Balanced Binary Search Trees

EECS2011 N & Z:
Fundamentals of Data Structures
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Learning Outcomes of this Lecture

This module is designed to help you understand:

- When the ***Worst-Case RT*** of a ***BST Search*** Occurs
- ***Height-Balance*** Property
- Performing ***Rotations*** to Restore Tree ***Balance***

Balanced Binary Search Trees: Motivation

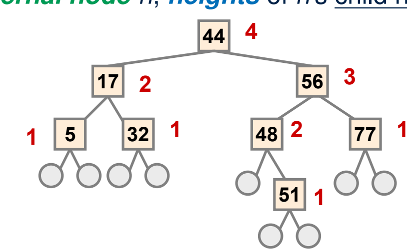
- After ***insertions*** into a BST, the ***worst-case RT*** of a ***search*** occurs when the ***height*** h is at its ***maximum***: ***O(n)***:
  - e.g., Entries were inserted in an decreasing order of their keys
    $$\langle 100, 75, 68, 60, 50, 1\rangle$$
    ⇒ ***One-path***, ***left-slanted*** BST
  - e.g., Entries were inserted in an increasing order of their keys
    $$\langle 1, 50, 60, 68, 75, 100\rangle$$
    ⇒ ***One-path***, ***right-slanted*** BST
  - e.g., Last entry's key is in-between keys of the previous two entries
    $$\langle 1, 100, 50, 75, 60, 68\rangle$$
    ⇒ ***One-path***, ***side-alternating*** BST
- To avoid the worst-case RT ($\because$ a ***ill-balanced tree***), we need to take actions ***as soon as*** the tree becomes ***unbalanced***.

Balanced Binary Search Trees: Definition

- Given a node p, the ***height*** of the subtree rooted at p is:

$$height(p) = \begin{cases} 0 & \text{if } p \text{ is } \textit{external} \\ 1 + \textbf{MAX}\ (\{\ height(c) \mid parent\ (c) = p\ \}) & \text{if } p \text{ is } \textit{internal} \end{cases}$$

- A ***balanced*** BST T satisfies the ***height-balance property***:
  For every ***internal node*** n, ***heights*** of n's child nodes differ $\leq$ ***1***.

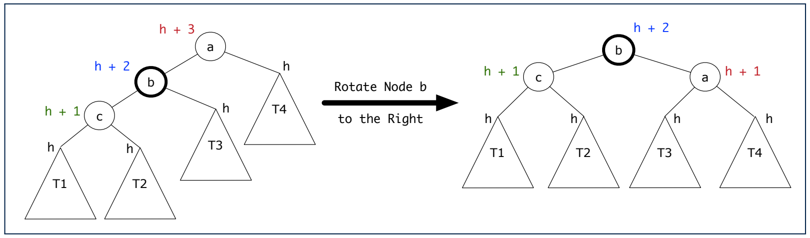


Q: Is the above tree a ***balanced BST***? ✓
Q: Will the tree remain ***balanced*** after inserting 55? ×
Q: Will the tree remain ***balanced*** after inserting 63? ✓

Fixing Unbalanced BST: Rotations

A tree **rotation** is performed:

- When the latest **insertion**/**deletion** creates ***unbalanced*** nodes, along the ***ancestor path*** of the node being inserted/deleted.
- To change the ***shape*** of tree, ***restoring*** the ***height-balance property***



Q. An ***in-order traversal*** on the resulting tree?
A. Still produces a sequence of ***sorted keys*** $\langle T_1, c, T_2, b, T_3, a, T_4 \rangle$

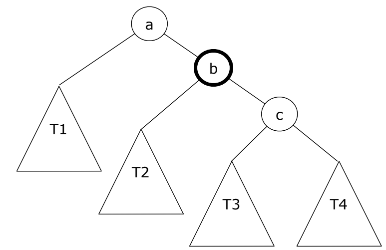
- After **rotating** node *b* to the right:
  - Heights of ***descendants*** (b, c, T_1, T_2, T_3) and ***sibling*** (T_4) stay ***unchanged***.
  - Height of ***parent*** (a) is ***decreased by 1***.
  - ⇒ ***Balance*** of node *a* was ***restored*** by the ***rotation***.

After Insertions: Trinode Restructuring via Rotation(s)

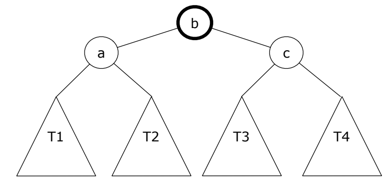
After ***inserting*** a new node *n*:

- **Case 1**: Nodes on *n*'s ***ancestor path*** remain ***balanced***.
  ⇒ No rotations needed
- **Case 2**: At least one of *n*'s ***ancestors*** becomes ***unbalanced***.
  1. Get the **first**/**lowest** ***unbalanced*** node *a* on *n*'s ***ancestor path***.
  2. Get *a*'s child node *b* in *n*'s ***ancestor path***.
  3. Get *b*'s child node *c* in *n*'s ***ancestor path***.
  4. Perform rotation(s) based on the ***alignment*** of *a*, *b*, and *c*:
     - Slanted the ***same*** way ⇒ ***single*** rotation on the **middle** node *b*
     - Slanted ***different*** ways ⇒ ***double*** rotations on the **lower** node *c*

