \; d \in \mathbb{N}$

axm0_2 $\{ \; d > 0$

inv0_1 $\{ \; n \in \mathbb{N}$

inv0_2 $\{ \; n \leq d$

inv1_1 $\{ \; a \in \mathbb{N}$

inv1_2 $\{ \; b \in \mathbb{N}$

inv1_3 $\{ \; c \in \mathbb{N}$

inv1_4 $\{ \; a + b + c = n$

inv1_5 $\{ \; a = 0 \lor c = 0$

Concrete guards of *ML_out* $\begin{cases} a + b < d \\ c = 0 \end{cases}$

$\vdash$

Concrete invariant **inv1_4**
with *ML_out*'s effect in the post-state $\{ \; (a + 1) + b + c = (n + 1)$

ML_out/inv1_4/INV

INV PO of m_1: ML_in/inv1_5/INV

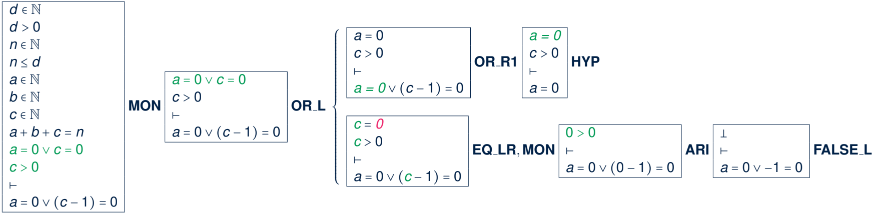
axm0_1	$\{\ d \in \mathbb{N}$
axm0_2	$\{\ d > 0$
inv0_1	$\{\ n \in \mathbb{N}$
inv0_2	$\{\ n \leq d$
inv1_1	$\{\ a \in \mathbb{N}$
inv1_2	$\{\ b \in \mathbb{N}$
inv1_3	$\{\ c \in \mathbb{N}$
inv1_4	$\{\ a + b + c = n$
inv1_5	$\{\ a = 0 \lor c = 0$
Concrete guards of *ML_in*	$\{\ c > 0$
	$\vdash$
Concrete invariant **inv1_5** with *ML_in*'s effect in the post-state	$\{\ a = 0 \lor (c - 1) = 0$

ML_in/inv1_5/INV

Proving Refinement: ML_out/inv1_4/INV

$$\boxed{\begin{array}{l} d \in \mathbb{N} \\ d > 0 \\ n \in \mathbb{N} \\ n \leq d \\ a \in \mathbb{N} \\ b \in \mathbb{N} \\ c \in \mathbb{N} \\ a + b + c = n \\ a = 0 \lor c = 0 \\ a + b < d \\ c = 0 \\ \vdash \\ (a + 1) + b + c = (n + 1) \end{array}} \;\textbf{MON}\; \boxed{\begin{array}{l} a + b + c = n \\ \vdash \\ (a + 1) + b + c = (n + 1) \end{array}} \;\textbf{ARI}\; \boxed{\begin{array}{l} a + b + c = n \\ \vdash \\ a + b + c + 1 = n + 1 \end{array}} \;\textbf{EQ\_LR, MON}\; \boxed{\begin{array}{l} \vdash \\ n + 1 = n + 1 \end{array}} \;\textbf{EQ}$$

Proving Refinement: ML_in/inv1_5/INV



$d \in \mathbb{N}$
$d > 0$
$n \in \mathbb{N}$
$n \le d$
$a \in \mathbb{N}$
$b \in \mathbb{N}$
$c \in \mathbb{N}$
$a + b + c = n$
$a = 0 \lor c = 0$
$c > 0$
$\vdash$
$a = 0 \lor (c - 1) = 0$

MON

$a = 0 \lor c = 0$
$c > 0$
$\vdash$
$a = 0 \lor (c - 1) = 0$

OR_L

$a = 0$
$c > 0$
$\vdash$
$a = 0 \lor (c - 1) = 0$

OR_R1

$a = 0$
$c > 0$
$\vdash$
$a = 0$

HYP

$c = 0$
$c > 0$
$\vdash$
$a = 0 \lor (c - 1) = 0$

EQ_LR, MON

$0 > 0$
$\vdash$
$a = 0 \lor (0 - 1) = 0$

ARI

$\perp$
$\vdash$
$a = 0 \lor -1 = 0$

FALSE_L

Initializing the Refined System m_1

- Discharging the **twelve sequents** proved that:
  - ***concrete invariants*** preserved by `ML_out` & `ML_in`
  - ***concrete guards*** of *ML_out* & *ML_in* entail their ***abstract*** counterparts
- What's left is the specification of how the ***ASM***'s ***initial state*** looks like:

```
init
    begin
        a := 0
        b := 0
        c := 0
    end
```

✓ No cars on bridge (heading either way) and island

✓ Initialization always possible: guard is ***true***.

✓ There is no ***pre-state*** for *init*.

$\therefore$ The RHS of := must not involve variables.

$\therefore$ The RHS of := may only involve constants.

✓ There is only the ***post-state*** for *init*.

$\therefore$ Before-***After Predicate***: $a' = 0 \land b' = 0 \land c' = 0$

PO of m_1 Concrete Invariant Establishment

- Some (new) formal components are needed:
  - $K(c)$: effect of ***abstract init***'s actions:

    e.g., $K(\langle d \rangle)$ of *init* $\widehat{=}$ $\langle 0 \rangle$
  - $v' = K(c)$: ***before-after predicate*** formalizing ***abstract*** *init*'s actions

    e.g., BAP of *init*: $\langle \boldsymbol{n'} \rangle = \langle 0 \rangle$
  - $L(c)$: effect of ***concrete init***'s actions:

    e.g., $K(\langle d \rangle)$ of *init* $\widehat{=}$ $\langle 0, 0, 0 \rangle$
  - $w' = L(c)$: ***before-after predicate*** formalizing ***concrete*** *init*'s actions

    e.g., BAP of *init*: $\langle \boldsymbol{a'}, \boldsymbol{b'}, \boldsymbol{c'} \rangle = \langle 0, 0, 0 \rangle$
- Accordingly, PO of ***invariant establishment*** is formulated as a **sequent**:

$$\boxed{\begin{array}{l} \text{Axioms} \\ \vdash \\ \textit{Concrete Invariants} \text{ Satisfied at } \underline{\text{Post}}\text{-State} \end{array}} \ \underline{\text{INV}} \qquad \boxed{\begin{array}{l} A(c) \\ \vdash \\ J_i(c, \mathbf{K(c)}, \mathbf{L(c)}) \end{array}} \ \underline{\text{INV}}$$

Discharging PO of m_1
Concrete Invariant Establishment

- How many ***sequents*** to be proved? [# ***concrete*** invariants]
- Two (of the five) sequents generated for ***concrete*** *init* of m_1:

$$\boxed{\begin{array}{l} d \in \mathbb{N} \\ d > 0 \\ \vdash \\ 0 + 0 + 0 = 0 \end{array}} \quad \text{init/}\textbf{inv1\_4}\text{/INV} \qquad \boxed{\begin{array}{l} d \in \mathbb{N} \\ d > 0 \\ \vdash \\ 0 = 0 \lor 0 = 0 \end{array}} \quad \text{init/}\textbf{inv1\_5}\text{/INV}$$

- Can we discharge the ***PO*** $\boxed{\text{init/}\textbf{inv1\_4}\text{/INV}}$?

$$\boxed{\begin{array}{l} d \in \mathbb{N} \\ d > 0 \\ \vdash \\ 0 + 0 + 0 = 0 \end{array}} \quad \textbf{ARI, MON} \quad \boxed{\vdash \top} \quad \textbf{TRUE\_R}$$

∴ ***init/inv1_4/INV*** succeeds in being discharged.

- Can we discharge the ***PO*** $\boxed{\text{init/}\textbf{inv1\_5}\text{/INV}}$?

$$\boxed{\begin{array}{l} d \in \mathbb{N} \\ d > 0 \\ \vdash \\ 0 = 0 \lor 0 = 0 \end{array}} \quad \textbf{ARI, MON} \quad \boxed{\vdash \top} \quad \textbf{TRUE\_R}$$

∴ ***init/inv1_5/INV*** succeeds in being discharged.

Model m_1: New, Concrete Events

- The system acts as an **Abstract State Machine (ASM)**: it ***evolves*** as ***actions of enabled events*** change values of variables, subject to ***invariants***.
- Considered ***concrete/refined events*** already existing in m_0: *ML_out* & *ML_in*
- ***New event*** *IL_in*:

```
IL_in
  when
    ??
  then
    ??
  end
```

  - *IL_in* denotes a car entering the island (getting off the bridge).
  - *IL_in* ***enabled*** only when:
    - The bridge's current traffic flows to the island.

      Q. Limited number of cars on the bridge and the island?

      A. Ensured when the earlier *ML_out* (of same car) occurred

- ***New event*** *IL_out*:

```
IL_out
  when
    ??
  then
    ??
  end
```

  - *IL_out* denotes a car exiting the island (getting on the bridge).
  - *IL_out* ***enabled*** only when:
    - There is some car on the island.
    - The bridge's current traffic flows to the mainland.

Model m_1: BA Predicates of Multiple Actions

Consider ***actions*** of m_1's two ***new*** events:

```
IL_in
  when
    $a > 0$
  then
    $a := a - 1$
    $b := b + 1$
  end
```

```
IL_out
  when
    $b > 0$
    $a = 0$
  then
    $b := b - 1$
    $c := c + 1$
  end
```

- What is the ***BAP*** of *ML_in*'s ***actions***?

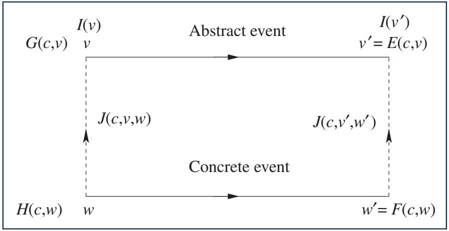
$$a' = a - 1 \land b' = b + 1 \land c' = c$$

- What is the ***BAP*** of *ML_in*'s ***actions***?

$$a' = a \land b' = b - 1 \land c' = c + 1$$

Visualizing Inv. Preservation in Refinement

- Recall how a ***concrete*** event is ***simulated*** by its ***abstract*** counterpart:



- For each ***new*** event:
    - Strictly speaking, it does **not** have an ***abstract*** counterpart.
    - It is ***simulated by*** a special ***abstract*** event (transforming v to v'):

```
skip
  begin

  end
```

- *skip* is a “dummy” event: non-guarded and does nothing
- **Q**. ***BAP*** of the skip event?

