Overview of Compilation

Readings: EAC2 Chapter 1

EECS4302 M:
Compilers and Interpreters
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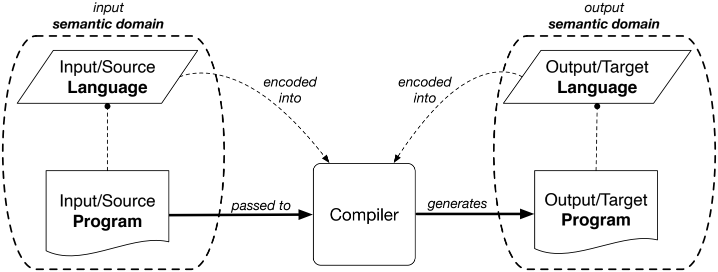
What is a Compiler? (1)

A **software system** that ***automatically*** ***translates/transforms*** **input/source** programs (written in one language) to **output/target** programs (written in another language).



- *Semantic Domain* : context with its own vocabulary and meanings
  e.g., OO, database, predicates
- Source and target may be in **different** ***semantic domains***.
  e.g., Java programs to SQL relational database schemas/queries
  e.g., C procedural programs to MISP assembly instructions

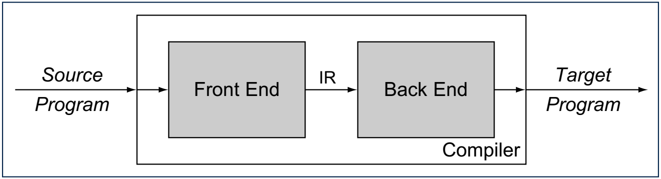
What is a Compiler? (2)

- The idea about a compiler is extremely powerful:

  You can turn anything to anything else,

  as long as the following are ***clear*** about them:
  - SYNTAX [***specifiable*** as CFGs]
  - SEMANTICS [***programmable*** as mapping functions]
- Construction of a compiler should conform to good *software engineering principles*.
  - Modularity & Information Hiding [interacting components]
  - Single Choice Principle
  - Design Patterns (e.g., composite, visitor)
  - Regression Testing at different levels: e.g., Unit & Acceptance

Compiler: Typical Infrastructure (1)



Source Program

Front End

IR

Back End

Compiler

Target Program

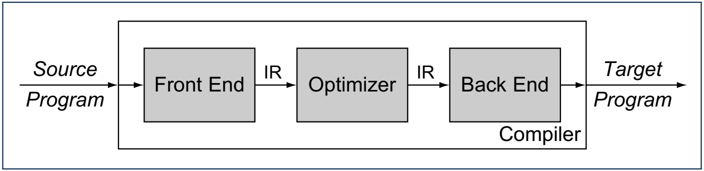
- **FRON END**:
  - Encodes: knowledge of the **source** language
  - Transforms: from the **source** to some ***IR*** (*intermediate representation*)
  - Principle: *meaning* of the source must be *preserved* in the ***IR***.
- **BACK END**:
  - Encodes knowledge of the **target** language
  - Transforms: from the ***IR*** to the **target**

Q. How many ***IRs*** needed for building a number of compilers:
JAVA-TO-C, EIFFEL-TO-C, JAVA-TO-PYTHON, EIFFEL-TO-PYTHON?

A. Two IRs suffice: One for OO; one for procedural.

⇒ IR should be as *language-independent* as possible.

Compiler: Typical Infrastructure (2)



Source Program → Front End → IR → Optimizer → IR → Back End → Target Program

Compiler

OPTIMIZER:

- An ***IR**-to-**IR*** transformer that aims at "improving" the **output** of front end, before passing it as **input** of the back end.
- Think of this transformer as attempting to discover an "***optimal***" solution to some computational problem.
  e.g., runtime performance, static design

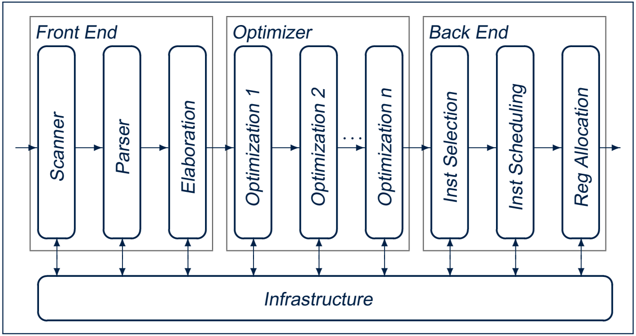
Q. Behaviour of the **target** program predicated upon?

1. ***Meaning*** of the **source** preserved in **IR**?
2. **IR**-to-**IR** transformation of the optimizer ***semantics-preserving***?
3. ***Meaning*** of **IR** preserved in the generated **target**?

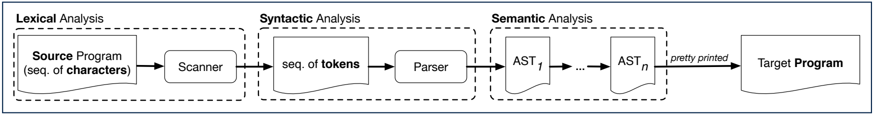
(1) – (3) necessary & sufficient for the *soundness* of a compiler.

Example Compiler One

- Consider a conventional compiler which turns a **C-like program** into executable **machine instructions.**
- The ***source*** (C-like program) and ***target*** (machine instructions) are at different levels of *abstraction*:
  - C-like program is like "high-level" ***specification***.
  - Macine instructions are the low-level, efficient ***implementation***.



Example Compiler One: Scanner vs. Parser vs. Optimizer



- The same input program may be treated differently:
  1. As a ***character sequence*** [subject to ***lexical*** analysis]
  2. As a ***token sequence*** [subject to ***syntactic*** analysis]
  3. As a *abstract syntax tree (AST)* [subject to *semantic* analysis]
- ***(1)*** & ***(2)*** are routine tasks of lexical/grammar rule specification.
- *(3)* is where the most fun is about writing a compiler:

  A series of *semantics-preserving* AST-to-AST transformations.

Example Compiler One: Scanner

- The source program is treated as a sequence of ***characters***.
- A scanner performs *lexical analysis* on the input character sequence and produces a sequence of ***tokens***.
- ANALOGY: Tokens are like individual ***words*** in an essay.

  ⇒ Invalid tokens ≈ Misspelt words

  e.g., a token for a useless delimiter: e.g., space, tab, new line

  e.g., a token for a useful delimiter: e.g., `(`, `)`, `{`, `}`, `,`

  e.g., a token for an identifier (for e.g., a variable, a function)

  e.g., a token for a keyword (e.g,. `int`, `char`, `if`, `for`, `while`)

  e.g., a token for a number (for e.g., `1.23`, `2.46`)

  Q. How to specify such pattern pattern of characters?

  A. ***Regular Expressions*** (***REs***)

  e.g., RE for keyword `while` [`while`]

  e.g., RE for an identifier [`[a-zA-Z][a-zA-Z0-9_]*`]

  e.g., RE for a white space [`[ \t\r]+`]

Example Compiler One: Parser

- A parser's input is a sequence of ***tokens*** (by some scanner).
- A parser performs *syntactic analysis* on the input token sequence and produces an ***abstract syntax tree (AST)***.
- ANALOGY: ASTs are like individual ***sentences*** in an essay.
  - ⇒ Tokens not ***parseable*** into a valid AST ≈ Grammatical errors

Q. An essay with no speling and grammatical errors good enough?
A. No, it may talk about non-sense (sentences in wrong contexts).
⇒ An input program with no lexical/syntactic errors should still be subject to *semantic analysis* (e.g., type checking, code optimization).

Q.: How to specify such pattern pattern of tokens?
A.: ***Context-Free Grammars*** (***CFGs***)

e.g., CFG (with **terminals** and **non-terminals**) for a while-loop:

WhileLoop ::= `WHILE` `LPAREN` *BoolExpr* `RPAREN` `LCBRAC` *Impl* `RCBRAC`
Impl ::=
| *Instruction* `SEMICOL` *Impl*

Example Compiler One: Optimizer

- Consider an input *AST* which has the pretty printing:

```
b := ... ; c := ... ; a := ...
across i |..| n is i
  loop
    read d
    a := a * 2 * b * c * d
  end
```

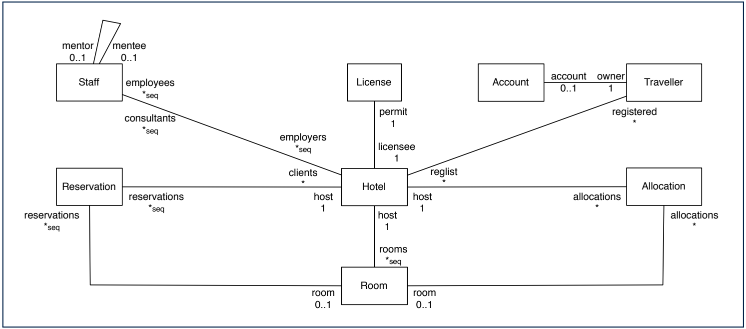
Q. AST of above program ***optimized*** for performance?
A. No ∵ values of `2`, `b`, `c` stay invariant within the loop.

