Review of Math

MEB: Chapter 9

EECS3342 Z: System Specification and Refinement
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Learning Outcomes of this Lecture

This module is designed to help you **review**:

- Propositional Logic
- Predicate Logic
- Sets, Relations, and Functions

Propositional Logic (1)

- A *proposition* is a statement of claim that must be of either *true* or *false*, but not both.
- Basic logical operands are of type Boolean: *true* and *false*.
- We use logical operators to construct compound statements.
  - Unary logical operator: negation ($\neg$)

p	$\neg p$
true	*false*
false	*true*

  - Binary logical operators: conjunction ($\wedge$), disjunction ($\vee$), implication ($\Rightarrow$), equivalence ($\equiv$), and if-and-only-if ($\iff$).

p	q	$p \wedge q$	$p \vee q$	$p \Rightarrow q$	$p \iff q$	$p \equiv q$
true	*true*	*true*	*true*	*true*	*true*	*true*
true	*false*	*false*	*true*	*false*	*false*	*false*
false	*true*	*false*	*true*	*true*	*false*	*false*
false	*false*	*false*	*false*	*true*	*true*	*true*

Propositional Logic: Implication (1)

- Written as $p \Rightarrow q$ [pronounced as "p implies q"]
  - We call p the antecedent, assumption, or premise.
  - We call q the consequence or conclusion.
- Compare the *truth* of $p \Rightarrow q$ to whether a contract is *honoured*:
  - antecedent/assumption/premise $p \approx$ promised terms [e.g., salary]
  - consequence/conclusion $q \approx$ obligations [e.g., duties]
- When the promised terms are met, then the contract is:
  - *honoured* if the obligations fulfilled. [$(true \Rightarrow true) \iff true$]
  - *breached* if the obligations violated. [$(true \Rightarrow false) \iff false$]
- When the promised terms are not met, then:
  - Fulfilling the obligation (q) or not ($\neg q$) does *not breach* the contract.

p	q	$p \Rightarrow q$
false	*true*	*true*
false	*false*	*true*

Propositional Logic: Implication (2)

There are alternative, equivalent ways to expressing $p \Rightarrow q$:

- *q* **if** *p*

  q is *true* if *p* is *true*
- *p* **only if** *q*

  If *p* is *true*, then for $p \Rightarrow q$ to be *true*, it can only be that *q* is also *true*.
  Otherwise, if *p* is *true* but *q* is *false*, then $(true \Rightarrow false) \equiv false$.

  Note. To prove $p \equiv q$, prove $p \iff q$ (pronounced: "p if and only if q"):
  - *p* **if** *q* [$q \Rightarrow p$]
  - *p* **only if** *q* [$p \Rightarrow q$]
- *p* is **sufficient** for *q*

  For *q* to be *true*, it is sufficient to have *p* being *true*.
- *q* is **necessary** for *p* [similar to *p* **only if** *q*]

  If *p* is *true*, then it is necessarily the case that *q* is also *true*.
  Otherwise, if *p* is *true* but *q* is *false*, then $(true \Rightarrow false) \equiv false$.
- *q* **unless** $\neg p$ [When is $p \Rightarrow q$ *true*?]

  If *q* is *true*, then $p \Rightarrow q$ *true* regardless of *p*.
  If *q* is *false*, then $p \Rightarrow q$ cannot be *true* unless *p* is *false*.

Propositional Logic: Implication (3)

Given an implication $p \Rightarrow q$, we may construct its:

- **Inverse**: $\neg p \Rightarrow \neg q$ [negate antecedent and consequence]
- **Converse**: $q \Rightarrow p$ [swap antecedent and consequence]
- **Contrapositive**: $\neg q \Rightarrow \neg p$ [inverse of converse]

Propositional Logic (2)

- **Axiom**: Definition of $\Rightarrow$

$$p \Rightarrow q \equiv \neg p \vee q$$

- **Theorem**: Identity of $\Rightarrow$

$$true \Rightarrow p \equiv p$$

- **Theorem**: Zero of $\Rightarrow$

$$false \Rightarrow p \equiv true$$

- **Axiom**: De Morgan

$$\begin{aligned} \neg(p \wedge q) &\equiv \neg p \vee \neg q \\ \neg(p \vee q) &\equiv \neg p \wedge \neg q \end{aligned}$$

- **Axiom**: Double Negation

$$p \equiv \neg\,(\neg\, p)$$

- **Theorem**: Contrapositive

$$p \Rightarrow q \equiv \neg q \Rightarrow \neg p$$

Predicate Logic (1)