Trinode Restructuring: Single, Left Rotation



After a ***left rotation*** on the middle node *b*:

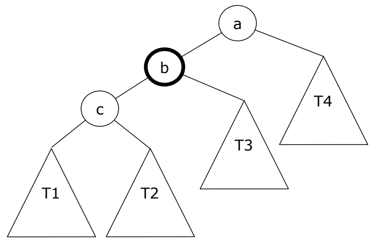


BST property maintained? $\langle T_1, a, T_2, b, T_3, c, T_4 \rangle$

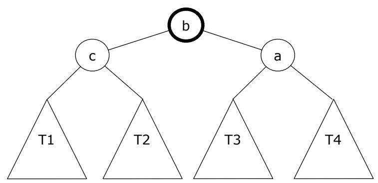
Left Rotation

- ***Insert*** the following sequence of nodes into an empty BST:
  $\langle 44, 17, 78, 32, 50, 88, 95 \rangle$
- Is the BST now ***balanced***?
- ***Insert*** 100 into the BST.
- Is the BST still ***balanced***?
- Perform a ***left rotation*** on the appropriate node.
- Is the BST again ***balanced***?

Trinode Restructuring: Single, Right Rotation



After a ***right rotation*** on the middle node *b*:

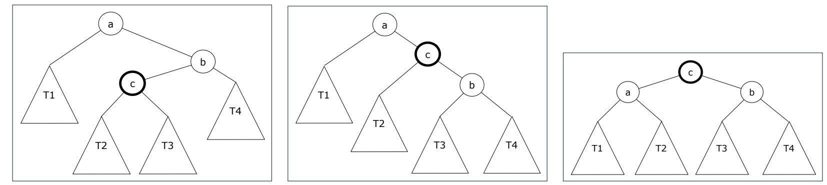


BST property maintained? $\langle T_1,\ a,\ T_2,\ b,\ T_3,\ c,\ T_4 \rangle$

Right Rotation

- ***Insert*** the following sequence of nodes into an empty BST:
  $\langle 44, 17, 78, 32, 50, 88, 48 \rangle$
- Is the BST now ***balanced***?
- ***Insert*** 46 into the BST.
- Is the BST still ***balanced***?
- Perform a ***right rotation*** on the appropriate node.
- Is the BST again ***balanced***?

Trinode Restructuring: Double, R-L Rotations



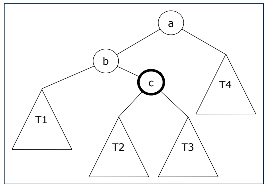
Perform a ***Right Rotation*** on Node *c* | Perform a ***Left Rotation*** on Node *c* | After Right-Left Rotations

BST property maintained? $\langle T_1,\ a,\ T_2,\ c,\ T_3,\ b,\ T_4 \rangle$

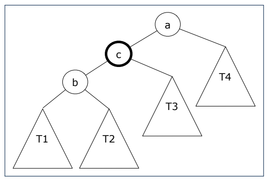
R-L Rotations

- ***Insert*** the following sequence of nodes into an empty BST:
  $\langle 44, 17, 78, 32, 50, 88, 82, 95 \rangle$
- Is the BST now ***balanced***?
- ***Insert*** 85 into the BST.
- Is the BST still ***balanced***?
- Perform the ***R-L rotations*** on the appropriate node.
- Is the BST again ***balanced***?

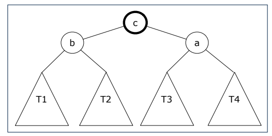
Trinode Restructuring: Double, L-R Rotations



Perform a ***Left Rotation*** on Node c



Perform a ***Right Rotation*** on Node c



After Left-Right Rotations

BST property maintained? $\langle T_1,\ b,\ T_2,\ c,\ T_3,\ a,\ T_4 \rangle$

L-R Rotations

- ***Insert*** the following sequence of nodes into an empty BST:

  $\langle 44, 17, 78, 32, 50, 88, 48, 62 \rangle$
- Is the BST now ***balanced***?
- ***Insert*** 54 into the BST.
- Is the BST still ***balanced***?
- Perform the ***L-R rotations*** on the appropriate node.
- Is the BST again ***balanced***?

After Deletions: Continuous Trinode Restructuring

- *Recall*: ***Deletion*** from a BST results in

  removing a node with zero or one ***internal*** child node.
- After ***deleting*** an existing node, say its child is n:

  Case 1: Nodes on n's ***ancestor path*** remain ***balanced***. ⇒ No rotations

  Case 2: At least one of n's ***ancestors*** becomes ***unbalanced***.