- An ***optimizer*** may *transform* AST like above into:

```
b := ... ; c := ... ; a := ...
temp := 2 * b * c
across i |..| n is i
  loop
    read d
    a := a * d
  end
```

Example Compiler Two

- Consider a compiler which turns a ***Domain-Specific Language (DSL)*** of classes & predicates into a **SQL database.**
- The input/source contains 2 parts:
  - **DATA MODEL**: classes and associations (client-supplier relations) e.g., data model of a Hotel Reservation System:



  - **BEHAVIOURAL MODEL**: update methods specified as predicates

Example Compiler Two: Mapping Data

```
class A {
  attributes
    s: string
    as: set(A . b) [*] }
```

```
class B {
  attributes
    is: set (int)
    b: B . as }
```

- Each class is turned into a ***class table***:
  - Column `oid` stores the object reference. [PRIMARY KEY]
  - Implementation strategy for attributes:

	SINGLE-VALUED	MULTI-VALUED
PRIMITIVE-TYPED	column in ***class table***	***collection table***
REFERENCE-TYPED	***association table***	

- Each ***collection table***:
  - Column `oid` stores the context object.
  - 1 column stores the corresponding primitive value or `oid`.
- Each ***association table***:
  - Column `oid` stores the association reference.
  - 2 columns store `oid`'s of both association ends. [FOREIGN KEY]

Example Compiler Two: Input/Source

- Consider a **valid** input/source program:

```
class Account {
 attributes
   owner: Traveller . account
   balance: int
}
```

```
class Traveller {
 attributes
   name: string
   reglist: set(Hotel . registered)[*]
}
```

```
class Hotel {
 attributes
   name: string
   registered: set(Traveller . reglist)[*]
 methods
   register {
       t? : extent(Traveller)
     & t? /: registered
     ==>
        registered := registered \/ {t?}
     || t?.reglist := t?.reglist \/ {this}
   }
}
```

- How do you specify the ***scanner*** and ***parser*** accordingly?

Example Compiler Two: Output/Target

- Class associations are compiled into ***database schemas***.

```
CREATE TABLE `Account`(
  `oid` INTEGER AUTO_INCREMENT,`balance` INTEGER,
  PRIMARY KEY (`oid`));
CREATE TABLE `Traveller`(
  `oid` INTEGER AUTO_INCREMENT,`name` CHAR(30),
  PRIMARY KEY (`oid`));
CREATE TABLE `Hotel`(
  `oid` INTEGER AUTO_INCREMENT,`name` CHAR(30),
  PRIMARY KEY (`oid`));
CREATE TABLE `Account_owner_Traveller_account`(
  `oid` INTEGER AUTO_INCREMENT, `owner` INTEGER, `account` INTEGER,
  PRIMARY KEY (`oid`));
CREATE TABLE `Traveller_reglist_Hotel_registered`(
  `oid` INTEGER AUTO_INCREMENT, `reglist` INTEGER, `registered` INTEGER,
  PRIMARY KEY (`oid`));
```

- Predicate methods are compiled into ***stored procedures***.

```
CREATE PROCEDURE `Hotel_register`(IN `this?` INTEGER, IN `t?` INTEGER)
  BEGIN
    ...
  END
```

Example Compiler Two: Mapping Behaviour

- Challenge: Transform the OO dot notation into table queries.

  e.g., The AST corresponding to the following dot notation

  (in context of class `Account`, retrieving the owner's list of registrations)

```
this.owner.reglist
```

  may be transformed into the following (nested) table lookups:

```
SELECT (VAR ‘reglist‘)
       (TABLE ‘Hotel_registered_Traveller_reglist‘)
       (VAR ‘registered‘ = (SELECT (VAR ‘owner‘)
                                   (TABLE ‘Account_owner_Traveller_account‘)
                                   (VAR ‘owner‘ = VAR ‘this‘)))
```

- At the database level:
  - Maintaining a large amount of data is ***efficient***
  - Specifying **data** and **updates** is ***tedious*** & ***error-prone***.
  - RESOLUTIONS:
    - Define a DSL supporting the right level of ***abstraction*** for specification
    - Implement a DSL-TO-SQL compiler.

Beyond this lecture . . .

- Read Chapter 1 of EAC2 to find out more about Example Compiler One
- Read this paper to find out more about Example Compiler Two:

`http://dx.doi.org/10.4204/EPTCS.105.8`

Index (1)

Index (2)

Scanner: Lexical Analysis

Readings: EAC2 Chapter 2

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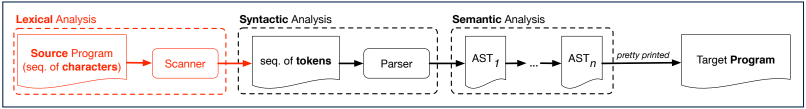
Scanner in Context

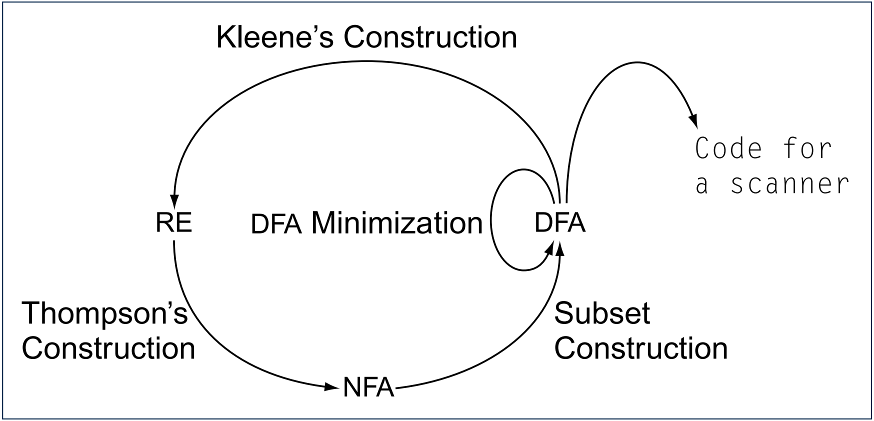
- Recall:



- Treats the input programas as a ***a sequence of characters***
- Applies rules ***recognizing*** character sequences as ***tokens***

  [***lexical*** analysis]
- Upon termination:
  - Reports character sequences not recognizable as tokens
  - Produces a ***a sequence of tokens***
- Only part of compiler touching ***every character*** in input program.
- Tokens *recognizable* by scanner constitute a *regular language*.

Scanner: Formulation & Implementation



Alphabets

- An *alphabet* is a *finite*, *nonempty* set of symbols.
  - The convention is to write Σ, possibly with a informative subscript, to denote the alphabet in question.

    e.g., $\Sigma_{eng} = \{a, b, \dots, z, A, B, \dots, Z\}$ [the English alphabet]

    e.g., $\Sigma_{bin} = \{0, 1\}$ [the binary alphabet]

    e.g., $\Sigma_{dec} = \{d \mid 0 \leq d \leq 9\}$ [the decimal alphabet]

    e.g., Σ_{key} [the keyboard alphabet]
- Use either a *set enumeration* or a *set comprehension* to define your own alphabet.