  A. $n' = n$

Refinement Rule: Invariant Preservation

- The new events *IL_in* and *IL_out* do not exist in $\mathbf{m_0}$, but:
  - They **exist** in $\mathbf{m_1}$ and may impact upon the ***concrete*** state space.
  - They ***preserve*** the ***concrete invariants***, just as *ML_out* & *ML_in* do.
- Recall the ***PO/VC Rule of Invariant Preservation for Refinement***:

$$\begin{array}{|l|}\hline A(c)\\ I(c,\mathbf{v})\\ J(c,\mathbf{v},\mathbf{w})\\ H(c,\mathbf{w})\\ \vdash\\ J_i(c,E(c,\mathbf{v}),F(c,\mathbf{w}))\\ \hline\end{array}\ \underline{\text{INV}}$$

where J_i denotes a single ***concrete invariant***

  - How many ***sequents*** to be proved? [# ***new*** evts × # ***concrete*** invariants]
  - Here are two (of the ten) ***sequents*** generated:

$$\begin{array}{|l|}\hline d\in\mathbb{N}\\ d>0\\ n\in\mathbb{N}\\ n\le d\\ a\in\mathbb{N}\\ b\in\mathbb{N}\\ c\in\mathbb{N}\\ a+b+c=n\\ a=0\lor c=0\\ a>0\\ \vdash\\ (a-1)+(b+1)+c=n\\ \hline\end{array}\ \underline{\text{IL\_in/inv1\_4/INV}}\qquad\begin{array}{|l|}\hline d\in\mathbb{N}\\ d>0\\ n\in\mathbb{N}\\ n\le d\\ a\in\mathbb{N}\\ b\in\mathbb{N}\\ c\in\mathbb{N}\\ a+b+c=n\\ a=0\lor c=0\\ a>0\\ \vdash\\ (a-1)=0\lor c=0\\ \hline\end{array}\ \underline{\text{IL\_in/inv1\_5/INV}}$$

- **Exercises**. Specify and prove other **eight** ***POs of Invariant Preservation***.

INV PO of m_1: IL_in/inv1_4/INV

$$\begin{array}{rl}
\textbf{axm0\_1} & \{\; d \in \mathbb{N} \\
\textbf{axm0\_2} & \{\; d > 0 \\
\textbf{inv0\_1} & \{\; n \in \mathbb{N} \\
\textbf{inv0\_2} & \{\; n \leq d \\
\textbf{inv1\_1} & \{\; a \in \mathbb{N} \\
\textbf{inv1\_2} & \{\; b \in \mathbb{N} \\
\textbf{inv1\_3} & \{\; c \in \mathbb{N} \\
\textbf{inv1\_4} & \{\; a + b + c = n \\
\textbf{inv1\_5} & \{\; a = 0 \lor c = 0 \\
\textit{Guards} \text{ of } \mathit{IL\_in} & \{\; a > 0 \\
 & \vdash \\
\textit{Concrete} \text{ invariant } \textbf{inv1\_4} & \\
\text{with } \mathit{IL\_in}\text{'s effect in the } \underline{\text{post}}\text{-state} & \{\; (a - 1) + (b + 1) + c = n
\end{array}$$

IL_in/inv1_4/INV

INV PO of m_1: IL_in/inv1_5/INV

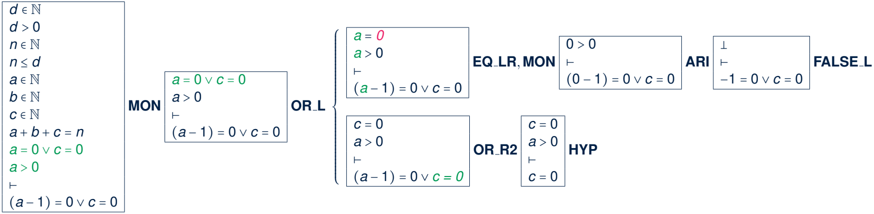
axm0_1	$\{\; d \in \mathbb{N}$
axm0_2	$\{\; d > 0$
inv0_1	$\{\; n \in \mathbb{N}$
inv0_2	$\{\; n \leq d$
inv1_1	$\{\; a \in \mathbb{N}$
inv1_2	$\{\; b \in \mathbb{N}$
inv1_3	$\{\; c \in \mathbb{N}$
inv1_4	$\{\; a + b + c = n$
inv1_5	$\{\; a = 0 \lor c = 0$
Guards of *IL_in*	$\{\; a > 0$
	$\vdash$
Concrete invariant **inv1_5** with *IL_in*'s effect in the post-state	$\{\; (a - 1) = 0 \lor c = 0$

IL_in/inv1_5/INV

Proving Refinement: IL_in/inv1_4/INV

$$
\boxed{\begin{array}{l}
d \in \mathbb{N} \\
d > 0 \\
n \in \mathbb{N} \\
n \leq d \\
a \in \mathbb{N} \\
b \in \mathbb{N} \\
c \in \mathbb{N} \\
a + b + c = n \\
a = 0 \lor c = 0 \\
a > 0 \\
\vdash \\
(a - 1) + (b + 1) + c = n
\end{array}}
\;\textbf{MON}\;
\boxed{\begin{array}{l}
a + b + c = n \\
\vdash \\
(a - 1) + (b + 1) + c = n
\end{array}}
\;\textbf{ARI}\;
\boxed{\begin{array}{l}
a + b + c = n \\
\vdash \\
a + b + c = n
\end{array}}
\;\textbf{HYP}
$$

Proving Refinement: IL_in/inv1_5/INV



$d \in \mathbb{N}$
$d > 0$
$n \in \mathbb{N}$
$n \leq d$
$a \in \mathbb{N}$
$b \in \mathbb{N}$
$c \in \mathbb{N}$
$a + b + c = n$
$a = 0 \lor c = 0$
$a > 0$
$\vdash$
$(a - 1) = 0 \lor c = 0$

MON

$a = 0 \lor c = 0$
$a > 0$
$\vdash$
$(a - 1) = 0 \lor c = 0$

OR_L

$a = 0$
$a > 0$
$\vdash$
$(a - 1) = 0 \lor c = 0$

EQ_LR, MON

$0 > 0$
$\vdash$
$(0 - 1) = 0 \lor c = 0$

ARI

$\bot$
$\vdash$
$-1 = 0 \lor c = 0$

FALSE_L

$c = 0$
$a > 0$
$\vdash$
$(a - 1) = 0 \lor c = 0$

OR_R2

$c = 0$
$a > 0$
$\vdash$
$c = 0$

HYP

Livelock Caused by New Events Diverging

- An alternative m_1 (with **inv1_4**, **inv1_5**, and **guards** of new events removed):

constants: d

axioms:
axm0_1 : $d \in \mathbb{N}$
axm0_2 : $d > 0$

variables: a, b, c

invariants:
inv1_1 : $a \in \mathbb{Z}$
inv1_2 : $b \in \mathbb{Z}$
inv1_3 : $c \in \mathbb{Z}$

```
ML_out
  when
    a + b < d
    c = 0
  then
    a := a + 1
  end
```

```
ML_in
  when
    c > 0
  then
    c := c - 1
  end
```

```
IL_in
  begin
    a := a - 1
    b := b + 1
  end
```

```
IL_out
  begin
    b := b - 1
    c := c + 1
  end
```

Concrete invariants are **under-specified**: only typing constraints.

Exercises : Show that **Invariant Preservation** is provable, but **Guard Strengthening** is not.

- Say this alternative m_1 is implemented as is:
  IL_in and *IL_out* **always** ***enabled*** and may occur ***indefinitely***, preventing other "old" events (*ML_out* and *ML_in*) from ever happening:

  $$\langle init, IL\_in, IL\_out, IL\_in, IL\_out, \dots \rangle$$

  Q: What are the corresponding ***abstract*** transitions?

  A: $\langle init, skip, skip, skip, skip, \dots \rangle$ [≈ executing `while(true);`]

- We say that these two ***new*** events *diverge*, creating a *livelock*:
  - Different from a ***deadlock*** ∵ **always** an event occurring (*IL_in* or *IL_out*).
  - But their ***indefinite*** occurrences contribute **nothing** useful.

PO of Convergence of New Events

The PO/VC rule for *non-divergence/livelock freedom* consists of two parts:

- Interleaving of ***new*** events charactered as an integer expression: ***variant***.
- A variant $V(c, w)$ may refer to constants and/or ***concrete*** variables.
- In the original m_1, let's try $\boxed{\textbf{variants} : 2 \cdot a + b}$

1. ***Variant Stays Non-Negative***

$$
\boxed{\begin{array}{l} A(c) \\ I(c, v) \\ J(c, v, w) \\ H(c, w) \\ \vdash \\ V(c, w) \in \mathbb{N} \end{array}} \quad \underline{\text{NAT}}
$$

- Variant $V(c, w)$ measures <u>how many more times</u> the ***new*** events can occur.
- If a ***new*** event is ***enabled***, then $V(c, w) > 0$.
- When $V(c, w)$ <u>reaches</u> 0, some "old" events must happen s.t. $V(c, w)$ goes back <u>above</u> 0.

2. ***A New Event Occurrence Decreases Variant***

$$
\boxed{\begin{array}{l} A(c) \\ I(c, v) \\ J(c, v, w) \\ H(c, w) \\ \vdash \\ V(c, F(c, w)) < V(c, w) \end{array}} \quad \underline{\text{VAR}}
$$

- If a ***new*** event is ***enabled*** and occurs, the value of $V(c, w) \downarrow$.

PO of Convergence of New Events: NAT

- Recall: PO related to ***Variant Stays Non-Negative***:

$$\boxed{\begin{array}{l} A(c) \\ I(c,v) \\ J(c,v,w) \\ H(c,w) \\ \vdash \\ V(c,w) \in \mathbb{N} \end{array}} \quad \underline{\text{NAT}}$$

How many ***sequents*** to be proved?

[# ***new*** events]

- For the ***new*** event *IL_in*:

$$\boxed{\begin{array}{lll} d \in \mathbb{N} & d > 0 & \\ n \in \mathbb{N} & n \leq d & \\ a \in \mathbb{N} & b \in \mathbb{N} & c \in \mathbb{N} \\ a + b + c = n & a = 0 \lor c = 0 & \\ a > 0 & & \\ \vdash & & \\ 2 \cdot a + b \in \mathbb{N} & & \end{array}} \quad \underline{\text{IL\_in/NAT}}$$

Exercises: Prove **IL_in/NAT** and Formulate/Prove **IL_out/NAT**.

PO of Convergence of New Events: VAR

- Recall: PO related to ***A New Event Occurrence Decreases Variant***

$$\boxed{\begin{array}{l} A(c) \\ I(c,v) \\ J(c,v,w) \\ H(c,w) \\ \vdash \\ V(c,F(c,w)) < V(c,w) \end{array}} \quad \underline{\text{VAR}}$$

How many ***sequents*** to be proved?

[# ***new*** events]

- For the ***new*** event *IL_in*:

$$\boxed{\begin{array}{l} \begin{array}{lll} d \in \mathbb{N} & d > 0 & \\ n \in \mathbb{N} & n \leq d & \\ a \in \mathbb{N} & b \in \mathbb{N} & c \in \mathbb{N} \\ a + b + c = n & a = 0 \lor c = 0 & \\ a > 0 & & \end{array} \\ \vdash \\ 2 \cdot (a - 1) + (b + 1) < 2 \cdot a + b \end{array}} \quad \underline{\text{IL\_in/VAR}}$$

Exercises: Prove **IL_in/VAR** and Formulate/Prove **IL_out/VAR**.

Convergence of New Events: Exercise

Given the original **m_1**, what if the following ***variant*** expression is used:

$$\boxed{\textbf{variants} : a + b}$$

Are the formulated sequents still ***provable***?