- A *predicate* is a ***universal*** or ***existential*** statement about objects in some universe of disclosure.
- Unlike propositions, predicates are typically specified using ***variables***, each of which declared with some ***range*** of values.
- We use the following symbols for common numerical ranges:
  - $\mathbb{Z}$: the set of integers $[\,-\infty, \ldots, -1, 0, 1, \ldots, +\infty\,]$
  - $\mathbb{N}$: the set of natural numbers $[\,0, 1, \ldots, +\infty\,]$
- Variable(s) in a predicate may be ***quantified***:
  - *Universal quantification* :
    All values that a variable may take satisfy certain property.
    e.g., Given that *i* is a natural number, *i* is *always* non-negative.
  - *Existential quantification* :
    Some value that a variable may take satisfies certain property.
    e.g., Given that *i* is an integer, *i can be* negative.

Predicate Logic (2.1): Universal Q. ($\forall$)

- A *universal quantification* has the form $(\forall X \bullet R \Rightarrow P)$
  - X is a comma-separated list of variable names
  - R is a *constraint on types/ranges* of the listed variables
  - P is a *property* to be satisfied
- ***For all*** (combinations of) values of variables listed in X that satisfies R, it is the case that P is satisfied.
  - $\forall i \bullet i \in \mathbb{N} \Rightarrow i \geq 0$ [*true*]
  - $\forall i \bullet i \in \mathbb{Z} \Rightarrow i \geq 0$ [*false*]
  - $\forall i, j \bullet i \in \mathbb{Z} \land j \in \mathbb{Z} \Rightarrow i < j \lor i > j$ [*false*]
- *Proof Strategies*
  1. How to prove $(\forall X \bullet R \Rightarrow P)$ ***true***?
     - **Hint.** When is $R \Rightarrow P$ ***true***? [*true* $\Rightarrow$ *true*, *false* $\Rightarrow$ _]
     - Show that for all instances of $x \in X$ s.t. $R(x)$, $P(x)$ holds.
     - Show that for all instances of $x \in X$ it is the case $\neg R(x)$.
  2. How to prove $(\forall X \bullet R \Rightarrow P)$ ***false***?
     - **Hint.** When is $R \Rightarrow P$ ***false***? [*true* $\Rightarrow$ *false*]
     - Give a ***witness/counterexample*** of $x \in X$ s.t. $R(x)$, $\neg P(x)$ holds.

Predicate Logic (2.2): Existential Q. ($\exists$)

- An *existential quantification* has the form $(\exists X \bullet R \wedge P)$
  - X is a comma-separated list of variable names
  - R is a *constraint on types/ranges* of the listed variables
  - P is a *property* to be satisfied
- ***There exist*** (a combination of) values of variables listed in X that satisfy both R and P.
  - $\exists i \bullet i \in \mathbb{N} \wedge i \geq 0$ [*true*]
  - $\exists i \bullet i \in \mathbb{Z} \wedge i \geq 0$ [*true*]
  - $\exists i, j \bullet i \in \mathbb{Z} \wedge j \in \mathbb{Z} \wedge i < j \vee i > j$ [*true*]
- *Proof Strategies*
  1. How to prove $(\exists X \bullet R \wedge P)$ ***true***?
     - **Hint.** When is $R \wedge P$ ***true***? [*true* $\wedge$ *true*]
     - Give a ***witness*** of $x \in X$ s.t. $R(x)$, $P(x)$ holds.
  2. How to prove $(\exists X \bullet R \wedge P)$ ***false***?
     - **Hint.** When is $R \wedge P$ ***false***? [*true* $\wedge$ *false*, *false* $\wedge$ _]
     - Show that for all instances of $x \in X$ s.t. $R(x)$, $\neg P(x)$ holds.
     - Show that for all instances of $x \in X$ it is the case $\neg R(x)$.

Predicate Logic (3): Exercises

- Prove or disprove: $\forall x \bullet (x \in \mathbb{Z} \wedge 1 \leq x \leq 10) \Rightarrow x > 0$.

  All 10 integers between 1 and 10 are greater than 0.
- Prove or disprove: $\forall x \bullet (x \in \mathbb{Z} \wedge 1 \leq x \leq 10) \Rightarrow x > 1$.

  Integer 1 (a witness/counterexample) in the range between 1 and 10 is *not* greater than 1.
- Prove or disprove: $\exists x \bullet (x \in \mathbb{Z} \wedge 1 \leq x \leq 10) \wedge x > 1$.

  Integer 2 (a witness) in the range between 1 and 10 is greater than 1.
- Prove or disprove that $\exists x \bullet (x \in \mathbb{Z} \wedge 1 \leq x \leq 10) \wedge x > 10$?