Strings (1)

- A *string* or a *word* is *finite* sequence of symbols chosen from some *alphabet*.

  e.g., Oxford is a string from the English alphabet Σ_{eng}
  e.g., 01010 is a string from the binary alphabet Σ_{bin}
  e.g., 01010.01 is *not* a string from Σ_{bin}
  e.g., 57 is a string from the binary alphabet Σ_{dec}
- It is not correct to say, e.g., $01010 \in \Sigma_{bin}$ [Why?]
- The *length* of a string w, denoted as $|w|$, is the number of characters it contains.
  - e.g., $|Oxford| = 6$
  - ϵ is the *empty string* ($|\epsilon| = 0$) that may be from any alphabet.
- Given two strings x and y, their *concatenation*, denoted as xy, is a new string formed by a copy of x followed by a copy of y.
  - e.g., Let $x = 01101$ and $y = 110$, then $xy = 01101110$
  - The empty string ϵ is the *identity for concatenation*:

    $\epsilon w = w = w\epsilon$ for any string w

Strings (2)

- Given an *alphabet* Σ, we write Σ^k, where $k \in \mathbb{N}$, to denote the *set of strings of length k from* Σ

$$\Sigma^k = \{w \mid w \texttt{ is from } \Sigma \wedge |w| = k\}$$

  - e.g., $\{0, 1\}^2 = \{00, 01, 10, 11\}$
  - Σ^0 is $\{\epsilon\}$ for any alphabet Σ
- Σ^+ is the set of *nonempty* strings from alphabet Σ

$$\Sigma^+ = \Sigma^1 \cup \Sigma^2 \cup \Sigma^3 \cup \ldots = \{w \mid w \in \Sigma^k \wedge k > 0\} = \bigcup_{k>0} \Sigma^k$$

- Σ^* is the set of strings of *all possible lengths* from alphabet Σ

$$\Sigma^* = \Sigma^+ \cup \{\epsilon\}$$

Review Exercises: Strings

1. What is $|\{a, b, \ldots, z\}^5|$?
2. Enumerate, in a systematic manner, the set $\{a, b, c\}^4$.
3. Explain the difference between Σ and Σ^1.

   Σ is a set of *symbols*; Σ^1 is a set of *strings* of length 1.
4. Prove or disprove: $\Sigma_1 \subseteq \Sigma_2 \Rightarrow \Sigma_1^* \subseteq \Sigma_2^*$

Languages

- *A language L over Σ* (where $|\Sigma|$ is finite) is a set of strings s.t.

$$L \subseteq \Sigma^*$$

- When useful, include an informative subscript to denote the *language L* in question.
  - e.g., The language of *valid* Java programs

$$L_{Java} = \{prog \mid prog \in \Sigma^*_{key} \land prog \texttt{ compiles in Eclipse}\}$$

  - e.g., The language of strings with n 0's followed by n 1's ($n \geq 0$)

$$\{\epsilon, 01, 0011, 000111, \dots\} = \{0^n 1^n \mid n \geq 0\}$$

  - e.g., The language of strings with an equal number of 0's and 1's

$$\begin{array}{rl} & \{\epsilon, 01, 10, 0011, 0101, 0110, 1100, 1010, 1001, \dots\} \\ = & \{w \mid \texttt{\# of 0's in } w \texttt{ = \# of 1's in } w\} \end{array}$$

Review Exercises: Languages

1. Use set comprehensions to define the following languages. Be as *formal* as possible.
   - A language over $\{0, 1\}$ consisting of strings beginning with some 0's (possibly none) followed by at least as many 1's.
   - A language over $\{a, b, c\}$ consisting of strings beginning with some a's (possibly none), followed by some b's and then some c's, s.t. the # of a's is at least as many as the sum of #'s of b's and c's.
2. Explain the difference between the two languages $\{\epsilon\}$ and $\varnothing$.
3. Justify that Σ^*, $\varnothing$, and $\{\epsilon\}$ are all languages over Σ.
4. Prove or disprove: If L is a language over Σ, and $\Sigma_2 \supseteq \Sigma$, then L is also a language over Σ_2.
   Hint: Prove that $\Sigma \subseteq \Sigma_2 \wedge L \subseteq \Sigma^* \Rightarrow L \subseteq \Sigma_2^*$
5. Prove or disprove: If L is a language over Σ, and $\Sigma_2 \subseteq \Sigma$, then L is also a language over Σ_2.
   Hint: Prove that $\Sigma_2 \subseteq \Sigma \wedge L \subseteq \Sigma^* \Rightarrow L \subseteq \Sigma_2^*$

Problems

- Given a *language* L over some *alphabet* Σ, a *problem* is the *decision* on whether or not a given *string* w is a member of L.

$$w \in L$$

  Is this equivalent to deciding $w \in \Sigma^*$? [*No*]

- e.g., The Java compiler solves the problem of *deciding* if the *string of symbols* typed in the Eclipse editor is a *member* of L_{Java} (i.e., set of Java programs with no syntax and type errors).

Regular Expressions (RE): Introduction

- *Regular expressions* (RegExp's) are:
  - A type of *language-defining* notation
    - This is *similar* to the equally-expressive *DFA*, *NFA*, and *ϵ-NFA*.
  - *Textual* and look just like a programming language
    - e.g., 01* + 10* denotes $L = \{0x \mid x \in \{1\}^*\} \cup \{1x \mid x \in \{0\}^*\}$
    - e.g., (0*10*10*)*10* denotes $L = \{w \mid w \texttt{ has odd \# of } 1\texttt{'s}\}$
    - This is *dissimilar* to the diagrammatic *DFA*, *NFA*, and *ϵ-NFA*.
    - RegExp's can be considered as a "user-friendly" alternative to *NFA* for describing software components. [e.g., text search]
    - Writing a RegExp is like writing an algebraic expression, using the defined operators, e.g., ((4 + 3) * 5) % 6
- Despite the programming convenience they provide, RegExp's, *DFA*, *NFA*, and *ϵ-NFA* are all *provably equivalent*.
  - They are capable of defining *all* and *only* regular languages.

RE: Language Operations (1)

- Given Σ of input alphabets, the simplest RegExp is $s \in \Sigma^1$.
  - e.g., Given $\Sigma = \{a, b, c\}$, expression a denotes the language consisting of a single string a.
- Given two languages $L, M \in \Sigma^*$, there are 3 operators for building a *larger language* out of them:

1. **Union**

   $$L \cup M = \{w \mid w \in L \vee w \in M\}$$

   In the textual form, we write $+$ for union.
2. **Concatenation**

   $$LM = \{xy \mid x \in L \wedge y \in M\}$$

   In the textual form, we write either `.` or nothing at all for concatenation.

RE: Language Operations (2)

3. **Kleene Closure** (or **Kleene Star**)

$$L^* = \bigcup_{i \geq 0} L^i$$

where

$$\begin{array}{lcl} L^0 & = & \{\epsilon\} \\ L^1 & = & L \\ L^2 & = & \{x_1 x_2 \mid x_1 \in L \land x_2 \in L\} \\ \ldots & & \\ L^i & = & \{ \underbrace{x_1 x_2 \ldots x_i}_{i \texttt{ repetations}} \mid x_j \in L \land 1 \leq j \leq i\} \\ \ldots & & \end{array}$$

In the textual form, we write $\star$ for closure.

Question: What is $|L^i|$ ($i \in \mathbb{N}$)? [$|L|^i$]

Question: Given that $L = \{0\}^*$, what is L^*? [L]

RE: Construction (1)

We may build *regular expressions* *recursively*:

- Each (*basic* or *recursive*) form of regular expressions denotes a language (i.e., a set of strings that it accepts).
- **Base Case**:
  - Constants ϵ and $\varnothing$ are regular expressions.

$$\begin{array}{lcl} L(\ \epsilon\ ) & = & \{\epsilon\} \\ L(\ \varnothing\ ) & = & \varnothing \end{array}$$

  - An input symbol $a \in \Sigma$ is a regular expression.

$$L(\ a\ ) = \{a\}$$

    If we want a regular expression for the language consisting of only the string $w \in \Sigma^*$, we write w as the regular expression.
  - Variables such as L, M, *etc.*, might also denote languages.