PO of Refinement: Deadlock Freedom

- Recall:
  - We proved that the initial model m_0 is deadlock free (see **DLF**).
  - We proved, according to ***guard strengthening***, that if a ***concrete*** event is enabled, then its ***abstract*** counterpart is enabled.
- PO of ***relative deadlock freedom*** for a ***refinement*** model:

$$\boxed{\begin{array}{l} A(c) \\ I(c,v) \\ J(c,v,w) \\ G_1(c,v) \vee \cdots \vee G_m(c,v) \\ \vdash \\ H_1(c,w) \vee \cdots \vee H_n(c,w) \end{array}} \quad \underline{\text{DLF}}$$

If an ***abstract*** state does not *deadlock* (i.e., $G_1(c,v) \vee \cdots \vee G_m(c,v)$), then its ***concrete*** counterpart does not *deadlock* (i.e., $H_1(c,w) \vee \cdots \vee H_n(c,w)$).

- Another way to think of the above PO:

  The ***refinement*** does **not** introduce, in the ***concrete***, any "new" *deadlock* scenarios **not** existing in the ***abstract*** state.

PO Rule: Relative Deadlock Freedom m_1

$$
\begin{array}{r l l}
\textbf{axm0\_1} & \{\ d \in \mathbb{N} & \\
\textbf{axm0\_2} & \{\ d > 0 & \\
\textbf{inv0\_1} & \{\ n \in \mathbb{N} & \\
\textbf{inv0\_2} & \{\ n \leq d & \\
\textbf{inv1\_1} & \{\ a \in \mathbb{N} & \\
\textbf{inv1\_2} & \{\ b \in \mathbb{N} & \\
\textbf{inv1\_3} & \{\ c \in \mathbb{N} & \\
\textbf{inv1\_4} & \{\ a + b + c = n & \\
\textbf{inv1\_5} & \{\ a = 0 \lor c = 0 & \\
\text{Disjunction of } \textit{\textbf{abstract}} \text{ guards} & \left\{ \begin{array}{l r} & n < d\ \} \\ \lor & n > 0\ \} \end{array} \right. & \begin{array}{l} \textbf{guards of } \textit{ML\_out} \textbf{ in } m_0 \\ \textbf{guards of } \textit{ML\_in} \textbf{ in } m_0 \end{array} \\
 & \vdash & \\
\text{Disjunction of } \textit{\textbf{concrete}} \text{ guards} & \left\{ \begin{array}{l r} & a + b < d \land c = 0\ \} \\ \lor & c > 0\ \} \\ \lor & a > 0\ \} \\ \lor & b > 0 \land a = 0\ \} \end{array} \right. & \begin{array}{l} \textbf{guards of } \textit{ML\_out} \textbf{ in } m_1 \\ \textbf{guards of } \textit{ML\_in} \textbf{ in } m_1 \\ \textbf{guards of } \textit{IL\_in} \textbf{ in } m_1 \\ \textbf{guards of } \textit{IL\_out} \textbf{ in } m_1 \end{array}
\end{array}
$$

DLF

Example Inference Rules (6)

$$\frac{H, \neg P \vdash Q}{H \vdash P \vee Q} \quad \textbf{OR\_R}$$

To prove a ***disjunctive goal***,
it suffices to prove one of the disjuncts,
with the the negation of the the other disjunct
serving as an additional hypothesis.

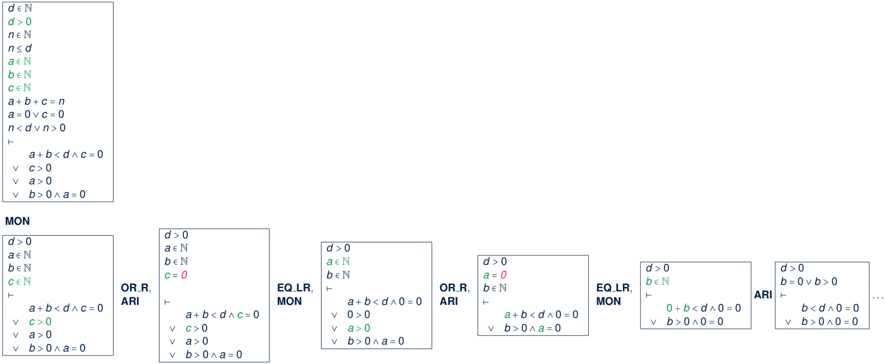
$$\frac{H, P, Q \vdash R}{H, P \wedge Q \vdash R} \quad \textbf{AND\_L}$$

To prove a goal with a ***conjunctive hypothesis***,
it suffices to prove the same goal,
with the the two conjuncts
serving as two separate hypotheses.

$$\frac{H \vdash P \qquad H \vdash Q}{H \vdash P \wedge Q} \quad \textbf{AND\_R}$$

To prove a goal with a ***conjunctive goal***,
it suffices to prove each conjunct
as a separate goal.

Proving Refinement: DLF of m_1



$$\begin{array}{l} d \in \mathbb{N} \\ d > 0 \\ n \in \mathbb{N} \\ n \le d \\ a \in \mathbb{N} \\ b \in \mathbb{N} \\ c \in \mathbb{N} \\ a + b + c = n \\ a = 0 \lor c = 0 \\ n < d \lor n > 0 \\ \vdash \\ \quad a + b < d \land c = 0 \\ \lor \quad c > 0 \\ \lor \quad a > 0 \\ \lor \quad b > 0 \land a = 0 \end{array}$$

MON

$$\begin{array}{l} d > 0 \\ a \in \mathbb{N} \\ b \in \mathbb{N} \\ c \in \mathbb{N} \\ \vdash \\ \quad a + b < d \land c = 0 \\ \lor \quad c > 0 \\ \lor \quad a > 0 \\ \lor \quad b > 0 \land a = 0 \end{array}$$

OR_R, ARI

$$\begin{array}{l} d > 0 \\ a \in \mathbb{N} \\ b \in \mathbb{N} \\ c = 0 \\ \vdash \\ \quad a + b < d \land c = 0 \\ \lor \quad c > 0 \\ \lor \quad a > 0 \\ \lor \quad b > 0 \land a = 0 \end{array}$$

EQ_LR, MON

$$\begin{array}{l} d > 0 \\ a \in \mathbb{N} \\ b \in \mathbb{N} \\ \vdash \\ \quad a + b < d \land 0 = 0 \\ \lor \quad 0 > 0 \\ \lor \quad a > 0 \\ \lor \quad b > 0 \land a = 0 \end{array}$$

OR_R, ARI

$$\begin{array}{l} d > 0 \\ a = 0 \\ b \in \mathbb{N} \\ \vdash \\ \quad a + b < d \land 0 = 0 \\ \lor \quad b > 0 \land a = 0 \end{array}$$

EQ_LR, MON

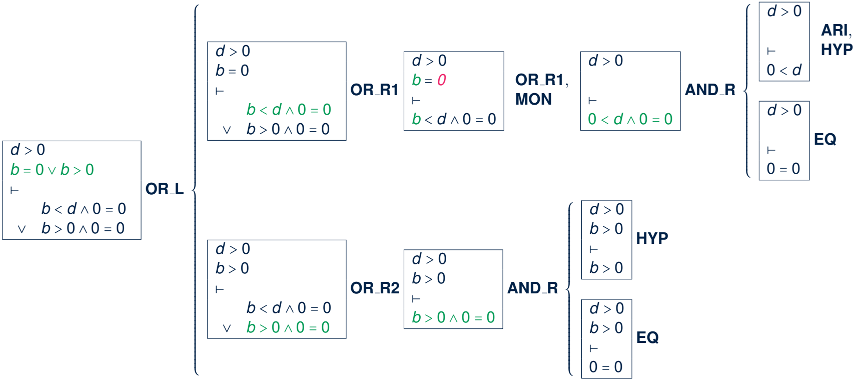
$$\begin{array}{l} d > 0 \\ b \in \mathbb{N} \\ \vdash \\ \quad 0 + b < d \land 0 = 0 \\ \lor \quad b > 0 \land 0 = 0 \end{array}$$

ARI

$$\begin{array}{l} d > 0 \\ b = 0 \lor b > 0 \\ \vdash \\ \quad b < d \land 0 = 0 \\ \lor \quad b > 0 \land 0 = 0 \end{array}$$

...

Proving Refinement: DLF of m_1 (continued)



$d > 0$
$b = 0 \lor b > 0$
$\vdash$
$b < d \land 0 = 0$
$\lor \quad b > 0 \land 0 = 0$

OR_L

$d > 0$
$b = 0$
$\vdash$
$b < d \land 0 = 0$
$\lor \quad b > 0 \land 0 = 0$

OR_R1

$d > 0$
$b = 0$
$\vdash$
$b < d \land 0 = 0$

OR_R1, MON

$d > 0$
$\vdash$
$0 < d \land 0 = 0$

AND_R

$d > 0$
$\vdash$
$0 < d$

ARI, HYP

$d > 0$
$\vdash$
$0 = 0$

EQ

$d > 0$
$b > 0$
$\vdash$
$b < d \land 0 = 0$
$\lor \quad b > 0 \land 0 = 0$

OR_R2

$d > 0$
$b > 0$
$\vdash$
$b > 0 \land 0 = 0$

AND_R

$d > 0$
$b > 0$
$\vdash$
$b > 0$

HYP

$d > 0$
$b > 0$
$\vdash$
$0 = 0$

EQ

First Refinement: Summary

- The final version of our ***first refinement*** m_1 is ***provably correct*** w.r.t.:
  - Establishment of ***Concrete Invariants*** [*init*]
  - Preservation of ***Concrete Invariants*** [old & new events]
  - Strengthening of ***guards*** [old events]
  - ***Convergence*** (a.k.a. livelock freedom, non-divergence) [new events]
  - Relative ***Deadlock*** Freedom
- Here is the final specification of m_1:

```
constants:  $d$
```

```
axioms:
  axm0_1 : $d \in \mathbb{N}$
  axm0_2 : $d > 0$
```

```
variables:  $a, b, c$
```

```
invariants:
  inv1_1 : $a \in \mathbb{N}$
  inv1_2 : $b \in \mathbb{N}$
  inv1_3 : $c \in \mathbb{N}$
  inv1_4 :  $a + b + c = n$
  inv1_5 :  $a = 0 \lor c = 0$
```

```
variants:
  $2 \cdot a + b$
```

```
init
  begin
    $a := 0$
    $b := 0$
    $c := 0$
  end
```

```
ML_out
  when
    $a + b < d$
    $c = 0$
  then
    $a := a + 1$
  end
```

```
ML_in
  when
    $c > 0$
  then
    $c := c - 1$
  end
```

```
IL_in
  when
    $a > 0$
  then
    $a := a - 1$
    $b := b + 1$
  end
```

```
IL_out
  when
    $b > 0$
    $a = 0$
  then
    $b := b - 1$
    $c := c + 1$
  end
```

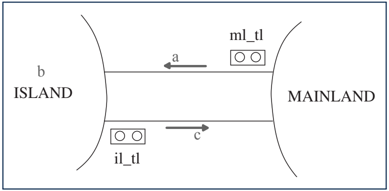
Model m_2: "More Concrete" Abstraction

- 2nd ***refinement*** has even more ***concrete*** perception of the bridge controller:
    - We "**zoom in**" by observing the system from **even closer to the ground**, so that the one-way traffic of the bridge is controlled via:

ml_tl: a traffic light for exiting the ML

il_tl: a traffic light for exiting the IL

abstract variables ***a***, ***b***, ***c*** from m_1 still used (instead of being replaced)



    - Nonetheless, sensors remain ***abstracted*** away!
- That is, we focus on these three ***environment constraints***:

ENV1	The system is equipped with two traffic lights with two colors: green and red.
ENV2	The traffic lights control the entrance to the bridge at both ends of it.
ENV3	Cars are not supposed to pass on a red traffic light, only on a green one.