  All integers in the range between 1 and 10 are *not* greater than 10.

Predicate Logic (4): Switching Quantifications

Conversions between ∀ and ∃:

$$(\forall X \bullet R \Rightarrow P) \iff \neg(\exists X \bullet R \land \neg P)$$
$$(\exists X \bullet R \land P) \iff \neg(\forall X \bullet R \Rightarrow \neg P)$$

Sets: Definitions and Membership

- A **set** is a collection of objects.
  - Objects in a set are called its *elements* or *members*.
  - *Order* in which elements are arranged does not matter.
  - An element can appear *at most once* in the set.
- We may define a set using:
  - ***Set Enumeration***: Explicitly list all members in a set.
    e.g., $\{1, 3, 5, 7, 9\}$
  - ***Set Comprehension***: Implicitly specify the condition that all members satisfy.
    e.g., $\{x \mid 1 \le x \le 10 \land x \texttt{ is an odd number}\}$
- An empty set (denoted as $\{\}$ or $\varnothing$) has no members.
- We may check if an element is a *member* of a set:
  e.g., $5 \in \{1, 3, 5, 7, 9\}$ [*true*]
  e.g., $4 \notin \{x \mid x \le 1 \le 10, x \texttt{ is an odd number}\}$ [*true*]
- The number of elements in a set is called its *cardinality*.
  e.g., $|\varnothing| = 0$, $|\{x \mid x \le 1 \le 10, x \texttt{ is an odd number}\}| = 5$

Set Relations

Given two sets S_1 and S_2:

- S_1 is a ***subset*** of S_2 if every member of S_1 is a member of S_2.

$$S_1 \subseteq S_2 \iff (\forall x \bullet x \in S1 \Rightarrow x \in S2)$$

- S_1 and S_2 are ***equal*** iff they are the subset of each other.

$$S_1 = S_2 \iff S_1 \subseteq S_2 \wedge S_2 \subseteq S_1$$

- S_1 is a ***proper subset*** of S_2 if it is a strictly smaller subset.

$$S_1 \subset S_2 \iff S_1 \subseteq S_2 \wedge |S1| < |S2|$$

Set Relations: Exercises

$? \subseteq S$ always holds	[$\varnothing$ and S]
$? \subset S$ always fails	[S]
$? \subset S$ holds for some S and fails for some S	[$\varnothing$]
$S_1 = S_2 \Rightarrow S_1 \subseteq S_2$?	[Yes]
$S_1 \subseteq S_2 \Rightarrow S_1 = S_2$?	[No]

Set Operations

Given two sets S_1 and S_2:

- ***Union*** of S_1 and S_2 is a set whose members are in either.

$$S_1 \cup S_2 = \{x \mid x \in S_1 \lor x \in S_2\}$$

- ***Intersection*** of S_1 and S_2 is a set whose members are in both.

$$S_1 \cap S_2 = \{x \mid x \in S_1 \land x \in S_2\}$$

- ***Difference*** of S_1 and S_2 is a set whose members are in S_1 but not S_2.

$$S_1 \setminus S_2 = \{x \mid x \in S_1 \land x \notin S_2\}$$

Power Sets

The *power set* of a set S is a *set* of all S's *subsets*.

$$\mathbb{P}(S) = \{s \mid s \subseteq S\}$$

The power set contains subsets of *cardinalities* 0, 1, 2, ..., $|S|$.
e.g., $\mathbb{P}(\{1,2,3\})$ is a set of sets, where each member set s has cardinality 0, 1, 2, or 3:

$$\left\{\begin{array}{l} \varnothing, \\ \{1\},\ \{2\},\ \{3\}, \\ \{1,2\},\ \{2,3\},\ \{3,1\}, \\ \{1,2,3\} \end{array}\right\}$$

Exercise: What is $\mathbb{P}(\{1,2,3,4,5\}) \setminus \mathbb{P}(\{1,2,3\})$?

Set of Tuples

Given n sets $S_1, S_2, \ldots, S_n$, a *cross/Cartesian product* of theses sets is a set of n-tuples.