RE: Construction (2)

- **Recursive Case** Given that E and F are regular expressions:
  - The union $E + F$ is a regular expression.

$$L(\ E + F\ ) = L(E) \cup L(F)$$

  - The concatenation EF is a regular expression.

$$L(\ EF\ ) = L(E)L(F)$$

  - Kleene closure of E is a regular expression.

$$L(\ E^*\ ) = (L(E))^*$$

  - A parenthesized E is a regular expression.

$$L(\ (E)\ ) = L(E)$$

RE: Construction (3)

Exercises:

- $\varnothing L$ [$\varnothing L = \varnothing = L\varnothing$]
- $\varnothing^*$

$$\begin{array}{rcl}\varnothing^* & = & \varnothing^0 \cup \varnothing^1 \cup \varnothing^2 \cup \ldots \\ & = & \{\epsilon\} \cup \varnothing \cup \varnothing \cup \ldots \\ & = & \{\epsilon\}\end{array}$$

- $\varnothing^* L$ [$\varnothing^* L = L = L\varnothing^*$]
- $\varnothing + L$ [$\varnothing + L = L = \varnothing + L$]

RE: Construction (4)

Write a regular expression for the following language

$$\{ w \mid w \texttt{ has alternating } 0\texttt{'s and } 1\texttt{'s} \}$$

- Would $(01)^*$ work? [alternating 10's?]
- Would $(01)^* + (10)^*$ work? [starting and ending with 1?]
- $0(10)^* + (01)^* + (10)^* + 1(01)^*$
- It seems that:
  - 1st and 3rd terms have $(10)^*$ as the common factor.
  - 2nd and 4th terms have $(01)^*$ as the common factor.
- Can we simplify the above regular expression?
- $(\epsilon + 0)(10)^* + (\epsilon + 1)(01)^*$

RE: Review Exercises

Write the regular expressions to describe the following languages:

- $\{ w \mid w \texttt{ ends with } 01 \}$
- $\{ w \mid w \texttt{ contains } 01 \texttt{ as a substring} \}$
- $\{ w \mid w \texttt{ contains no more than three consecutive } 1'\texttt{s} \}$
- $\{ w \mid w \texttt{ ends with } 01 \lor w \texttt{ has an odd \# of } 0'\texttt{s} \}$
-
$$\left\{ sx.y \;\middle|\; \begin{array}{ll} & s \in \{+, -, \epsilon\} \\ \land & x \in \Sigma_{dec}^{*} \\ \land & y \in \Sigma_{dec}^{*} \\ \land & \neg(x = \epsilon \land y = \epsilon) \end{array} \right\}$$
-
$$\left\{ xy \;\middle|\; \begin{array}{ll} & x \in \{0,1\}^{*} \land y \in \{0,1\}^{*} \\ \land & x \texttt{ has alternating } 0'\texttt{s and } 1'\texttt{s} \\ \land & y \texttt{ has an odd \# } 0'\texttt{s and an odd \# } 1'\texttt{s} \end{array} \right\}$$

RE: Operator Precedence

- In an order of *decreasing precedence*:
  - Kleene star operator
  - Concatenation operator
  - Union operator
- When necessary, use *parentheses* to force the intended order of evaluation.
- e.g.,
  - 10^* vs. $(10)^*$ [10^* is equivalent to $1(0^*)$]
  - $01^* + 1$ vs. $0(1^* + 1)$ [$01^* + 1$ is equivalent to $(0(1^*)) + (1)$]
  - $0 + 1^*$ vs. $(0 + 1)^*$ [$0 + 1^*$ is equivalent to $(0) + (1^*)$]

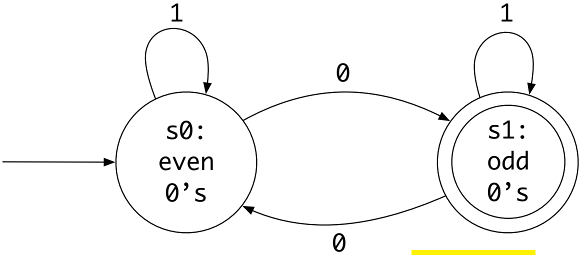
DFA: Deterministic Finite Automata (1.1)

- A *deterministic finite automata (DFA)* is a *finite state machine* (*FSM*) that *accepts* (or recognizes) a pattern of behaviour.
  - For our purpose of this course, we study patterns of *strings* (i.e., how *alphabet symbols* are ordered).
  - Unless otherwise specified, we consider strings in $\{0, 1\}^*$
  - Each pattern contains the set of satisfying strings.
  - We describe the patterns of strings using *set comprehensions*:
    - $\{ w \mid w \texttt{ has an odd number of 0's} \}$
    - $\{ w \mid w \texttt{ has an even number of 1's} \}$
    - $\left\{ w \,\middle|\, \begin{array}{ll} & w \neq \epsilon \\ \wedge & w \texttt{ has equal \# of alternating 0's and 1's} \end{array} \right\}$
    - $\{ w \mid w \texttt{ contains } 01 \texttt{ as a substring} \}$
    - $\left\{ w \,\middle|\, \begin{array}{ll} & w \texttt{ has an even number of 0's} \\ \wedge & w \texttt{ has an odd number of 1's} \end{array} \right\}$
- Given a pattern description, we design a DFA that accepts it.
  - The resulting DFA can be transformed into an executable program.

DFA: Deterministic Finite Automata (1.2)

The *transition diagram* below defines a DFA which *accepts* exactly the language

$$\{ w \mid w \texttt{ has an odd number of 0's} \}$$

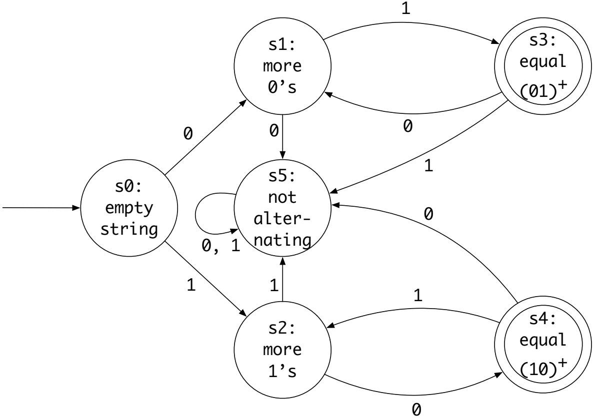


- Each *incoming* or *outgoing* arc (called a *transition*) corresponds to an input alphabet symbol.
- s_0 with an unlabelled *incoming* transition is the *start state*.
- s_3 drawn as a double circle is a *final state*.
- All states have *outgoing* transitions covering $\{0, 1\}$.

DFA: Deterministic Finite Automata (1.3)

The *transition diagram* below defines a DFA which *accepts* exactly the language

$$\left\{ w \;\middle|\; \begin{array}{ll} & w \neq \epsilon \\ \wedge & w \texttt{ has equal \# of alternating 0's and 1's} \end{array} \right\}$$



Review Exercises: Drawing DFAs

Draw the transition diagrams for DFAs which accept other example string patterns:

- $\{\, w \mid w \text{ has an even number of 1's} \,\}$
- $\{\, w \mid w \text{ contains } 01 \text{ as a substring} \,\}$
- $\left\{ w \;\middle|\; \begin{array}{ll} & w \text{ has an even number of 0's} \\ \wedge & w \text{ has an odd number of 1's} \end{array} \right\}$

DFA: Deterministic Finite Automata (2.1)

A *deterministic finite automata (DFA)* is a 5-tuple

$$M = (Q,\ \Sigma,\ \delta,\ q_0,\ F)$$

- Q is a finite set of *states*.
- Σ is a finite set of *input symbols* (i.e., the *alphabet*).
- $\delta : (Q \times \Sigma) \rightarrow Q$ is a *transition function*
  δ takes as arguments a state and an input symbol and returns a state.
- $q_0 \in Q$ is the *start state*.
- $F \subseteq Q$ is a set of *final* or *accepting states*.

DFA: Deterministic Finite Automata (2.2)

- Given a DFA $M = (Q, \Sigma, \delta, q_0, F)$:
  - We write *L(M)* to denote the *language of M*: the set of strings that M *accepts*.
  - A string is *accepted* if it results in a sequence of transitions: beginning from the *start* state and ending in a *final* state.

$$L(M) = \left\{ \begin{array}{l} a_1 a_2 \dots a_n \mid \\ \quad 1 \le i \le n \wedge a_i \in \Sigma \wedge \delta(q_{i-1}, a_i) = q_i \wedge q_n \in F \end{array} \right\}$$

  - M *rejects* any string $w \notin L(M)$.
- We may also consider *L(M)* as *concatenations of labels* from the set of all valid *paths* of *M*'s transition diagram; each such path starts with q_0 and ends in a state in F.