- We are ***obliged to prove*** this ***added concreteness*** is ***consistent*** with m_1.

Model m_2: Refined, Concrete State Space

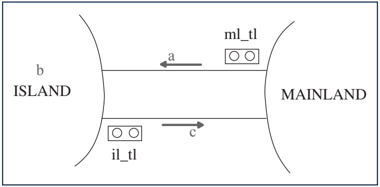
1. The **static** part introduces the notion of traffic light colours:

sets: $COLOR$

constants: $red, green$

axioms:
axm2_1 : $COLOR = \{green, red\}$
axm2_2 : $green \neq red$

2. The **dynamic** part shows the ***superposition refinement*** scheme:



- ***Abstract*** variables ***a***, ***b***, ***c*** from m_1 are still in use in ***m_2***.
- Two new, ***concrete*** variables are introduced: ***ml_tl*** and ***il_tl***
- **Constrast**: In m_1, ***abstract*** variable n is replaced by ***concrete*** variables a, b, c.

variables:
a, b, c
ml_tl
il_tl

invariants:
inv2_1 : $ml\_tl \in COLOUR$
inv2_2 : $il\_tl \in COLOUR$
inv2_3 : ??
inv2_4 : ??

◇ **inv2_1** & **inv2_2**: typing constraints

◇ **inv2_3**: being allowed to exit ML **means** cars within limit and no opposite traffic

◇ **inv2_4**: being allowed to exit IL **means** some car in IL and no opposite traffic

Model m_2: Refining Old, Abstract Events

- The system acts as an **ABSTRACT STATE MACHINE (ASM)**: it ***evolves*** as ***actions of enabled events*** change values of variables, subject to ***invariants***.
- ***Concrete/Refined*** version of ***event*** *ML_out*:

```
ML_out
  when
    ??
  then
    $a := a + 1$
  end
```

  - Recall the ***abstract*** guard of *ML_out* in m_1: $(c = 0) \wedge (a + b < d)$

    ⇒ Unrealistic as drivers should **not** know about a, b, c!
  - *ML_out* is ***refined***: a car exits the ML (to the bridge) only when:
    - the traffic light *ml_tl* allows
- ***Concrete/Refined*** version of ***event*** *IL_out*:

```
IL_out
  when
    ??
  then
    $b := b - 1$
    $c := c + 1$
  end
```

  - Recall the ***abstract*** guard of *IL_out* in m_1: $(a = 0) \wedge (b > 0)$

    ⇒ Unrealistic as drivers should **not** know about a, b, c!
  - *IL_out* is ***refined***: a car exits the IL (to the bridge) only when:
    - the traffic light *il_tl* allows

Q1. How about the other two "old" ***events*** *IL_in* and *ML_in*?
A1. No need to ***refine*** as already ***guarded*** by *ML_out* and *IL_out*.
Q2. What if the driver disobeys *ml_tl* or *il_tl*? [**A2**. **ENV3**]

Model m_2: New, Concrete Events

- The system acts as an **ABSTRACT STATE MACHINE (ASM)** : it ***evolves*** as ***actions of enabled events*** change values of variables, subject to ***invariants***.
- Considered ***events*** already existing in m_1:
  - *ML_out* & *IL_out* [REFINED]
  - *IL_in* & *ML_in* [UNCHANGED]
- ***New event*** *ML_tl_green*:

```
ML_tl_green
  when
    ??
  then
    ml_tl := green
  end
```

  - *ML_tl_green* denotes the traffic light *ml_tl* turning green.
  - *ML_tl_green* ***enabled*** only when:
    - the traffic light not already green
    - limited number of cars on the bridge and the island
    - No opposite traffic

[⇒ *ML_out*'s ***abstract*** guard in m_1]

- ***New event*** *IL_tl_green*:

```
IL_tl_green
  when
    ??
  then
    il_tl := green
  end
```

  - *IL_tl_green* denotes the traffic light *il_tl* turning green.
  - *IL_tl_green* ***enabled*** only when:
    - the traffic light not already green
    - some cars on the island (i.e., island not empty)
    - No opposite traffic

[⇒ *IL_out*'s ***abstract*** guard in m_1]

Invariant Preservation in Refinement m_2

constants: d

sets: $COLOR$

axioms:
- **axm0_1** : $d \in \mathbb{N}$
- **axm0_2** : $d > 0$
- **axm2_1** : $COLOR = \{green, red\}$
- **axm2_2** : $green \neq red$

variables:
a, b, c
ml_tl
il_tl

invariants:
- **inv2_1** : $ml\_tl \in COLOUR$
- **inv2_2** : $il\_tl \in COLOUR$
- **inv2_3** : $ml\_tl = green \Rightarrow a + b < d \land c = 0$
- **inv2_4** : $il\_tl = green \Rightarrow b > 0 \land a = 0$

```
ML_tl_green
  when
    $ml\_tl = red$
    $a + b < d$
    $c = 0$
  then
    $ml\_tl := green$
  end
```

```
ML_out
  when
    $ml\_tl = green$
  then
    $a := a + 1$
  end
```

```
IL_in
  when
    $a > 0$
  then
    $a := a - 1$
    $b := b + 1$
  end
```

```
IL_tl_green
  when
    $il\_tl = red$
    $b > 0$
    $a = 0$
  then
    $il\_tl := green$
  end
```

```
IL_out
  when
    $il\_tl = green$
  then
    $b := b - 1$
    $c := c + 1$
  end
```

```
ML_in
  when
    $c > 0$
  then
    $c := c - 1$
  end
```

Recall the ***PO/VC Rule of Invariant Preservation for Refinement***:

$$\boxed{\begin{array}{l} A(c) \\ I(c, \mathbf{v}) \\ J(c, \mathbf{v}, \mathbf{w}) \\ H(c, \mathbf{w}) \\ \vdash \\ J_i(c, E(c, \mathbf{v}), F(c, \mathbf{w})) \end{array}} \quad \underline{\text{INV}}$$

where J_i denotes a single ***concrete invariant***

- How many ***sequents*** to be proved? [# *concrete* evts × # *concrete* invariants = 6 × 4]
- We discuss two sequents: *ML_out*/**inv2_4**/*INV* and *IL_out*/**inv2_3**/*INV*

Exercises. Specify and prove (some of) other twenty-two ***POs of Invariant Preservation***.

INV PO of m_2: ML_out/inv2_4/INV

axm0_1	$\{\ d \in \mathbb{N}$
axm0_2	$\{\ d > 0$
axm2_1	$\{\ COLOUR = \{green, red\}$
axm2_2	$\{\ green \neq red$
inv0_1	$\{\ n \in \mathbb{N}$
inv0_2	$\{\ n \leq d$
inv1_1	$\{\ a \in \mathbb{N}$
inv1_2	$\{\ b \in \mathbb{N}$
inv1_3	$\{\ c \in \mathbb{N}$
inv1_4	$\{\ a + b + c = n$
inv1_5	$\{\ a = 0 \lor c = 0$
inv2_1	$\{\ ml\_tl \in COLOUR$
inv2_2	$\{\ il\_tl \in COLOUR$
inv2_3	$\{\ ml\_tl = green \Rightarrow a + b < d \land c = 0$
inv2_4	$\{\ il\_tl = green \Rightarrow b > 0 \land a = 0$
Concrete guards of *ML_out*	$\{\ ml\_tl = green$
	$\vdash$
Concrete invariant **inv2_4** with *ML_out*'s effect in the post-state	$\{\ il\_tl = green \Rightarrow b > 0 \land (a + 1) = 0$

ML_out/inv2_4/INV

INV PO of m_2: IL_out/inv2_3/INV

axm0_1	$d \in \mathbb{N}$
axm0_2	$d > 0$
axm2_1	$COLOUR = \{green, red\}$
axm2_2	$green \neq red$
inv0_1	$n \in \mathbb{N}$
inv0_2	$n \leq d$
inv1_1	$a \in \mathbb{N}$
inv1_2	$b \in \mathbb{N}$
inv1_3	$c \in \mathbb{N}$
inv1_4	$a + b + c = n$
inv1_5	$a = 0 \lor c = 0$
inv2_1	$ml\_tl \in COLOUR$
inv2_2	$il\_tl \in COLOUR$
inv2_3	$ml\_tl = green \Rightarrow a + b < d \land c = 0$
inv2_4	$il\_tl = green \Rightarrow b > 0 \land a = 0$
Concrete guards of *IL_out*	$il\_tl = green$
	$\vdash$
Concrete invariant **inv2_3** with *ML_out*'s effect in the post-state	$ml\_tl = green \Rightarrow a + (b - 1) < d \land (c + 1) = 0$

IL_out/inv2_3/INV

Example Inference Rules (7)

$$\frac{H, P, Q \vdash R}{H, P, P \Rightarrow Q \vdash R} \quad \textbf{IMP\_L}$$

If a hypothesis ***P*** matches the assumption of another ***implicative hypothesis P ⇒ Q***, then the conclusion ***Q*** of the ***implicative hypothesis*** can be used as a new hypothesis for the sequent.

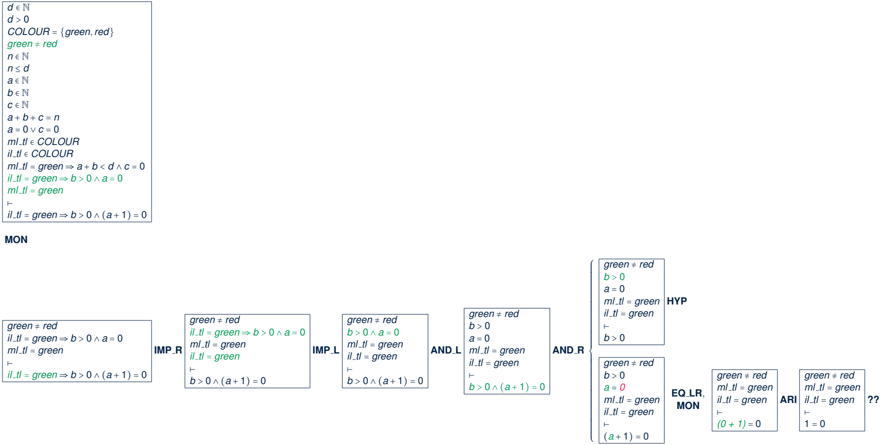
$$\frac{H, P \vdash Q}{H \vdash P \Rightarrow Q} \quad \textbf{IMP\_R}$$

To prove an ***implicative goal P ⇒ Q***, it suffices to prove its conclusion ***Q***, with its assumption ***P*** serving as a new hypotheses.

$$\frac{H, \neg Q \vdash P}{H, \neg P \vdash Q} \quad \textbf{NOT\_L}$$

To prove a goal ***Q*** with a ***negative hypothesis*** $\neg$ ***P***, it suffices to prove the negated hypothesis $\neg(\neg P) \equiv$ ***P*** with the negated original goal $\neg$ ***Q*** serving as a new hypothesis.