Each *n-tuple* $(e_1, e_2, \ldots, e_n)$ contains n elements, each of which a member of the corresponding set.

$$S_1 \times S_2 \times \cdots \times S_n = \{(e_1, e_2, \ldots, e_n) \mid e_i \in S_i \wedge 1 \leq i \leq n\}$$

e.g., $\{a, b\} \times \{2, 4\} \times \{\$, \&\}$ is a set of triples:

$$\begin{aligned} & \{a, b\} \times \{2, 4\} \times \{\$, \&\} \\ = \; & \{\, (e_1, e_2, e_3) \mid e_1 \in \{a, b\} \wedge e_2 \in \{2, 4\} \wedge e_3 \in \{\$, \&\} \,\} \\ = \; & \left\{ \begin{array}{l} (a, 2, \$), (a, 2, \&), (a, 4, \$), (a, 4, \&), \\ (b, 2, \$), (b, 2, \&), (b, 4, \$), (b, 4, \&) \end{array} \right\} \end{aligned}$$

Relations (1): Constructing a Relation

A *relation* is a set of mappings, each being an ***ordered pair*** that maps a member of set S to a member of set T.

e.g., Say $S = \{1, 2, 3\}$ and $T = \{a, b\}$

- $\varnothing$ is an empty relation.
- $\boxed{S \times T}$ is the *maximum* relation (say r_1) between S and T, mapping from each member of S to each member in T:

$$\{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$$

- $\{(x, y) \mid (x, y) \in S \times T \land x \neq 1\}$ is a relation (say r_2) that maps only some members in S to every member in T:

$$\{(2, a), (2, b), (3, a), (3, b)\}$$

Relations (2.1): Set of Possible Relations

- We use the power set operator to express the set of *all possible relations* on S and T:

$$\mathbb{P}(S \times T)$$

  Each member in $\mathbb{P}(S \times T)$ is a relation.

- To declare a relation variable r, we use the colon (**:**) symbol to mean *set membership*:

$$r : \mathbb{P}(S \times T)$$

- Or alternatively, we write:

$$r : S \leftrightarrow T$$

  where the set $S \leftrightarrow T$ is synonymous to the set $\mathbb{P}(S \times T)$

Relations (2.2): Exercise

Enumerate $\{a, b\} \leftrightarrow \{1, 2, 3\}$.

- **Hints**:
  - You may enumerate all relations in $\mathbb{P}(\{a, b\} \times \{1, 2, 3\})$ via their *cardinalities*: 0, 1, ..., $|\{a, b\} \times \{1, 2, 3\}|$.
  - What's the *maximum* relation in $\mathbb{P}(\{a, b\} \times \{1, 2, 3\})$?

$$\{\ (a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\ \}$$

- The answer is a set containing ***all*** of the following relations:
  - Relation with cardinality 0: $\varnothing$
  - How many relations with cardinality 1? $[\ \binom{|\{a,b\}\times\{1,2,3\}|}{1} = 6\ ]$
  - How many relations with cardinality 2? $[\ \binom{|\{a,b\}\times\{1,2,3\}|}{2} = \frac{6\times 5}{2!} = 15\ ]$

  ...

  - Relation with cardinality $|\{a, b\} \times \{1, 2, 3\}|$:

$$\{\ (a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\ \}$$

Relations (3.1): Domain, Range, Inverse

Given a relation

r = {(a, 1), (b, 2), (c, 3), (a, 4), (b, 5), (c, 6), (d, 1), (e, 2), (f, 3)}

- ***domain*** **of** r: set of first-elements from r
  - Definition: $\text{dom}(r) = \{\ d \mid (d, r') \in r\ \}$
  - e.g., $\text{dom}(r) = \{a, b, c, d, e, f\}$
  - ASCII syntax: `dom(r)`
- ***range*** **of** r: set of second-elements from r
  - Definition: $\text{ran}(r) = \{\ r' \mid (d, r') \in r\ \}$
  - e.g., $\text{ran}(r) = \{1, 2, 3, 4, 5, 6\}$
  - ASCII syntax: `ran(r)`
- ***inverse*** **of** r: a relation like r with elements swapped
  - Definition: $r^{-1} = \{\ (r', d) \mid (d, r') \in r\ \}$
  - e.g., $r^{-1} = \{(1, a), (2, b), (3, c), (4, a), (5, b), (6, c), (1, d), (2, e), (3, f)\}$
  - ASCII syntax: `r~`

Relations (3.2): Image

Given a relation

$$r = \{(a, 1), (b, 2), (c, 3), (a, 4), (b, 5), (c, 6), (d, 1), (e, 2), (f, 3)\}$$

relational image of r over set s : sub-range of r mapped by s.