DFA: Deterministic Finite Automata (2.3)

- Given a *DFA* $M = (Q, \Sigma, \delta, q_0, F)$, we may simplify the definition of *L(M)* by extending δ (which takes an input symbol) to $\hat{\delta}$ (which takes an input string).

  $$\hat{\delta} : (Q \times \Sigma^*) \to Q$$

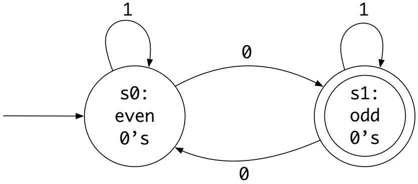
  We may define $\hat{\delta}$ recursively, using δ!

  $$\begin{array}{rcl} \hat{\delta}(q, \epsilon) & = & q \\ \hat{\delta}(q, xa) & = & \delta(\ \hat{\delta}(q, x), a\ ) \end{array}$$

  where $q \in Q$, $x \in \Sigma^*$, and $a \in \Sigma$
- A neater definition of $L(M)$: the set of strings $w \in \Sigma^*$ such that $\hat{\delta}(q_0, w)$ is an *accepting state*.

  $$L(M) = \{w \mid w \in \Sigma^* \wedge \hat{\delta}(q_0, w) \in F\}$$
- A language L is said to be a *regular language*, if there is some *DFA* M such that $L = L(M)$.

DFA: Deterministic Finite Automata (2.4)



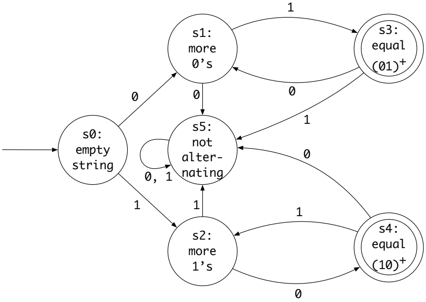
We formalize the above DFA as $M = (Q,\ \Sigma,\ \delta,\ q_0,\ F)$, where

- $Q = \{s_0, s_1\}$
- $\Sigma = \{0, 1\}$
- $\delta = \{((s_0, 0), s_1), ((s_0, 1), s_0), ((s_1, 0), s_0), ((s_1, 1), s_1)\}$

state \ input	0	1
s_0	s_1	s_0
s_1	s_0	s_1

- $q_0 = s_0$
- $F = \{s_1\}$

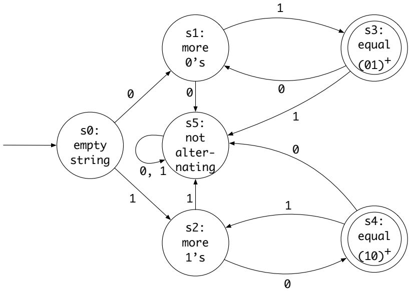
DFA: Deterministic Finite Automata (2.5.1)



We formalize the above DFA as $M = (Q,\ \Sigma,\ \delta,\ q_0,\ F)$, where

- $Q = \{s_0, s_1, s_2, s_3, s_4, s_5\}$
- $\Sigma = \{0, 1\}$
- $q_0 = s_0$
- $F = \{s_3, s_4\}$

DFA: Deterministic Finite Automata (2.5.2)



- δ =

state \ input	0	1
s_0	s_1	s_2
s_1	s_5	s_3
s_2	s_4	s_5
s_3	s_1	s_5
s_4	s_5	s_2
s_5	s_5	s_5

Review Exercises: Formalizing DFAs

Formalize DFAs (as 5-tuples) for the other example string patterns mentioned:

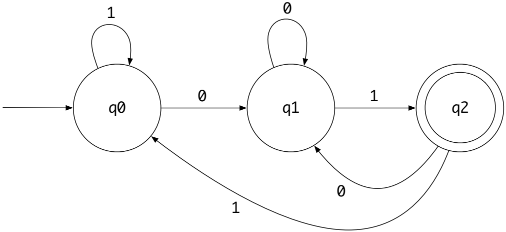
- $\{\, w \mid w \texttt{ has an even number of 0's} \,\}$
- $\{\, w \mid w \texttt{ contains } 01 \texttt{ as a substring} \,\}$
- $\left\{ w \;\middle|\; \begin{array}{cl} & w \texttt{ has an even number of 0's} \\ \wedge & w \texttt{ has an odd number of 1's} \end{array} \right\}$

NFA: Nondeterministic Finite Automata (1.1)

Problem: Design a DFA that accepts the following language:

$$L = \{ \, x01 \mid x \in \{0,1\}^* \, \}$$

That is, L is the set of strings of 0s and 1s ending with 01.



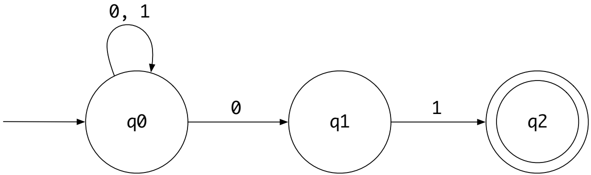
Given an input string w, we may simplify the above DFA by:

- *nondeterministically* treating state q_0 as both:
  - a state *ready* to read the last two input symbols from w
  - a state *not yet ready* to read the last two input symbols from w
- substantially reducing the outgoing transitions from q_1 and q_2

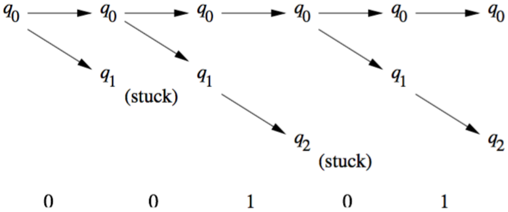
Compare the above DFA with the DFA in slide 39.

NFA: Nondeterministic Finite Automata (1.2)

- A *non-deterministic finite automata (NFA)* that accepts the same language:



- How an NFA determines if an input *00101* should be processed:



NFA: Nondeterministic Finite Automata (2)

- A *nondeterministic finite automata (NFA)*, like a *DFA*, is a *FSM* that *accepts* (or recognizes) a pattern of behaviour.
- An NFA being *nondeterministic* means that from a given state, the *same input label* might corresponds to *multiple transitions* that lead to *distinct states*.
  - Each such transition offers an *alternative path*.
  - Each alternative path is explored independently and in parallel.
  - If **there exists** an alternative path that *succeeds* in processing the input string, then we say the NFA *accepts* that input string.
  - If **all** alternative paths get stuck at some point and *fail* to process the input string, then we say the NFA *rejects* that input string.
- NFAs are often more succinct (i.e., fewer states) and easier to design than DFAs.
- However, NFAs are just as *expressive* as are DFAs.
  - We can **always** convert an NFA to a DFA.

NFA: Nondeterministic Finite Automata (3.1)

- A *nondeterministic finite automata (NFA)* is a 5-tuple

$$M = (Q,\ \Sigma,\ \delta,\ q_0,\ F)$$

  - Q is a finite set of *states*.
  - Σ is a finite set of *input symbols* (i.e., the *alphabet*).
  - $\delta : (Q \times \Sigma) \rightarrow \mathbb{P}(Q)$ is a *transition function*

    δ takes as arguments a state and an input symbol and returns a set of states.
  - $q_0 \in Q$ is the *start state*.
  - $F \subseteq Q$ is a set of *final* or *accepting states*.
- What is the difference between a *DFA* and an *NFA*?
  - The transition function δ of a *DFA* returns a *single* state.
  - The transition function δ of an *NFA* returns a *set* of states.

NFA: Nondeterministic Finite Automata (3.2)

- Given a *NFA* $M = (Q, \Sigma, \delta, q_0, F)$, we may simplify the definition of *L(M)* by extending δ (which takes an input symbol) to $\hat{\delta}$ (which takes an input string).

  $$\hat{\delta} : (Q \times \Sigma^*) \to \mathbb{P}(Q)$$

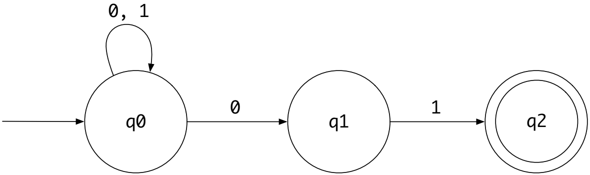
  We may define $\hat{\delta}$ recursively, using δ!

  $$\begin{array}{lcl} \hat{\delta}(q, \epsilon) & = & \{q\} \\ \hat{\delta}(q, xa) & = & \bigcup\{\delta(q', a) \mid q' \in \hat{\delta}(q, x)\} \end{array}$$

  where $q \in Q$, $x \in \Sigma^*$, and $a \in \Sigma$
- A neater definition of $L(M)$: the set of strings $w \in \Sigma^*$ such that $\hat{\delta}(q_0, w)$ contains **at least one** *accepting state*.

  $$L(M) = \{w \mid w \in \Sigma^* \land \hat{\delta}(q_0, w) \cap F \neq \varnothing\}$$

NFA: Nondeterministic Finite Automata (4)



Given an input string 00101:

- **Read 0**: $\delta(q_0, 0) = \{ q_0, q_1 \}$
- **Read 0**: $\delta(q_0, 0) \cup \delta(q_1, 0) = \{ q_0, q_1 \} \cup \varnothing = \{ q_0, q_1 \}$
- **Read 1**: $\delta(q_0, 1) \cup \delta(q_1, 1) = \{ q_0 \} \cup \{ q_2 \} = \{ q_0, q_2 \}$
- **Read 0**: $\delta(q_0, 0) \cup \delta(q_2, 0) = \{ q_0, q_1 \} \cup \varnothing = \{ q_0, q_1 \}$
- **Read 1**: $\delta(q_0, 1) \cup \delta(q_1, 1) = \{ q_0, q_1 \} \cup \{ q_2 \} = \{ q_0, q_1, q_2 \}$

  $\because \{ q_0, q_1, q_2 \} \cap \{ q_2 \} \neq \varnothing \therefore$ `00101 is` *accepted*

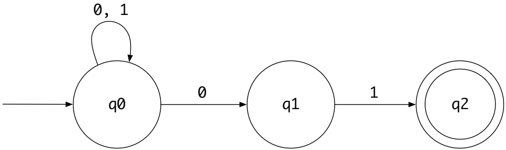
DFA ≡ NFA (1)

- For many languages, constructing an accepting *NFA* is easier than a *DFA*.
- From each state of an *NFA*:
  - Outgoing transitions need **not** cover the entire Σ.
  - An input symbol may *non-deterministically* lead to multiple states.
- In practice:
  - An *NFA* has just as many states as its equivalent *DFA* does.
  - An *NFA* often has fewer transitions than its equivalent *DFA* does.
- In the worst case:
  - While an *NFA* has n states, its equivalent *DFA* has 2^n states.
- Nonetheless, an *NFA* is still just as *expressive* as a *DFA*.
  - Every language accepted by some *NFA* can also be accepted by some *DFA*.

$$\forall N : NFA \bullet (\exists D : DFA \bullet L(D) = L(N))$$

DFA ≡ NFA (2.2): Lazy Evaluation (1)

Given an *NFA*:



Subset construction (with *lazy evaluation*) produces a *DFA* transition table:

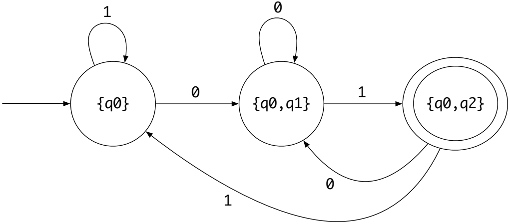
state \ input	0	1
$\{q_0\}$	$\delta(q_0, 0)$ $= \{q_0, q_1\}$	$\delta(q_0, 1)$ $= \{q_0\}$
$\{q_0, q_1\}$	$\delta(q_0, 0) \cup \delta(q_1, 0)$ $= \{q_0, q_1\} \cup \varnothing$ $= \{q_0, q_1\}$	$\delta(q_0, 1) \cup \delta(q_1, 1)$ $= \{q_0\} \cup \{q_2\}$ $= \{q_0, q_2\}$
$\{q_0, q_2\}$	$\delta(q_0, 0) \cup \delta(q_2, 0)$ $= \{q_0, q_1\} \cup \varnothing$ $= \{q_0, q_1\}$	$\delta(q_0, 1) \cup \delta(q_2, 1)$ $= \{q_0\} \cup \varnothing$ $= \{q_0\}$

DFA ≡ NFA (2.2): Lazy Evaluation (2)

Applying *subset construction* (with *lazy evaluation*), we arrive in a *DFA* transition table:

state \ input	0	1
$\{q_0\}$	$\{q_0, q_1\}$	$\{q_0\}$
$\{q_0, q_1\}$	$\{q_0, q_1\}$	$\{q_0, q_2\}$
$\{q_0, q_2\}$	$\{q_0, q_1\}$	$\{q_0\}$

We then draw the *DFA* accordingly:



Compare the above DFA with the DFA in slide 31.

DFA $\equiv$ NFA (2.2): Lazy Evaluation (3)

- Given an *NFA* $N = (Q_N, \Sigma_N, \delta_N, q_0, F_N)$, often only a small portion of the $|\mathbb{P}(Q_N)|$ subset states is *reachable* from $\{q_0\}$.

```
ALGORITHM: ReachableSubsetStates
  INPUT: $q_0 : Q_N$   ;   OUTPUT: $Reachable \subseteq \mathbb{P}(Q_N)$
PROCEDURE:
  Reachable := { {q_0} }
  ToDiscover := { {q_0} }
  while ( $ToDiscover \neq \varnothing$ ) {
    choose $S : \mathbb{P}(Q_N)$ such that $S \in ToDiscover$
    remove S from ToDiscover
    NotYetDiscovered :=
        ( $\{\delta_N(s,0) \mid s \in S\} \cup \{\delta_N(s,1) \mid s \in S\}$ ) $\setminus$ Reachable
    Reachable := $Reachable \cup NotYetDiscovered$
    ToDiscover := $ToDiscover \cup NotYetDiscovered$
  }
  return Reachable
```

- RT of *ReachableSubsetStates*? [$O(2^{|Q_N|})$]

ϵ-NFA: Examples (1)

Draw the NFA for the following two languages:

1.

$$\left\{ xy \;\middle|\; \begin{array}{ll} & x \in \{0,1\}^* \\ \wedge & y \in \{0,1\}^* \\ \wedge & x \texttt{ has alternating } 0'\texttt{s and } 1'\texttt{s} \\ \wedge & y \texttt{ has an odd \# } 0'\texttt{s and an odd \# } 1'\texttt{s} \end{array} \right\}$$

2.

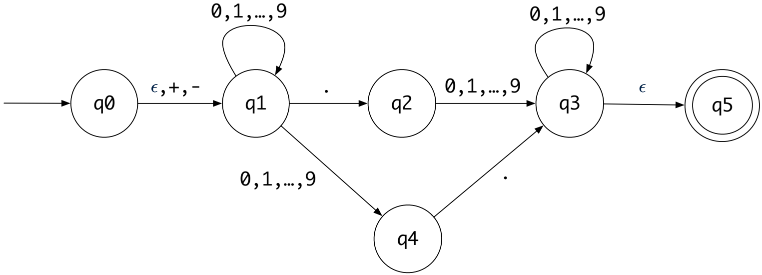
$$\left\{ w : \{0,1\}^* \;\middle|\; \begin{array}{ll} & w \texttt{ has alternating } 0'\texttt{s and } 1'\texttt{s} \\ \vee & w \texttt{ has an odd \# } 0'\texttt{s and an odd \# } 1'\texttt{s} \end{array} \right\}$$

3.

$$\left\{ sx.y \;\middle|\; \begin{array}{ll} & s \in \{+,-,\epsilon\} \\ \wedge & x \in \Sigma_{dec}^* \\ \wedge & y \in \Sigma_{dec}^* \\ \wedge & \neg(x = \epsilon \wedge y = \epsilon) \end{array} \right\}$$

ϵ-NFA: Examples (2)

$$\left\{ sx.y \;\middle|\; \begin{array}{ll} & s \in \{+, -, \epsilon\} \\ \wedge & x \in \Sigma_{dec}^{*} \\ \wedge & y \in \Sigma_{dec}^{*} \\ \wedge & \neg(x = \epsilon \wedge y = \epsilon) \end{array} \right\}$$



From q_0 to q_1, reading a sign is **optional**: a *plus* or a *minus*, or *nothing at all* (i.e., ϵ).

ϵ-NFA: Formalization (1)

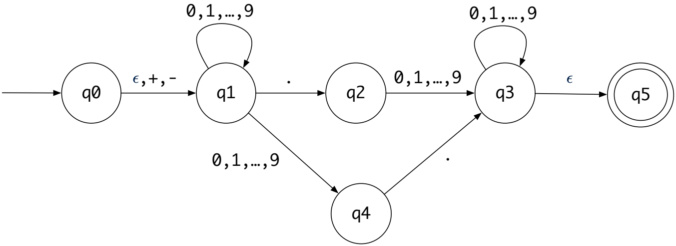
An *ϵ-NFA* is a 5-tuple

$$M = (Q,\ \Sigma,\ \delta,\ q_0,\ F)$$

- Q is a finite set of *states*.
- Σ is a finite set of *input symbols* (i.e., the *alphabet*).
- $\delta : (Q \times (\Sigma \cup \{\epsilon\})) \rightarrow \mathbb{P}(Q)$ is a *transition function*

  δ takes as arguments a state and an input symbol, or *an empty string* ϵ, and returns a set of states.
- $q_0 \in Q$ is the *start state*.
- $F \subseteq Q$ is a set of *final* or *accepting states*.