Proving ML_out/inv2_4/INV: First Attempt



$$\begin{array}{l} d \in \mathbb{N} \\ d > 0 \\ COLOUR = \{green, red\} \\ green \neq red \\ n \in \mathbb{N} \\ n \leq d \\ a \in \mathbb{N} \\ b \in \mathbb{N} \\ c \in \mathbb{N} \\ a + b + c = n \\ a = 0 \lor c = 0 \\ ml\_tl \in COLOUR \\ il\_tl \in COLOUR \\ ml\_tl = green \Rightarrow a + b < d \land c = 0 \\ il\_tl = green \Rightarrow b > 0 \land a = 0 \\ ml\_tl = green \\ \vdash \\ il\_tl = green \Rightarrow b > 0 \land (a + 1) = 0 \end{array}$$

MON

$$\begin{array}{l} green \neq red \\ il\_tl = green \Rightarrow b > 0 \land a = 0 \\ ml\_tl = green \\ \vdash \\ il\_tl = green \Rightarrow b > 0 \land (a + 1) = 0 \end{array}$$

IMP_R

$$\begin{array}{l} green \neq red \\ il\_tl = green \Rightarrow b > 0 \land a = 0 \\ ml\_tl = green \\ il\_tl = green \\ \vdash \\ b > 0 \land (a + 1) = 0 \end{array}$$

IMP_L

$$\begin{array}{l} green \neq red \\ b > 0 \land a = 0 \\ ml\_tl = green \\ il\_tl = green \\ \vdash \\ b > 0 \land (a + 1) = 0 \end{array}$$

AND_L

$$\begin{array}{l} green \neq red \\ b > 0 \\ a = 0 \\ ml\_tl = green \\ il\_tl = green \\ \vdash \\ b > 0 \land (a + 1) = 0 \end{array}$$

AND_R

$$\begin{array}{l} green \neq red \\ b > 0 \\ a = 0 \\ ml\_tl = green \\ il\_tl = green \\ \vdash \\ b > 0 \end{array}$$

HYP

$$\begin{array}{l} green \neq red \\ b > 0 \\ a = 0 \\ ml\_tl = green \\ il\_tl = green \\ \vdash \\ (a + 1) = 0 \end{array}$$

EQ_LR, MON

$$\begin{array}{l} green \neq red \\ ml\_tl = green \\ il\_tl = green \\ \vdash \\ (0 + 1) = 0 \end{array}$$

ARI

$$\begin{array}{l} green \neq red \\ ml\_tl = green \\ il\_tl = green \\ \vdash \\ 1 = 0 \end{array}$$

??

Proving IL_out/inv2_3/INV: First Attempt

$d \in \mathbb{N}$
$d > 0$
$COLOUR = \{green, red\}$
$green \neq red$
$n \in \mathbb{N}$
$n \leq d$
$a \in \mathbb{N}$
$b \in \mathbb{N}$
$c \in \mathbb{N}$
$a + b + c = n$
$a = 0 \vee c = 0$
$ml\_tl \in COLOUR$
$il\_tl \in COLOUR$
$ml\_tl = green \Rightarrow a + b < d \wedge c = 0$
$il\_tl = green \Rightarrow b > 0 \wedge a = 0$
$il\_tl = green$
$\vdash$
$ml\_tl = green \Rightarrow a + (b - 1) < d \wedge (c + 1) = 0$

MON

$green \neq red$
$ml\_tl = green \Rightarrow a + b < d \wedge c = 0$
$il\_tl = green$
$\vdash$
$ml\_tl = green \Rightarrow a + (b - 1) < d \wedge (c + 1) = 0$

IMP_R

$green \neq red$
$ml\_tl = green \Rightarrow a + b < d \wedge c = 0$
$il\_tl = green$
$ml\_tl = green$
$\vdash$
$a + (b - 1) < d \wedge (c + 1) = 0$

IMP_L

$green \neq red$
$a + b < d \wedge c = 0$
$il\_tl = green$
$ml\_tl = green$
$\vdash$
$a + (b - 1) < d \wedge (c + 1) = 0$

AND_L

$green \neq red$
$a + b < d$
$c = 0$
$il\_tl = green$
$ml\_tl = green$
$\vdash$
$a + (b - 1) < d \wedge (c + 1) = 0$

AND_R

$green \neq red$
$a + b < d$
$c = 0$
$il\_tl = green$
$ml\_tl = green$
$\vdash$
$a + (b - 1) < d$

MON

$a + b < d$
$\vdash$
$a + (b - 1) < d$

ARI

$green \neq red$
$a + b < d$
$c = 0$
$il\_tl = green$
$ml\_tl = green$
$\vdash$
$(c + 1) = 0$

EQ_LR, MON

$green \neq red$
$il\_tl = green$
$ml\_tl = green$
$\vdash$
$(0 + 1) = 0$

ARI

$green \neq red$
$il\_tl = green$
$ml\_tl = green$
$\vdash$
$1 = 0$

??

Failed: ML_out/inv2_4/INV, IL_out/inv2_3/INV

- Our first attempts of proving ***ML_out/inv2_4/INV*** and ***IL_out/inv2_3/INV*** both failed the 2nd case (resulted from applying IR **AND_R**):

$$\boxed{green \neq red \land il\_tl = green \land ml\_tl = green \ \vdash\ 1 = 0}$$

- This ***unprovable*** sequent gave us a good hint:
  - Goal $\boxed{1 = 0 \equiv \textbf{false}}$ suggests that the ***safety requirements*** $a = 0$ (for **inv2_4**) and $c = 0$ (for **inv2_3**) ***contradict*** with the current m_2.
  - Hyp. $\boxed{il\_tl = green = ml\_tl}$ suggests a ***possible, dangerous state*** of m_2, where two cars heading different directions are on the one-way bridge:

$$\langle \underbrace{init}_{\substack{d=2\\a'=0\\b'=0\\c'=0\\ml\_tl'=red\\il\_tl'=red}},\ \underbrace{ML\_tl\_green}_{\substack{d=2\\a'=0\\b'=0\\c'=0\\\mathbf{ml\_tl'=green}\\il\_tl'=red}},\ \underbrace{ML\_out}_{\substack{d=2\\\mathbf{a'=1}\\b'=0\\c'=0\\ml\_tl'=green\\il\_tl'=red}},\ \underbrace{IL\_in}_{\substack{d=2\\\mathbf{a'=0}\\\mathbf{b'=1}\\c'=0\\ml\_tl'=green\\il\_tl'=red}},\ \underbrace{IL\_tl\_green}_{\substack{d=2\\a'=0\\b'=1\\c'=0\\ml\_tl'=green\\\mathbf{il\_tl'=green}}},\ \underbrace{IL\_out}_{\substack{d=2\\a'=0\\\mathbf{b'=0}\\\mathbf{c'=1}\\ml\_tl'=green\\il\_tl'=green}},\ \underbrace{ML\_out}_{\substack{d=2\\\mathbf{a'=1}\\b'=0\\c'=1\\ml\_tl'=green\\il\_tl'=green}} \rangle$$

Fixing m_2: Adding an Invariant

- Having understood the failed proofs, we add a proper ***invariant*** to m_2:

invariants:

$\ldots$

$\textbf{inv2\_5} : ml\_tl = red \lor il\_tl = red$

- We have effectively resulted in an improved m_2 more faithful w.r.t. **REQ3**:

REQ3	The bridge is one-way or the other, not both at the same time.

- Having added this new invariant ***inv2_5***:
  - Original 6 × 4 generated sequents to be updated: **inv2_5** a new hypothesis e.g., Are ***ML_out/inv2_4/INV*** and ***IL_out/inv2_3/INV*** now ***provable***?
  - Additional 6 × 1 sequents to be generated due to this new invariant e.g., Are ***ML_tl_green/inv2_5/INV*** and ***IL_tl_green/inv2_5/INV*** ***provable***?

INV PO of m_2: ML_out/inv2_4/INV – Updated

axm0_1	$d \in \mathbb{N}$
axm0_2	$d > 0$
axm2_1	$COLOUR = \{green, red\}$
axm2_2	$green \neq red$
inv0_1	$n \in \mathbb{N}$
inv0_2	$n \leq d$
inv1_1	$a \in \mathbb{N}$
inv1_2	$b \in \mathbb{N}$
inv1_3	$c \in \mathbb{N}$
inv1_4	$a + b + c = n$
inv1_5	$a = 0 \vee c = 0$
inv2_1	$ml\_tl \in COLOUR$
inv2_2	$il\_tl \in COLOUR$
inv2_3	$ml\_tl = green \Rightarrow a + b < d \wedge c = 0$
inv2_4	$il\_tl = green \Rightarrow b > 0 \wedge a = 0$
inv2_5	$ml\_tl = red \vee il\_tl = red$
Concrete guards of *ML_out*	$ml\_tl = green$
	$\vdash$
Concrete invariant **inv2_4** with *ML_out*'s effect in the post-state	$il\_tl = green \Rightarrow b > 0 \wedge (a + 1) = 0$

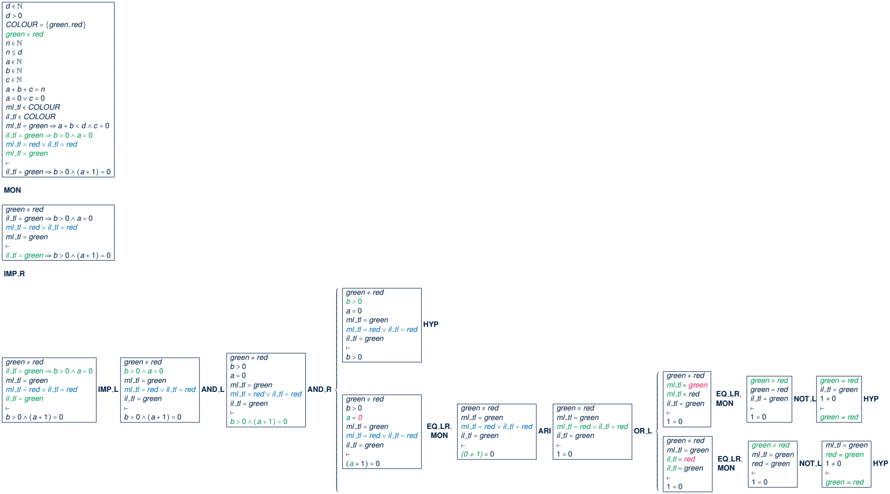
ML_out/inv2_4/INV

INV PO of m_2: IL_out/inv2_3/INV – Updated

axm0_1	$\{\ d \in \mathbb{N}$
axm0_2	$\{\ d > 0$
axm2_1	$\{\ COLOUR = \{green, red\}$
axm2_2	$\{\ green \neq red$
inv0_1	$\{\ n \in \mathbb{N}$
inv0_2	$\{\ n \leq d$
inv1_1	$\{\ a \in \mathbb{N}$
inv1_2	$\{\ b \in \mathbb{N}$
inv1_3	$\{\ c \in \mathbb{N}$
inv1_4	$\{\ a + b + c = n$
inv1_5	$\{\ a = 0 \vee c = 0$
inv2_1	$\{\ ml\_tl \in COLOUR$
inv2_2	$\{\ il\_tl \in COLOUR$
inv2_3	$\{\ ml\_tl = green \Rightarrow a + b < d \wedge c = 0$
inv2_4	$\{\ il\_tl = green \Rightarrow b > 0 \wedge a = 0$
inv2_5	$\{\ ml\_tl = red \vee il\_tl = red$
Concrete guards of *IL_out*	$\{\ il\_tl = green$
	$\vdash$
Concrete invariant **inv2_3** with *ML_out*'s effect in the post-state	$\{\ ml\_tl = green \Rightarrow a + (b - 1) < d \wedge (c + 1) = 0$