- Definition: $r[s] = \{\ r' \mid (d, r') \in r \land d \in s\ \}$
- e.g., $r[\{a, b\}] = \{1, 2, 4, 5\}$
- ASCII syntax: `r[s]`

Relations (3.3): Restrictions

Given a relation

r = {(a, 1), (b, 2), (c, 3), (a, 4), (b, 5), (c, 6), (d, 1), (e, 2), (f, 3)}

- ***domain restriction* of *r* over set *ds***: sub-relation of *r* with domain *ds*.
  - Definition: $ds \lhd r = \{ (d, r') \mid (d, r') \in r \land d \in ds \}$
  - e.g., $\{a, b\} \lhd r = \{(\mathbf{a}, 1), (\mathbf{b}, 2), (\mathbf{a}, 4), (\mathbf{b}, 5)\}$
  - ASCII syntax: `ds <| r`
- ***range restriction* of *r* over set *rs***: sub-relation of *r* with range *rs*.
  - Definition: $r \rhd rs = \{ (d, r') \mid (d, r') \in r \land r' \in rs \}$
  - e.g., $r \rhd \{1, 2\} = \{(a, \mathbf{1}), (b, \mathbf{2}), (d, \mathbf{1}), (e, \mathbf{2})\}$
  - ASCII syntax: `r |> rs`

Relations (3.4): Subtractions

Given a relation

r = {(a, 1), (b, 2), (c, 3), (a, 4), (b, 5), (c, 6), (d, 1), (e, 2), (f, 3)}

- ***domain subtraction*** of *r* over set *ds* : sub-relation of *r* with domain not *ds*.
  - Definition: $ds \lhd r = \{\ (d, r') \mid (d, r') \in r \land d \notin ds\ \}$
  - e.g., $\{a, b\} \lhd r = \{(\mathbf{c}, 3), (\mathbf{c}, 6), (\mathbf{d}, 1), (\mathbf{e}, 2), (\mathbf{f}, 3)\}$
  - ASCII syntax: `ds <<| r`
- ***range subtraction*** of *r* over set *rs* : sub-relation of *r* with range not *rs*.
  - Definition: $r \rhd rs = \{\ (d, r') \mid (d, r') \in r \land r' \notin rs\ \}$
  - e.g., $r \rhd \{1, 2\} = \{\{(c, \mathbf{3}), (a, \mathbf{4}), (b, \mathbf{5}), (c, \mathbf{6}), (f, \mathbf{3})\}\}$
  - ASCII syntax: `r |>> rs`

Relations (3.5): Overriding

Given a relation

$$r = \{(a, 1), (b, 2), (c, 3), (a, 4), (b, 5), (c, 6), (d, 1), (e, 2), (f, 3)\}$$

overriding of r with relation t: a relation which agrees with t within $\text{dom}(t)$, and agrees with r outside $\text{dom}(t)$

- Definition: $r \Leftarrow t = \{\ (d, r') \mid (d, r') \in t \lor ((d, r') \in r \land d \notin \text{dom}(t))\ \}$
- e.g.,

$$\begin{aligned} & r \Leftarrow \{(a,3),(c,4)\} \\ = \quad & \underbrace{\{(a,3),(c,4)\}}_{\{(d,r') \mid (d,r') \in t\}} \cup \underbrace{\{(b,2),(b,5),(d,1),(e,2),(f,3)\}}_{\{(d,r') \mid (d,r') \in r \land d \notin \text{dom}(t)\}} \\ = \quad & \{(a,3),(c,4),(b,2),(b,5),(d,1),(e,2),(f,3)\} \end{aligned}$$

- ASCII syntax: `r <+ t`

Relations (4): Exercises

1. *Define $r[s]$ in terms of other relational operations.*

   Answer: $r[s] = \text{ran}(s \lhd r)$

   e.g.,

$$r[\underbrace{\{a,b\}}_{s}] = \text{ran}(\underbrace{\{(\mathbf{a},1),(\mathbf{b},2),(\mathbf{a},4),(\mathbf{b},5)\}}_{\{a,b\}\lhd r}) = \{1,2,4,5\}$$

2. *Define $r \Lleftarrow t$ in terms of other relational operators.*

   Answer: $r \Lleftarrow t = t \cup (\text{dom}(t) \unlhd r)$

   e.g.,

$$\begin{aligned} & r \Lleftarrow \underbrace{\{(a,3),(c,4)\}}_{t} \\ = \quad & \underbrace{\{(a,3),(c,4)\}}_{t} \cup \underbrace{\{(b,2),(b,5),(d,1),(e,2),(f,3)\}}_{\underbrace{\text{dom}(t)}_{\{a,c\}} \unlhd r} \\ = \quad & \{(a,3),(c,4),(b,2),(b,5),(d,1),(e,2),(f,3)\} \end{aligned}$$

Functions (1): Functional Property

- A ***relation*** r on sets S and T (i.e., $r \in S \leftrightarrow T$) is also a *function* if it satisfies the ***functional property***:

  $$\mathit{isFunctional}(r) \iff \forall s, t_1, t_2 \bullet (s \in S \land t_1 \in T \land t_2 \in T) \Rightarrow ((s, t_1) \in r \land (s, t_2) \in r \Rightarrow t_1 = t_2)$$