ϵ-NFA: Formalization (2)



Draw a transition table for the above NFA's δ function:

	ϵ	+, -	.	0 .. 9
q_0	$\{q_1\}$	$\{q_1\}$	$\varnothing$	$\varnothing$
q_1	$\varnothing$	$\varnothing$	$\{q_2\}$	$\{q_1, q_4\}$
q_2	$\varnothing$	$\varnothing$	$\varnothing$	$\{q_3\}$
q_3	$\{q_5\}$	$\varnothing$	$\varnothing$	$\{q_3\}$
q_4	$\varnothing$	$\varnothing$	$\{q_3\}$	$\varnothing$
q_5	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$

ϵ-NFA: Epsilon-Closures (1)

- Given *ϵ-NFA N*

$$N = (Q,\ \Sigma,\ \delta,\ q_0,\ F)$$

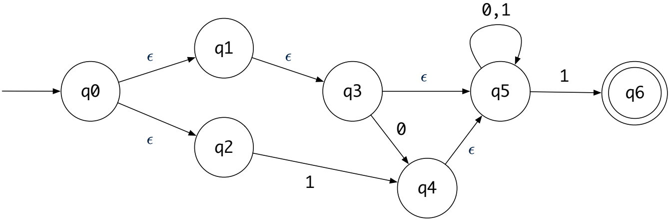
we define the *epsilon closure* (or *ϵ-closure*) as a function

$$\texttt{ECLOSE} : Q \rightarrow \mathbb{P}(Q)$$

- For any state $q \in Q$

$$\texttt{ECLOSE}(q) = \{q\} \cup \bigcup_{p \in \delta(q,\epsilon)} \texttt{ECLOSE}(p)$$

ϵ-NFA: Epsilon-Closures (2)



$$
\begin{array}{cl}
 & \mathrm{ECLOSE}(q_0) \\
= & \{\delta(q_0, \epsilon) = \{q_1, q_2\}\} \\
 & \{q_0\} \cup \mathrm{ECLOSE}(q_1) \cup \mathrm{ECLOSE}(q_2) \\
= & \{\mathit{ECLOSE}(q_1),\ \ \delta(q_1, \epsilon) = \{q_3\},\ \ \mathit{ECLOSE}(q_2),\ \ \delta(q_2, \epsilon) = \varnothing\} \\
 & \{q_0\} \cup (\ \{q_1\} \cup \mathit{ECLOSE}(q_3)\ ) \cup (\ \{q_2\} \cup \varnothing\ ) \\
= & \{\mathit{ECLOSE}(q_3),\ \ \delta(q_3, \epsilon) = \{q_5\}\} \\
 & \{q_0\} \cup (\ \{q_1\} \cup (\ \{q_3\} \cup \mathit{ECLOSE}(q_5)\ )\ ) \cup (\ \{q_2\} \cup \varnothing\ ) \\
= & \{\mathit{ECLOSE}(q_5),\ \ \delta(q_5, \epsilon) = \varnothing\} \\
 & \{q_0\} \cup (\ \{q_1\} \cup (\ \{q_3\} \cup (\ \{q_5\} \cup \varnothing\ )\ )\ ) \cup (\ \{q_2\} \cup \varnothing\ )
\end{array}
$$

ϵ-NFA: Formalization (3)

- Given a ϵ-*NFA* $M = (Q, \Sigma, \delta, q_0, F)$, we may simplify the definition of *L(M)* by extending δ (which takes an input symbol) to $\hat{\delta}$ (which takes an input string).

  $$\hat{\delta} : (Q \times \Sigma^*) \to \mathbb{P}(Q)$$

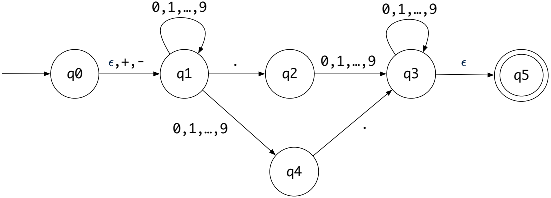
  We may define $\hat{\delta}$ recursively, using δ!

  $$\begin{array}{lcl} \hat{\delta}(q, \epsilon) & = & \texttt{ECLOSE}(q) \\ \hat{\delta}(q, xa) & = & \bigcup\{ \texttt{ECLOSE}(q'') \mid q'' \in \delta(q', a) \wedge q' \in \hat{\delta}(q, x) \} \end{array}$$

  where $q \in Q$, $x \in \Sigma^*$, and $a \in \Sigma$
- Then we define $L(M)$ as the set of strings $w \in \Sigma^*$ such that $\hat{\delta}(q_0, w)$ contains **at least one** *accepting state*.

  $$L(M) = \{w \mid w \in \Sigma^* \wedge \hat{\delta}(q_0, w) \cap F \neq \varnothing\}$$

ϵ-NFA: Formalization (4)



Given an input string 5.6:

$\hat{\delta}(q_0, \epsilon) = \text{ECLOSE}(q_0) = \{q_0, q_1\}$

- **Read 5**: $\delta(q_0, 5) \cup \delta(q_1, 5) = \emptyset \cup \{q_1, q_4\} = \{\ q_1, q_4\ \}$

  $\hat{\delta}(q_0, 5) = \text{ECLOSE}(q_1) \cup \text{ECLOSE}(q_4) = \{q_1\} \cup \{q_4\} = \{q_1, q_4\}$
- **Read .**: $\delta(q_1, .) \cup \delta(q_4, .) = \{q_2\} \cup \{q_3\} = \{\ q_2, q_3\ \}$

  $\hat{\delta}(q_0, 5.) = \text{ECLOSE}(q_2) \cup \text{ECLOSE}(q_3) = \{q_2\} \cup \{q_3, q_5\} = \{q_2, q_3, q_5\}$
- **Read 6**: $\delta(q_2, 6) \cup \delta(q_3, 6) \cup \delta(q_5, 6) = \{q_3\} \cup \{q_3\} \cup \emptyset = \{\ q_3\ \}$

  $\hat{\delta}(q_0, 5.6) = \text{ECLOSE}(q_3) = \{q_3, q_5\}$ [5.6 is *accepted*]

DFA $\equiv$ ϵ-NFA: Subset Construction (1)

Subset construction (with *lazy evaluation* and *epsilon closures*) produces a *DFA* transition table.

	$d \in 0..9$	$s \in \{+,-\}$	.
$\{q_0, q_1\}$	$\{q_1, q_4\}$	$\{q_1\}$	$\{q_2\}$
$\{q_1, q_4\}$	$\{q_1, q_4\}$	$\varnothing$	$\{q_2, q_3, q_5\}$
$\{q_1\}$	$\{q_1, q_4\}$	$\varnothing$	$\{q_2\}$
$\{q_2\}$	$\{q_3, q_5\}$	$\varnothing$	$\varnothing$
$\{q_2, q_3, q_5\}$	$\{q_3, q_5\}$	$\varnothing$	$\varnothing$
$\{q_3, q_5\}$	$\{q_3, q_5\}$	$\varnothing$	$\varnothing$

For example, $\delta(\{q_0, q_1\}, d)$ is calculated as follows: $[d \in 0..9]$

$$
\begin{array}{ll}
 & \bigcup\{\mathrm{ECLOSE}(q) \mid q \in \delta(q_0, d) \cup \delta(q_1, d)\} \\
= & \bigcup\{\mathrm{ECLOSE}(q) \mid q \in \varnothing \cup \{q_1, q_4\}\} \\
= & \bigcup\{\mathrm{ECLOSE}(q) \mid q \in \{q_1, q_4\}\} \\
= & \mathrm{ECLOSE}(q_1) \cup \mathrm{ECLOSE}(q_4) \\
= & \{q_1\} \cup \{q_4\} \\
= & \{q_1, q_4\}
\end{array}
$$

DFA $\equiv$ ϵ-NFA: Subset Construction (2)

- Given an *ϵ=NFA* $N = (Q_N, \Sigma_N, \delta_N, q_0, F_N)$, by applying the *extended* subset construction to it, the resulting *DFA* $D = (Q_D, \Sigma_D, \delta_D, q_{D_{start}}, F_D)$ is such that:

$$
\begin{array}{lcl}
\Sigma_D & = & \Sigma_N \\
Q_D & = & \{\ S \mid S \subseteq Q_N \land (\exists w : \Sigma^* \bullet S = \hat{\delta}_D(q_0, w))\ \} \\
q_{D_{start}} & = & \mathrm{ECLOSE}(q_0) \\
F_D & = & \{\ S \mid S \subseteq Q_N \land S \cap F_N \neq \varnothing\ \} \\
\delta_D(S, a) & = & \bigcup\{\ \mathrm{ECLOSE}(s') \mid s \in S \land s' \in \delta_N(s, a)\ \}
\end{array}
$$

Regular Expression to ϵ-NFA

- Just as we construct each complex *regular expression* recursively, we define its equivalent *ϵ-NFA* *recursively*.
- Given a regular expression R, we construct an ϵ-NFA E, such that $L(R) = L(E)$, with
  - Exactly **one** accept state.
  - No incoming arc to the start state.
  - No outgoing arc from the accept state.