IL_out/inv2_3/INV

Proving ML_out/inv2_4/INV: Second Attempt



$d \in \mathbb{N}$
$d > 0$
$COLOUR = \{green, red\}$
$green \neq red$
$n \in \mathbb{N}$
$n \leq d$
$a \in \mathbb{N}$
$b \in \mathbb{N}$
$c \in \mathbb{N}$
$a + b + c = n$
$a = 0 \lor c = 0$
$ml\_tl \in COLOUR$
$il\_tl \in COLOUR$
$ml\_tl = green \Rightarrow a + b < d \land c = 0$
$il\_tl = green \Rightarrow b > 0 \land a = 0$
$ml\_tl = red \lor il\_tl = red$
$ml\_tl = green$
$\vdash$
$il\_tl = green \Rightarrow b > 0 \land (a + 1) = 0$

MON

$green \neq red$
$il\_tl = green \Rightarrow b > 0 \land a = 0$
$ml\_tl = red \lor il\_tl = red$
$ml\_tl = green$
$\vdash$
$il\_tl = green \Rightarrow b > 0 \land (a + 1) = 0$

IMP_R

$green \neq red$
$il\_tl = green \Rightarrow b > 0 \land a = 0$
$ml\_tl = green$
$ml\_tl = red \lor il\_tl = red$
$il\_tl = green$
$\vdash$
$b > 0 \land (a + 1) = 0$

IMP_L

$green \neq red$
$b > 0 \land a = 0$
$ml\_tl = green$
$ml\_tl = red \lor il\_tl = red$
$il\_tl = green$
$\vdash$
$b > 0 \land (a + 1) = 0$

AND_L

$green \neq red$
$b > 0$
$a = 0$
$ml\_tl = green$
$ml\_tl = red \lor il\_tl = red$
$il\_tl = green$
$\vdash$
$b > 0 \land (a + 1) = 0$

AND_R

$green \neq red$
$b > 0$
$a = 0$
$ml\_tl = green$
$ml\_tl = red \lor il\_tl = red$
$il\_tl = green$
$\vdash$
$b > 0$

HYP

$green \neq red$
$b > 0$
$a = 0$
$ml\_tl = green$
$ml\_tl = red \lor il\_tl = red$
$il\_tl = green$
$\vdash$
$(a + 1) = 0$

EQ_LR, MON

$green \neq red$
$ml\_tl = green$
$ml\_tl = red \lor il\_tl = red$
$il\_tl = green$
$\vdash$
$(0 + 1) = 0$

ARI

$green \neq red$
$ml\_tl = green$
$ml\_tl = red \lor il\_tl = red$
$il\_tl = green$
$\vdash$
$1 = 0$

OR_L

$green \neq red$
$ml\_tl = green$
$ml\_tl = red$
$il\_tl = green$
$\vdash$
$1 = 0$

EQ_LR, MON

$green \neq red$
$green = red$
$il\_tl = green$
$\vdash$
$1 = 0$

NOT_L

$green = red$
$il\_tl = green$
$1 \neq 0$
$\vdash$
$green = red$

HYP

$green \neq red$
$ml\_tl = green$
$il\_tl = red$
$il\_tl = green$
$\vdash$
$1 = 0$

EQ_LR, MON

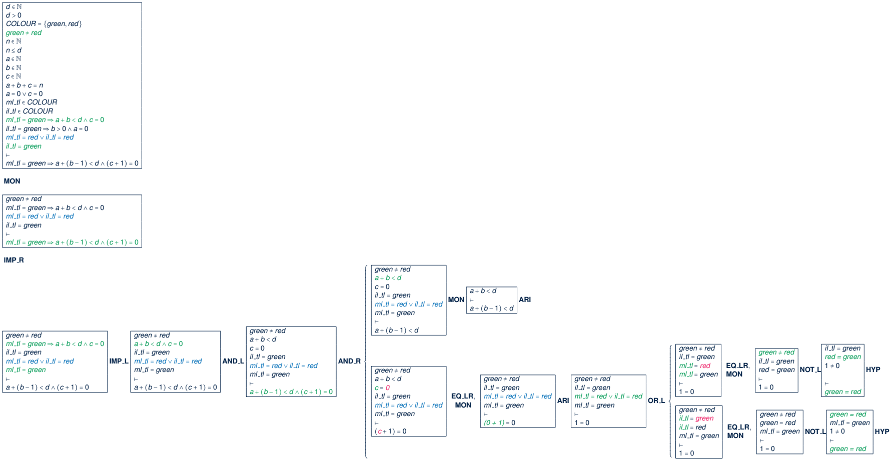
$green \neq red$
$ml\_tl = green$
$red = green$
$\vdash$
$1 = 0$

NOT_L

$ml\_tl = green$
$red = green$
$1 \neq 0$
$\vdash$
$green = red$

HYP

Proving IL_out/inv2_3/INV: Second Attempt



$d \in \mathbb{N}$
$d > 0$
$COLOUR = \{green, red\}$
$green \neq red$
$n \in \mathbb{N}$
$n \leq d$
$a \in \mathbb{N}$
$b \in \mathbb{N}$
$c \in \mathbb{N}$
$a + b + c = n$
$a = 0 \lor c = 0$
$ml\_tl \in COLOUR$
$il\_tl \in COLOUR$
$ml\_tl = green \Rightarrow a + b < d \land c = 0$
$il\_tl = green \Rightarrow b > 0 \land a = 0$
$ml\_tl = red \lor il\_tl = red$
$il\_tl = green$
$\vdash$
$ml\_tl = green \Rightarrow a + (b - 1) < d \land (c + 1) = 0$

MON

$green \neq red$
$ml\_tl = green \Rightarrow a + b < d \land c = 0$
$ml\_tl = red \lor il\_tl = red$
$il\_tl = green$
$\vdash$
$ml\_tl = green \Rightarrow a + (b - 1) < d \land (c + 1) = 0$

IMP_R

$green \neq red$
$ml\_tl = green \Rightarrow a + b < d \land c = 0$
$il\_tl = green$
$ml\_tl = red \lor il\_tl = red$
$ml\_tl = green$
$\vdash$
$a + (b - 1) < d \land (c + 1) = 0$

IMP_L

$green \neq red$
$a + b < d \land c = 0$
$il\_tl = green$
$ml\_tl = red \lor il\_tl = red$
$ml\_tl = green$
$\vdash$
$a + (b - 1) < d \land (c + 1) = 0$

AND_L

$green \neq red$
$a + b < d$
$c = 0$
$il\_tl = green$
$ml\_tl = red \lor il\_tl = red$
$ml\_tl = green$
$\vdash$
$a + (b - 1) < d \land (c + 1) = 0$

AND_R

$green \neq red$
$a + b < d$
$c = 0$
$il\_tl = green$
$ml\_tl = red \lor il\_tl = red$
$ml\_tl = green$
$\vdash$
$a + (b - 1) < d$

MON

$a + b < d$
$\vdash$
$a + (b - 1) < d$

ARI

$green \neq red$
$a + b < d$
$c = 0$
$il\_tl = green$
$ml\_tl = red \lor il\_tl = red$
$ml\_tl = green$
$\vdash$
$(c + 1) = 0$

EQ_LR, MON

$green \neq red$
$il\_tl = green$
$ml\_tl = red \lor il\_tl = red$
$ml\_tl = green$
$\vdash$
$(0 + 1) = 0$

ARI

$green \neq red$
$il\_tl = green$
$ml\_tl = red \lor il\_tl = red$
$ml\_tl = green$
$\vdash$
$1 = 0$

OR_L

$green \neq red$
$il\_tl = green$
$ml\_tl = red$
$ml\_tl = green$
$\vdash$
$1 = 0$

EQ_LR, MON

$green \neq red$
$il\_tl = green$
$red = green$
$\vdash$
$1 = 0$

NOT_L

$il\_tl = green$
$red = green$
$1 \neq 0$
$\vdash$
$green = red$

HYP

$green \neq red$
$il\_tl = green$
$il\_tl = red$
$ml\_tl = green$
$\vdash$
$1 = 0$

EQ_LR, MON

$green \neq red$
$green = red$
$ml\_tl = green$
$\vdash$
$1 = 0$

NOT_L

$green = red$
$ml\_tl = green$
$1 \neq 0$
$\vdash$
$green = red$

HYP

Fixing m_2: Adding Actions

- Recall that an ***invariant*** was added to m_2:

```
invariants:
  inv2_5 : $ml\_tl = red \lor il\_tl = red$
```

- Additional 6 × 1 sequents to be generated due to this new invariant:
  - e.g., ***ML_tl_green***/**inv2_5/INV** [for ***ML_tl_green*** to preserve **inv2_5**]
  - e.g., ***IL_tl_green***/**inv2_5/INV** [for ***IL_tl_green*** to preserve **inv2_5**]
- For the above ***sequents*** to be ***provable***, we need to revise the two events:

```
ML_tl_green
  when
    $ml\_tl = red$
    $a + b < d$
    $c = 0$
  then
    $ml\_tl := green$
    $il\_tl := red$
  end
```

```
IL_tl_green
  when
    $il\_tl = red$
    $b > 0$
    $a = 0$
  then
    $il\_tl := green$
    $ml\_tl := red$
  end
```

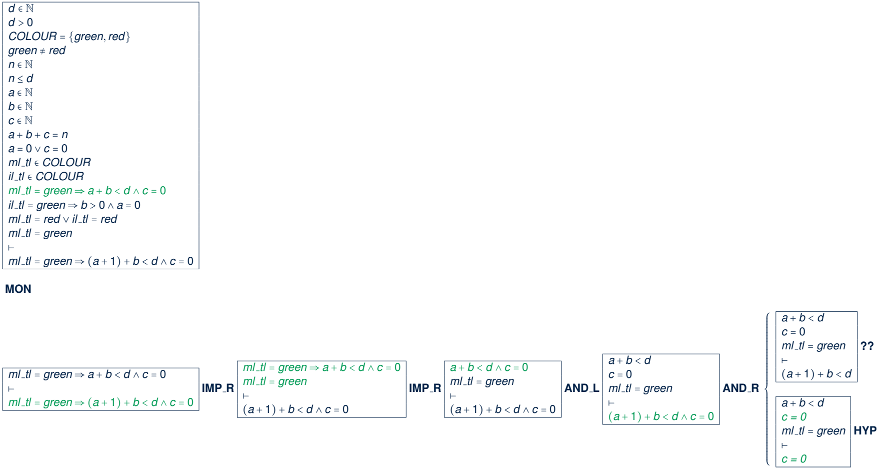
Exercise: Specify and prove ***ML_tl_green***/**inv2_5/INV** & ***IL_tl_green***/**inv2_5/INV**.