  - That is, in a ***function***, it is forbidden for a member of S to map to more than one members of T.
  - Equivalently, in a ***function***, two distinct members of T cannot be mapped by the same member of S.
- e.g., Say $S = \{1, 2, 3\}$ and $T = \{a, b\}$, which of the following ***relations*** satisfy the above ***functional property***?
  - $S \times T$ [No]
    Witness 1: $(1, a), (1, b)$; ***Witness*** 2: $(2, a), (2, b)$; ***Witness*** 3: $(3, a), (3, b)$.
  - $(S \times T) \setminus \{(x, y) \mid (x, y) \in S \times T \land x = 1\}$ [No]
    Witness 1: $(2, a), (2, b)$; ***Witness*** 2: $(3, a), (3, b)$
  - $\{(1, a), (2, b), (3, a)\}$ [Yes]
  - $\{(1, a), (2, b)\}$ [Yes]

Functions (2.1): Total vs. Partial

Given a **relation** $r \in S \leftrightarrow T$

- r is a ***partial function*** if it satisfies the ***functional property***:

  $$\boxed{r \in S \nrightarrow T} \iff (\texttt{isFunctional(r)} \wedge \mathrm{dom}(r) \subseteq S)$$

  Remark. $r \in S \nrightarrow T$ means there **may (or may not) be** $s \in S$ s.t. $r(s)$ is ***undefined***.

  - e.g., $\{\ \{(\mathbf{2}, a), (\mathbf{1}, b)\}, \{(\mathbf{2}, a), (\mathbf{3}, a), (\mathbf{1}, b)\}\ \} \subseteq \{1, 2, 3\} \nrightarrow \{a, b\}$
  - ASCII syntax: `r : +->`

- r is a ***total function*** if there is a mapping for each $s \in S$:

  $$\boxed{r \in S \rightarrow T} \iff (\texttt{isFunctional(r)} \wedge \mathrm{dom}(r) = S)$$

  Remark. $r \in S \rightarrow T$ implies $r \in S \nrightarrow T$, but not vice versa. Why?

  - e.g., $\{(\mathbf{2}, a), (\mathbf{3}, a), (\mathbf{1}, b)\} \in \{1, 2, 3\} \rightarrow \{a, b\}$
  - e.g., $\{(\mathbf{2}, a), (\mathbf{1}, b)\} \notin \{1, 2, 3\} \rightarrow \{a, b\}$
  - ASCII syntax: `r : -->`

Functions (2.2): Relation Image vs. Function Application

- Recall: A ***function*** is a ***relation***, but a ***relation*** is not necessarily a ***function***.
- Say we have a ***partial*** ***function*** $f \in \{1,2,3\} \nrightarrow \{a,b\}$:

$$f = \{(\mathbf{3}, a), (\mathbf{1}, b)\}$$

  - With f wearing the ***relation*** hat, we can invoke *relational images*:

$$\begin{array}{lcl} f[\{3\}] & = & \{a\} \\ f[\{1\}] & = & \{b\} \\ f[\{2\}] & = & \varnothing \end{array}$$

    Remark. Given that the inputs are **singleton** sets (e.g., $\{3\}$), so are the output sets (e.g., $\{a\}$). $\because$ Each member in the domain is mappe to at most one member in the range.

  - With f wearing the ***function*** hat, we can invoke *functional applications*:

$$\begin{array}{lcl} f(3) & = & a \\ f(1) & = & b \\ f(2) & \texttt{is} & \textit{undefined} \end{array}$$

Functions (2.3): Modelling Decision

An organization has a system for keeping **track** of its employees as to where they are on the premises (e.g., ``Zone A, Floor 23''). To achieve this, each employee is issued with an active badge which, when scanned, synchronizes their current positions to a central database.

Assume the following two sets:

- *Employee* denotes the **set** of all employees working for the organization.
- *Location* denotes the **set** of all valid locations in the organization.

1. Is it appropriate to *model/formalize* such a **track** functionality as a ***relation*** (i.e., $where\_is \in Employee \leftrightarrow Location$)?
   Answer. No – an employee cannot be at distinct locations simultaneously.
   e.g., $where\_is[Alan] = \{$ ``Zone A, Floor 23'', ``Zone C, Floor 46'' $\}$
2. How about a ***total function*** (i.e., $where\_is \in Employee \rightarrow Location$)?
   Answer. No – in reality, not necessarily all employees show up.
   e.g., $where\_is(Mark)$ should be ***undefined*** if Mark happens to be on vacation.
3. How about a ***partial function*** (i.e., $where\_is \in Employee \nrightarrow Location$)?
   Answer. Yes – this addresses the inflexibility of the total function.