  1. Get the **first**/**lowest** ***unbalanced*** node a on n's ***ancestor path***.

  2. Get a's ***taller*** child node b. [$b \notin n$'s ***ancestor path***]

  3. Choose b's child node c as follows:
    - b's two child nodes have ***different*** heights ⇒ c is the ***taller*** child
    - b's two child nodes have ***same*** height ⇒ a, b, c slant the ***same*** way

  4. Perform rotation(s) based on the ***alignment*** of a, b, and c:
    - Slanted the ***same*** way ⇒ ***single rotation*** on the **middle** node b
    - Slanted ***different*** ways ⇒ ***double rotations*** on the **lower** node c
- As n's ***unbalanced ancestors*** are found, keep applying **Case 2**,

  until **Case 1** is satisfied. [$O(h) = O(\log n)$ ***rotations***]

Single Trinode Restructuring Step

- ***Insert*** the following sequence of nodes into an empty BST:

  $\langle 44, 17, 62, 32, 50, 78, 48, 54, 88 \rangle$
- Is the BST now ***balanced***?
- ***Delete*** 32 from the BST.
- Is the BST still ***balanced***?
- Perform a ***left rotation*** on the appropriate node.
- Is the BST again ***balanced***?

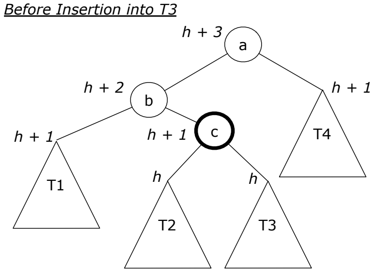
Multiple Trinode Restructuring Steps

- ***Insert*** the following sequence of nodes into an empty BST:

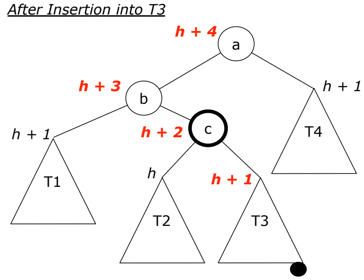
  $\langle 50, 25, 10, 30, 5, 15, 27, 1, 75, 60, 80, 55 \rangle$
- Is the BST now ***balanced***?
- ***Delete*** 80 from the BST.
- Is the BST still ***balanced***?
- Perform a ***right rotation*** on the appropriate node.
- Is the BST now ***balanced***?
- Perform another ***right rotation*** on the appropriate node.
- Is the BST again ***balanced***?

Restoring Balance from Insertions

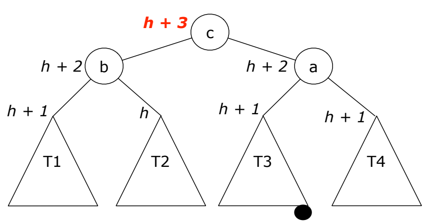
Before Insertion into T3



After Insertion into T3

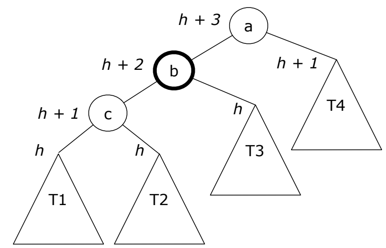


After Performing L-R Rotations on Node c: Height of Subtree Being Fixed Remains h + 3

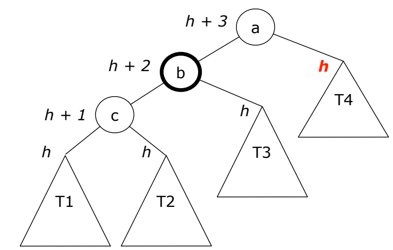


Restoring Balance from Deletions

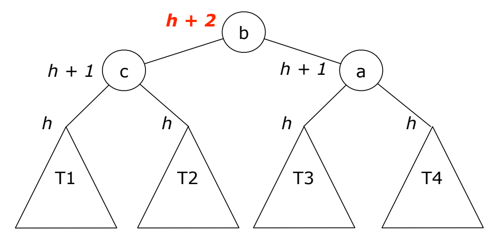
Before Deletion from T4



After Deletion from T4



After Performing Right Rotation on Node b: Height of Subtree Being Fixed Reduces its Height by 1!



Restoring Balance: Insertions vs. Deletions

- Each ***rotation*** involves only ***POs*** of setting parent-child references.

  $\Rightarrow$ ***O(1)*** running time for each tree ***rotation***
- After each ***insertion***, a ***trinode restructuring*** step can ***restore the balance*** of the subtree rooted at the first ***unbalanced*** node.

  $\Rightarrow$ ***O(1)*** rotations suffices to restore the balance of tree
- After each ***deletion***, one or more ***trinode restructuring*** steps may ***restore the balance*** of the subtree rooted at the first ***unbalanced*** node.

  $\Rightarrow$ May take ***O(log n)*** rotations to restore the balance of tree

Index (1)

Index (2)