INV PO of m_2: ML_out/inv2_3/INV

axm0_1	$\{\ d \in \mathbb{N}$
axm0_2	$\{\ d > 0$
axm2_1	$\{\ COLOUR = \{green, red\}$
axm2_2	$\{\ green \neq red$
inv0_1	$\{\ n \in \mathbb{N}$
inv0_2	$\{\ n \leq d$
inv1_1	$\{\ a \in \mathbb{N}$
inv1_2	$\{\ b \in \mathbb{N}$
inv1_3	$\{\ c \in \mathbb{N}$
inv1_4	$\{\ a + b + c = n$
inv1_5	$\{\ a = 0 \lor c = 0$
inv2_1	$\{\ ml\_tl \in COLOUR$
inv2_2	$\{\ il\_tl \in COLOUR$
inv2_3	$\{\ ml\_tl = green \Rightarrow a + b < d \land c = 0$
inv2_4	$\{\ il\_tl = green \Rightarrow b > 0 \land a = 0$
inv2_5	$\{\ ml\_tl = red \lor il\_tl = red$
Concrete guards of *ML_out*	$\{\ ml\_tl = green$
	$\vdash$
Concrete invariant **inv2_3** with *ML_out*'s effect in the post-state	$\{\ ml\_tl = green \Rightarrow (a + 1) + b < d \land c = 0$

ML_out/inv2_3/INV

Proving ML_out/inv2_3/INV: First Attempt



$d \in \mathbb{N}$
$d > 0$
$COLOUR = \{green, red\}$
$green \neq red$
$n \in \mathbb{N}$
$n \leq d$
$a \in \mathbb{N}$
$b \in \mathbb{N}$
$c \in \mathbb{N}$
$a + b + c = n$
$a = 0 \lor c = 0$
$ml\_tl \in COLOUR$
$il\_tl \in COLOUR$
$ml\_tl = green \Rightarrow a + b < d \land c = 0$
$il\_tl = green \Rightarrow b > 0 \land a = 0$
$ml\_tl = red \lor il\_tl = red$
$ml\_tl = green$
$\vdash$
$ml\_tl = green \Rightarrow (a + 1) + b < d \land c = 0$

MON

$ml\_tl = green \Rightarrow a + b < d \land c = 0$
$\vdash$
$ml\_tl = green \Rightarrow (a + 1) + b < d \land c = 0$

IMP_R

$ml\_tl = green \Rightarrow a + b < d \land c = 0$
$ml\_tl = green$
$\vdash$
$(a + 1) + b < d \land c = 0$

IMP_R

$a + b < d \land c = 0$
$ml\_tl = green$
$\vdash$
$(a + 1) + b < d \land c = 0$

AND_L

$a + b < d$
$c = 0$
$ml\_tl = green$
$\vdash$
$(a + 1) + b < d \land c = 0$

AND_R

$a + b < d$
$c = 0$
$ml\_tl = green$
$\vdash$
$(a + 1) + b < d$

??

$a + b < d$
$c = 0$
$ml\_tl = green$
$\vdash$
$c = 0$

HYP

Failed: ML_out/inv2_3/INV

- Our first attempt of proving ***ML_out/inv2_3/INV*** failed the 1st case (resulted from applying IR **AND_R**):

$$\boxed{a + b < d \land c = 0 \land ml\_tl = green \vdash (a + 1) + b < d}$$

- This ***unprovable*** sequent gave us a good hint:
  - Goal $\underbrace{(a+1)}_{a'} + \underbrace{b}_{b'} < d$ specifies the ***capacity requirement***.
  - Hypothesis $\boxed{c = 0 \land ml\_tl = green}$ assumes that it's safe to exit the ML.
  - Hypothesis $\boxed{a + b < d}$ is **not** strong enough to entail $(a + 1) + b < d$.

e.g., $d = 3, b = 0, a = 0$	[$(a + 1) + b < d$ evaluates to ***true***]
e.g., $d = 3, b = 1, a = 0$	[$(a + 1) + b < d$ evaluates to ***true***]
e.g., $d = 3, b = 0, a = 1$	[$(a + 1) + b < d$ evaluates to ***true***]
e.g., $d = 3, b = 0, a = 2$	[$(a + 1) + b < d$ evaluates to ***false***]
e.g., $d = 3, b = 1, a = 1$	[$(a + 1) + b < d$ evaluates to ***false***]
e.g., $d = 3, b = 2, a = 0$	[$(a + 1) + b < d$ evaluates to ***false***]

  - Therefore, $a + b < d$ (allowing one more car to exit ML) should be split:

$a + b + 1 \neq d$	[more later cars may exit ML, ***ml_tl*** remains ***green***]
$a + b + 1 = d$	[no more later cars may exit ML, ***ml_tl*** turns ***red***]

Fixing m_2: Splitting *ML_out* and *IL_out*

- Recall that ***ML_out/inv2_3/INV*** failed ∵ two cases not handled separately:

  $a + b + 1 \neq d$ [more later cars may exit ML, ***ml_tl*** remains ***green***]

  $a + b + 1 = d$ [no more later cars may exit ML, ***ml_tl*** turns ***red***]

- Similarly, ***IL_out/inv2_4/INV*** would fail ∵ two cases not handled separately:

  $b - 1 \neq 0$ [more later cars may exit IL, ***il_tl*** remains ***green***]

  $b - 1 = 0$ [no more later cars may exit IL, ***il_tl*** turns ***red***]

- Accordingly, we split ***ML_out*** and ***IL_out*** into two with corresponding guards.

```
ML_out_1
  when
    $ml\_tl = green$
    $a + b + 1 \neq d$
  then
    $a := a + 1$
  end
```

```
ML_out_2
  when
    $ml\_tl = green$
    $a + b + 1 = d$
  then
    $a := a + 1$
    $ml\_tl := red$
  end
```

```
IL_out_1
  when
    $il\_tl = green$
    $b \neq 1$
  then
    $b := b - 1$
    $c := c + 1$
  end
```

```
IL_out_2
  when
    $il\_tl = green$
    $b = 1$
  then
    $b := b - 1$
    $c := c + 1$
    $il\_tl := red$
  end
```

Exercise: Specify and prove ***ML_out*/inv2_3/INV** & ***IL_out*/inv2_4/INV**.
Exercise: Given the latest m_2, how many sequents to prove for ***invariant preservation***?
Exercise: Each split event (e.g., *ML_out_1*) refines its ***abstract*** counterpart (e.g., *ML_out*)?

m_2 Livelocks: New Events Diverging

- Recall that a system may ***livelock*** if the new events diverge.
- Current m_2's two new events ***ML_tl_green*** and ***IL_tl_green*** may *diverge*:

```
ML_tl_green
  when
    $ml\_tl = red$
    $a + b < d$
    $c = 0$
  then
    $ml\_tl := green$
    $il\_tl := red$
  end
```

```
IL_tl_green
  when
    $il\_tl = red$
    $b > 0$
    $a = 0$
  then
    $il\_tl := green$
    $ml\_tl := red$
  end
```

- *ML_tl_green* and *IL_tl_green* both ***enabled*** and may occur ***indefinitely***, preventing other "old" events (e.g., *ML_out*) from ever happening:

$$\langle \underbrace{init}_{\substack{d=2\\ a'=0\\ b'=0\\ c'=0\\ ml\_tl=red\\ il\_tl=red}}, \underbrace{ML\_tl\_green}_{\substack{d=2\\ a'=0\\ b'=0\\ c'=0\\ ml\_tl'=green\\ il\_tl'=red}}, \underbrace{ML\_out\_1}_{\substack{d=2\\ a'=1\\ b'=0\\ c'=0\\ ml\_tl'=green\\ il\_tl'=red}}, \underbrace{IL\_in}_{\substack{d=2\\ a'=0\\ b'=1\\ c'=0\\ ml\_tl'=green\\ il\_tl'=red}}, \underbrace{IL\_tl\_green}_{\substack{d=2\\ a'=0\\ b'=1\\ c'=0\\ ml\_tl'=red\\ il\_tl'=green}}, \underbrace{ML\_tl\_green}_{\substack{d=2\\ a'=0\\ b'=1\\ c'=0\\ ml\_tl'=green\\ il\_tl'=red}}, \underbrace{IL\_tl\_green}_{\substack{d=2\\ a'=0\\ b'=1\\ c'=0\\ ml\_tl'=red\\ il\_tl'=green}}, \dots \rangle$$

⇒ Two traffic lights keep changing colors so rapidly that **no** drivers can ever pass!

- **Solution**: Allow color changes between traffic lights in a disciplined way.

Fixing m_2: Regulating Traffic Light Changes

We introduce two variables/flags for regulating traffic light changes:

- ***ml_pass*** is ***1*** **if**, since ***ml_tl*** was last turned ***green***, at least one car exited the ML onto the bridge. Otherwise, ***ml_pass*** is ***0***.
- ***il_pass*** is ***1*** **if**, since ***il_tl*** was last turned ***green***, at least one car exited the IL onto the bridge. Otherwise, ***il_pass*** is ***0***.

variables: ml_pass, il_pass

invariants:
inv2_6 : $ml\_pass \in \{0, 1\}$
inv2_7 : $il\_pass \in \{0, 1\}$
inv2_8 : $ml\_tl = red \Rightarrow ml\_pass = 1$
inv2_9 : $il\_tl = red \Rightarrow il\_pass = 1$

```
ML_out_1
  when
    $ml\_tl = green$
    $a + b + 1 \neq d$
  then
    $a := a + 1$
    $ml\_pass := 1$
  end
```

```
ML_out_2
  when
    $ml\_tl = green$
    $a + b + 1 = d$
  then
    $a := a + 1$
    $ml\_tl := red$
    $ml\_pass := 1$
  end
```

```
IL_out_1
  when
    $il\_tl = green$
    $b \neq 1$
  then
    $b := b - 1$
    $c := c + 1$
    $il\_pass := 1$
  end
```

```
IL_out_2
  when
    $il\_tl = green$
    $b = 1$
  then
    $b := b - 1$
    $c := c + 1$
    $il\_tl := red$
    $il\_pass := 1$
  end
```

```
ML_tl_green
  when
    $ml\_tl = red$
    $a + b < d$
    $c = 0$
    $il\_pass = 1$
  then
    $ml\_tl := green$
    $il\_tl := red$
    $ml\_pass := 0$
  end
```

```
IL_tl_green
  when
    $il\_tl = red$
    $b > 0$
    $a = 0$
    $ml\_pass = 1$
  then
    $il\_tl := green$
    $ml\_tl := red$
    $il\_pass := 0$
  end
```

Fixing m_2: Measuring Traffic Light Changes

- Recall:
  - Interleaving of ***new*** events charactered as an integer expression: ***variant***.
  - A variant $V(c, w)$ may refer to constants and/or ***concrete*** variables.
  - In the latest m_2, let's try $\boxed{\textbf{variants} : ml\_pass + il\_pass}$
- Accordingly, for the ***new*** event *ML_tl_green*:

$$\begin{array}{lll}
d \in \mathbb{N} & d > 0 & \\
COLOUR = \{green, red\} & green \neq red & \\
n \in \mathbb{N} & n \leq d & \\
a \in \mathbb{N} & b \in \mathbb{N} & c \in \mathbb{N} \\
a + b + c = n & a = 0 \lor c = 0 & \\
ml\_tl \in COLOUR & il\_tl \in COLOUR & \\
ml\_tl = green \Rightarrow a + b < d \land c = 0 & il\_tl = green \Rightarrow b > 0 \land a = 0 & \\
ml\_tl = red \lor il\_tl = red & & \\
ml\_pass \in \{0, 1\} & il\_pass \in \{0, 1\} & \\
ml\_tl = red \Rightarrow ml\_pass = 1 & il\_tl = red \Rightarrow il\_pass = 1 & \\
ml\_tl = red & a + b < d & c = 0 \\
il\_pass = 1 & & \\
\vdash & & \\
0 + il\_pass < ml\_pass + il\_pass & &
\end{array}$$

ML_tl_green/VAR

Exercises: Prove **ML_tl_green/VAR** and Formulate/Prove **IL_tl_green/VAR**.