Functions (3.1): Injective Functions

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Given a ***function*** f (either partial or total):

- f is ***injective***/***one-to-one***/***an injection*** if f does **not** map more than one members of S to a single member of T.

  `isInjective(f)`

  $\iff$

  $$\forall s_1, s_2, t \bullet (s_1 \in S \land s_2 \in S \land t \in T) \Rightarrow ((s_1, t) \in f \land (s_2, t) \in f \Rightarrow s_1 = s_2)$$

- If f is a ***partial injection***, we write: $f \in S \rightarrowtail\!\!\!\!\!\!+ \; T$
  - e.g., $\{\ \varnothing, \{(1, \mathbf{a})\}, \{(2, \mathbf{a}), (3, \mathbf{b})\}\ \} \subseteq \{1, 2, 3\} \rightarrowtail\!\!\!\!\!\!+ \; \{a, b\}$
  - e.g., $\{(1, \mathbf{b}), (2, a), (3, \mathbf{b})\} \notin \{1, 2, 3\} \rightarrowtail\!\!\!\!\!\!+ \; \{a, b\}$ [total, not inj.]
  - e.g., $\{(1, \mathbf{b}), (3, \mathbf{b})\} \notin \{1, 2, 3\} \rightarrowtail\!\!\!\!\!\!+ \; \{a, b\}$ [partial, not inj.]
  - ASCII syntax: `f : >+>`
- If f is a ***total injection***, we write: $f \in S \rightarrowtail T$
  - e.g., $\{1, 2, 3\} \rightarrowtail \{a, b\} = \varnothing$
  - e.g., $\{(2, d), (1, a), (3, c)\} \in \{1, 2, 3\} \rightarrowtail \{a, b, c, d\}$
  - e.g., $\{(\mathbf{2}, d), (\mathbf{1}, c)\} \notin \{1, 2, 3\} \rightarrowtail \{a, b, c, d\}$ [not total, inj.]
  - e.g., $\{(2, \mathbf{d}), (1, c), (3, \mathbf{d})\} \notin \{1, 2, 3\} \rightarrowtail \{a, b, c, d\}$ [total, not inj.]
  - ASCII syntax: `f : >->`

Functions (3.2): Surjective Functions

Given a ***function*** f (either partial or total):

- f is ***surjective***/***onto***/***a surjection*** if f maps to all members of T.

$$\texttt{isSurjective}(f) \iff \text{ran}(f) = T$$

- If f is a ***partial surjection***, we write: $f \in S \twoheadrightarrow\!\!\!\!\!\!+\; T$
  - e.g., $\{\, \{(1,\mathbf{b}),(2,\mathbf{a})\}, \{(1,\mathbf{b}),(2,\mathbf{a}),(3,\mathbf{b})\} \,\} \subseteq \{1,2,3\} \twoheadrightarrow\!\!\!\!\!\!+\; \{a,b\}$
  - e.g., $\{(2,\mathbf{a}),(1,\mathbf{a}),(3,\mathbf{a})\,\} \notin \{1,2,3\} \twoheadrightarrow\!\!\!\!\!\!+\; \{a,b\}$ [total, not sur.]
  - e.g., $\{(2,\mathbf{b}),(1,\mathbf{b})\} \notin \{1,2,3\} \twoheadrightarrow\!\!\!\!\!\!+\; \{a,b\}$ [partial, not sur.]
  - ASCII syntax: `f : +->>`
- If f is a ***total surjection***, we write: $f \in S \twoheadrightarrow T$
  - e.g., $\{\, \{(2,a),(1,b),(3,a)\}, \{(2,b),(1,a),(3,b)\} \,\} \subseteq \{1,2,3\} \twoheadrightarrow \{a,b\}$
  - e.g., $\{(\mathbf{2},a),(\mathbf{3},b)\} \notin \{1,2,3\} \twoheadrightarrow \{a,b\}$ [not total, sur.]
  - e.g., $\{(2,\mathbf{a}),(3,\mathbf{a}),(1,\mathbf{a})\} \notin \{1,2,3\} \twoheadrightarrow \{a,b\}$ [total., not sur]
  - ASCII syntax: `f : -->>`

Functions (3.3): Bijective Functions

Given a function f:

f is ***bijective***/***a bijection***/*one-to-one correspondence* if f is ***total***, ***injective***, and ***surjective***.