PO Rule: Relative Deadlock Freedom of m_2

$$
\begin{array}{rl}
\textbf{axm0\_1} & \{\ d \in \mathbb{N} \\
\textbf{axm0\_2} & \{\ d > 0 \\
\textbf{axm2\_1} & \{\ COLOUR = \{green, red\} \\
\textbf{axm2\_2} & \{\ green \neq red \\
\textbf{inv0\_1} & \{\ n \in \mathbb{N} \\
\textbf{inv0\_2} & \{\ n \leq d \\
\textbf{inv1\_1} & \{\ a \in \mathbb{N} \\
\textbf{inv1\_2} & \{\ b \in \mathbb{N} \\
\textbf{inv1\_3} & \{\ c \in \mathbb{N} \\
\textbf{inv1\_4} & \{\ a + b + c = n \\
\textbf{inv1\_5} & \{\ a = 0 \lor c = 0 \\
\textbf{inv2\_1} & \{\ ml\_tl \in COLOUR \\
\textbf{inv2\_2} & \{\ il\_tl \in COLOUR \\
\textbf{inv2\_3} & \{\ ml\_tl = green \Rightarrow a + b < d \land c = 0 \\
\textbf{inv2\_4} & \{\ il\_tl = green \Rightarrow b > 0 \land a = 0 \\
\textbf{inv2\_5} & \{\ ml\_tl = red \lor il\_tl = red \\
\textbf{inv2\_6} & \{\ ml\_pass \in \{0, 1\} \\
\textbf{inv2\_7} & \{\ il\_pass \in \{0, 1\} \\
\textbf{inv2\_8} & \{\ ml\_tl = red \Rightarrow ml\_pass = 1 \\
\textbf{inv2\_9} & \{\ il\_tl = red \Rightarrow il\_pass = 1 \\
\text{Disjunction of } \textit{abstract} \text{ guards} & \left\{ \begin{array}{lrl}
 & a + b < d \land c = 0 & \}\ \textbf{guards of } ML\_out \textbf{ in } m_1 \\
\lor & c > 0 & \}\ \textbf{guards of } ML\_in \textbf{ in } m_1 \\
\lor & a > 0 & \}\ \textbf{guards of } IL\_in \textbf{ in } m_1 \\
\lor & b > 0 \land a = 0 & \}\ \textbf{guards of } IL\_out \textbf{ in } m_1
\end{array} \right. \\
 & \vdash \\
\text{Disjunction of } \textit{concrete} \text{ guards} & \left\{ \begin{array}{lrl}
 & ml\_tl = red \land a + b < d \land c = 0 \land il\_pass = 1 & \}\ \textbf{guards of } ML\_tl\_green \textbf{ in } m_2 \\
\lor & il\_tl = red \land b > 0 \land a = 0 \land ml\_pass = 1 & \}\ \textbf{guards of } IL\_tl\_green \textbf{ in } m_2 \\
\lor & ml\_tl = green \land a + b + 1 \neq d & \}\ \textbf{guards of } ML\_out\_1 \textbf{ in } m_2 \\
\lor & ml\_tl = green \land a + b + 1 = d & \}\ \textbf{guards of } ML\_out\_2 \textbf{ in } m_2 \\
\lor & il\_tl = green \land b \neq 1 & \}\ \textbf{guards of } IL\_out\_1 \textbf{ in } m_2 \\
\lor & il\_tl = green \land b = 1 & \}\ \textbf{guards of } IL\_out\_2 \textbf{ in } m_2 \\
\lor & a > 0 & \}\ \textbf{guards of } ML\_in \textbf{ in } m_2 \\
\lor & c > 0 & \}\ \textbf{guards of } IL\_in \textbf{ in } m_2
\end{array} \right.
\end{array}
$$

DLF

Proving Refinement: DLF of m_2

$$\boxed{\begin{array}{l}
d \in \mathbb{N} \\
d > 0 \\
COLOUR = \{green, red\} \\
green \neq red \\
n \in \mathbb{N} \\
n \leq d \\
a \in \mathbb{N} \\
b \in \mathbb{N} \\
c \in \mathbb{N} \\
a + b + c = n \\
a = 0 \lor c = 0 \\
ml\_tl \in COLOUR \\
il\_tl \in COLOUR \\
ml\_tl = green \Rightarrow a + b < d \land c = 0 \\
il\_tl = green \Rightarrow b > 0 \land a = 0 \\
ml\_tl = red \lor il\_tl = red \\
ml\_pass \in \{0, 1\} \\
il\_pass \in \{0, 1\} \\
ml\_tl = red \Rightarrow ml\_pass = 1 \\
il\_tl = red \Rightarrow il\_pass = 1 \\
\quad\;\; a + b < d \land c = 0 \\
\lor \;\; c > 0 \\
\lor \;\; a > 0 \\
\lor \;\; b > 0 \land a = 0 \\
\vdash \\
\quad\;\; ml\_tl = red \land a + b < d \land c = 0 \land il\_pass = 1 \\
\lor \;\; il\_tl = red \land b > 0 \land a = 0 \land ml\_pass = 1 \\
\lor \;\; ml\_tl = green \\
\lor \;\; il\_tl = green \\
\lor \;\; a > 0 \\
\lor \;\; c > 0
\end{array}}$$

⋮

$$\boxed{\begin{array}{l}
d \in \mathbb{N} \\
d > 0 \\
b \in \mathbb{N} \\
ml\_tl = red \\
il\_tl = red \\
ml\_tl = red \Rightarrow ml\_pass = 1 \\
il\_tl = red \Rightarrow il\_pass = 1 \\
\vdash \\
\quad\;\; b < d \land ml\_pass = 1 \land il\_pass = 1 \\
\lor \;\; b > 0 \land ml\_pass = 1 \land il\_pass = 1
\end{array}} \;\ldots\; \boxed{\begin{array}{l}
d \in \mathbb{N} \\
d > 0 \\
b \in \mathbb{N} \\
ml\_tl = red \\
il\_tl = red \\
ml\_pass = 1 \\
il\_pass = 1 \\
\vdash \\
\quad\;\; b < d \land ml\_pass = 1 \land il\_pass = 1 \\
\lor \;\; b > 0 \land ml\_pass = 1 \land il\_pass = 1
\end{array}} \;\ldots$$

$$\boxed{\begin{array}{l}
d > 0 \\
b \in \mathbb{N} \\
\vdash \\
b < d \lor b > 0
\end{array}} \;\textbf{ARI}\; \boxed{\begin{array}{l}
d > 0 \\
b > 0 \lor b = 0 \\
\vdash \\
b < d \lor b > 0
\end{array}} \;\textbf{OR\_L}\; \left\{ \begin{array}{l}
\boxed{\begin{array}{l}
d > 0 \\
b > 0 \\
\vdash \\
b < d \lor b > 0
\end{array}} \;\textbf{OR\_R2}\; \boxed{\begin{array}{l}
d > 0 \\
b > 0 \\
\vdash \\
b > 0
\end{array}} \;\textbf{HYP} \\[2ex]
\boxed{\begin{array}{l}
d > 0 \\
b = 0 \\
\vdash \\
b < d \lor b > 0
\end{array}} \;\textbf{EQ\_LR, MON}\; \boxed{\begin{array}{l}
d > 0 \\
\\
\vdash \\
0 < d \lor 0 > 0
\end{array}} \;\textbf{OR\_R1}\; \boxed{\begin{array}{l}
d > 0 \\
\\
\vdash \\
0 < d
\end{array}} \;\textbf{HYP}
\end{array} \right.$$

Second Refinement: Summary

- The final version of our ***second refinement*** m_2 is ***provably correct*** w.r.t.:
  - Establishment of ***Concrete Invariants*** [*init*]
  - Preservation of ***Concrete Invariants*** [old & new events]
  - Strengthening of ***guards*** [old events]
  - ***Convergence*** (a.k.a. livelock freedom, non-divergence) [new events]
  - Relative ***Deadlock*** Freedom
- Here is the final specification of m_2:

```
constants:  d
```

```
sets:  COLOR
```

```
axioms:
  axm0_1 : d ∈ ℕ
  axm0_2 : d > 0
  axm2_1 : COLOR = {green, red}
  axm2_2 : green ≠ red
```

```
variables:
  a
  b
  c
  ml_tl
  il_tl
  ml_pass
  il_pass
```

```
invariants:
  inv2_1 : ml_tl ∈ COLOUR
  inv2_2 : il_tl ∈ COLOUR
  inv2_3 : ml_tl = green ⇒ a + b < d ∧ c = 0
  inv2_4 : il_tl = green ⇒ b > 0 ∧ a = 0
  inv2_5 : ml_tl = red ∨ il_tl = red
  inv2_6 : ml_pass ∈ {0, 1}
  inv2_7 : il_pass ∈ {0, 1}
  inv2_8 : ml_tl = red ⇒ ml_pass = 1
  inv2_9 : il_tl = red ⇒ il_pass = 1
```

```
variants:
  ml_pass + il_pass
```

```
ML_tl_green
  when
    ml_tl = red
    a + b < d
    c = 0
    il_pass = 1
  then
    ml_tl := green
    il_tl := red
    ml_pass := 0
  end
```

```
IL_tl_green
  when
    il_tl = red
    b > 0
    a = 0
    ml_pass = 1
  then
    il_tl := green
    ml_tl := red
    il_pass := 0
  end
```

```
ML_out_1
  when
    ml_tl = green
    a + b + 1 ≠ d
  then
    a := a + 1
    ml_pass := 1
  end
```

```
ML_out_2
  when
    ml_tl = green
    a + b + 1 = d
  then
    a := a + 1
    ml_tl := red
    ml_pass := 1
  end
```

```
IL_out_1
  when
    il_tl = green
    b ≠ 1
  then
    b := b − 1
    c := c + 1
    il_pass := 1
  end
```

```
IL_out_2
  when
    il_tl = green
    b = 1
  then
    b := b − 1
    c := c + 1
    il_tl := red
    il_pass := 1
  end
```

```
ML_in
  when
    c > 0
  then
    c := c − 1
  end
```

```
IL_in
  when
    a > 0
  then
    a := a − 1
    b := b + 1
  end
```

Index (1)

Index (2)

Index (3)

Index (4)

Index (5)

Index (6)

Index (7)

Index (8)

Index (9)

Index (10)

Index (11)