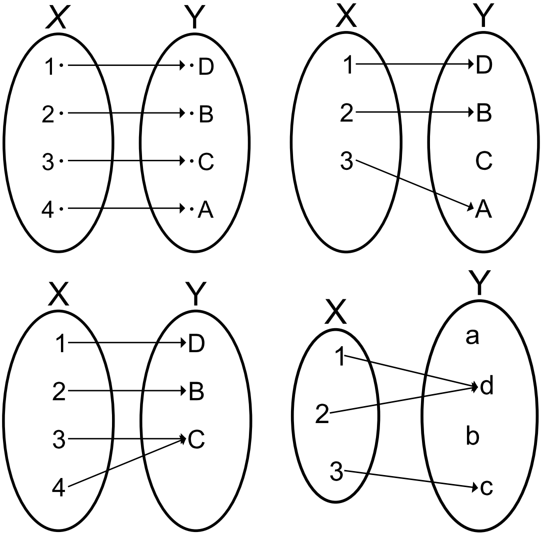
- e.g., $\{1,2,3\} \rightarrowtail\!\!\!\twoheadrightarrow \{a,b\} = \varnothing$
- e.g., $\{\ \{(1,a),(2,b),(3,c)\}, \{(2,a),(3,b),(1,c)\}\ \} \subseteq \{1,2,3\} \rightarrowtail\!\!\!\twoheadrightarrow \{a,b,c\}$
- e.g., $\{(\mathbf{2},b),(\mathbf{3},c),(\mathbf{4},a)\} \notin \{1,2,3,4\} \rightarrowtail\!\!\!\twoheadrightarrow \{a,b,c\}$

  [<u>not</u> total, inj., sur.]
- e.g., $\{(1,\mathbf{a}),(2,b),(3,c),(4,\mathbf{a})\} \notin \{1,2,3,4\} \rightarrowtail\!\!\!\twoheadrightarrow \{a,b,c\}$

  [total, <u>not</u> inj., sur.]
- e.g., $\{(1,\mathbf{a}),(2,\mathbf{c})\} \notin \{1,2\} \rightarrowtail\!\!\!\twoheadrightarrow \{a,b,c\}$

  [total, inj., <u>not</u> sur.]
- ASCII syntax: `f : >->>`

Functions (4.1): Exercises



Functions (4.2): Modelling Decisions

1. Should an array `a` declared as "`String[] a`" be *modelled/formalized* as a *partial* function (i.e., $a \in \mathbb{Z} \nrightarrow String$) or a *total* function (i.e., $a \in \mathbb{Z} \rightarrow String$)?

   Answer. $a \in \mathbb{Z} \rightarrow String$ is not appropriate as:
   - Indices are non-negative (i.e., $a(i)$, where $i < 0$, is ***undefined***).
   - Each array size is finite: not all positive integers are valid indices.
2. What does it mean if an **array** is *modelled/formalized* as a partial ***injection*** (i.e., $a \in \mathbb{Z} \nrightarrowtail String$)?

   Answer. It means that the array does **not** contain any duplicates.
3. Can an integer array "`int[] a`" be *modelled/formalized* as a partial ***surjection*** (i.e., $a \in \mathbb{Z} \twoheadrightarrow \mathbb{Z}$)?

   Answer. Yes, if `a` stores all 2^{32} integers (i.e., $[-2^{31}, 2^{31} - 1]$).
4. Can a string array "`String[] a`" be *modelled/formalized* as a partial ***surjection*** (i.e., $a \in \mathbb{Z} \twoheadrightarrow String$)?

   Answer. No $\because$ # possible strings is ∞.
5. Can an integer array "`int[]`" storing all 2^{32} values be *modelled/formalized* as a ***bijection*** (i.e., $a \in \mathbb{Z} \rightarrowtail\!\!\!\!\twoheadrightarrow \mathbb{Z}$)?

   Answer. No, because it cannot be ***total*** (as discussed earlier).

Beyond this lecture ...

- For the *where_is* $\in$ *Employee* $\nrightarrow$ *Location* model, what does it mean when it is:
  - ***Injective*** [*where_is* $\in$ *Employee* $\rightarrowtail\!\!\!\!\!\!+$ *Location*]
  - ***Surjective*** [*where_is* $\in$ *Employee* $\twoheadrightarrow\!\!\!\!\!\!+$ *Location*]
  - ***Bijective*** [*where_is* $\in$ *Employee* $\rightarrowtail\!\!\!\!\!\twoheadrightarrow$ *Location*]
- Review examples discussed in your earlier math courses on ***logic*** and ***set theory***.
- Ask questions in the Q&A sessions to clarify the reviewed concepts.

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