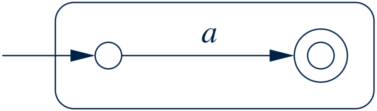
Regular Expression to ϵ-NFA

Base Cases:

- ϵ

- $\varnothing$

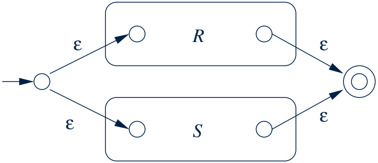
- a $[a \in \Sigma]$



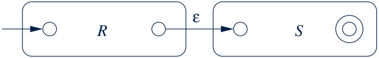
Regular Expression to ϵ-NFA

Recursive Cases: [*R* and *S* are RE's]

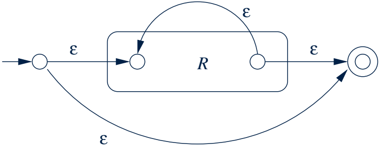
- $R + S$



- RS

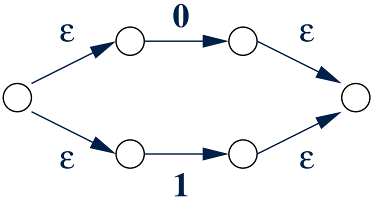


- R^*

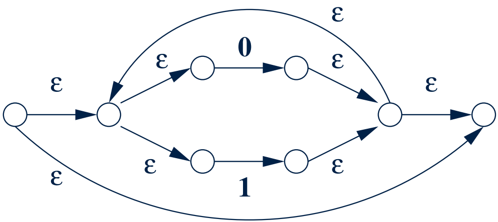


Regular Expression to ϵ-NFA: Examples (1.1)

- $0 + 1$

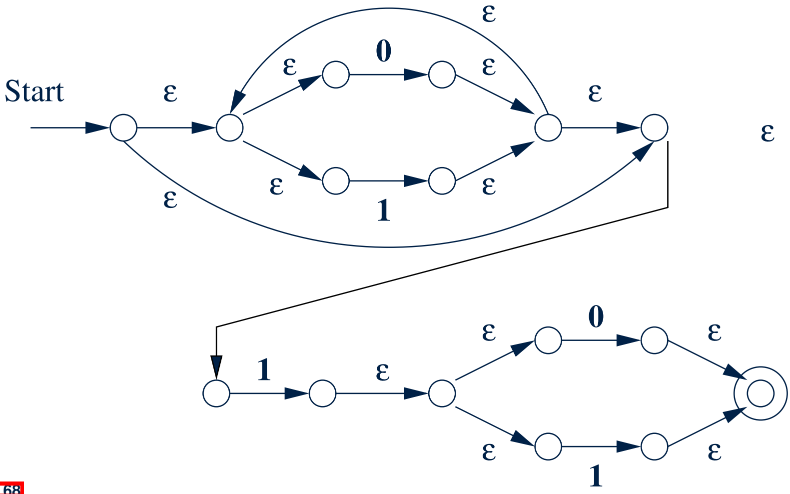


- $(0 + 1)^*$



Regular Expression to ϵ-NFA: Examples (1.2)

- $(0+1)^*1(0+1)$



Minimizing DFA: Motivation

- Recall: $\boxed{\text{Regular Expresion}} \longrightarrow \boxed{\epsilon\text{-NFA}} \longrightarrow \boxed{\text{DFA}}$
- DFA produced by the ***subset construction*** (with ***lazy evaluation***) may **not** be ***minimum*** on its size of state.
- When the required size of memory is sensitive

  (e.g., processor's cache memory),

  the fewer number of DFA states, the better.

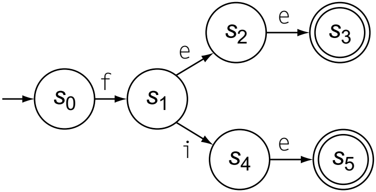
Minimizing DFA: Algorithm

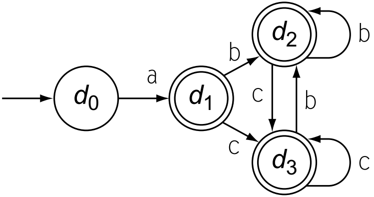
```
ALGORITHM: MinimizeDFAStates
  INPUT: DFA $M = (Q, \Sigma, \delta, q_0, F)$
  OUTPUT: $M'$ s.t. minimum |Q| and equivalent behaviour as M
PROCEDURE:
  P := ∅ /* refined partition so far */
  T := { F, Q - F } /* last refined partition */
  while (P ≠ T):
    P := T
    T := ∅
    for(p ∈ P s.t. |p| > 1):
      find the maximal S ⊆ p s.t. splittable(p, S)
      if S ≠ ∅ then
        T := T ∪ {S, p - S}
      else
        T := T ∪ {p}
      end
```

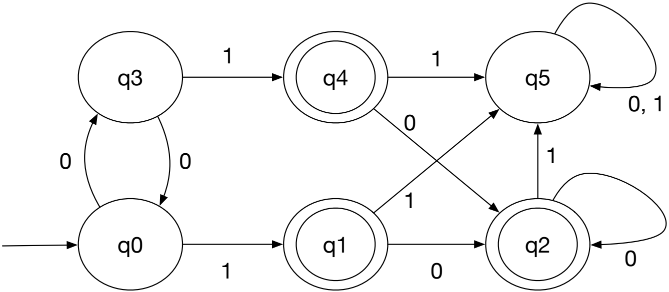
splittable(p, S) holds iff there is $c \in \Sigma$ s.t.

- Transition c leads all $s \in S$ to states in the ***same partition*** $p1$.
- Transition c leads some $s \in p - S$ to a ***different partition*** $p2$ ($p2 \neq p1$).

Minimizing DFA: Examples







Exercises: Minimize the DFA from **here**; Q1 & Q2, p59, EAC2.

Exercise:
Regular Expression to Minimized DFA

Given regular expression `r[0..9]+` which specifies the pattern of a register name, derive the equivalent DFA with the minimum number of states. Show all steps.

Implementing DFA as Scanner

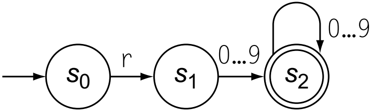
- The source language has a list of ***syntactic categories***:
  e.g., keyword `while` [`while`]
  e.g., identifiers [`[a-zA-Z][a-zA-Z0-9_]*`]
  e.g., white spaces [`[ \t\r]+`]
- A compiler's *scanner* must recognize ***words*** from **all** syntactic categories of the source language.
  - Each syntactic category is specified via a ***regular expression***.

$$\underbrace{r_1}_{\texttt{syn. cat. } 1} + \underbrace{r_1}_{\texttt{syn. cat. } 2} + \ldots + \underbrace{r_n}_{\texttt{syn. cat. } n}$$

  - Overall, a scanner should be implemented based on the *minimized DFA* accommodating all syntactic categories.
- Principles of a scanner:
  - Returns one ***word*** at a time
  - Each returned word is the ***longest possible*** that matches a ***pattern***
  - A **priority** may be specified among patterns (e.g., `new` is a keyword, not identifier)

Implementing DFA: Table-Driven Scanner (1)

- Consider the ***syntactic category*** of `register` names.
- Specified as a *regular expression*: `r[0..9]+`
- Afer conversion to ϵ-NFA, then to DFA, then to ***minimized DFA***:



- The following tables encode knowledge about the above DFA:

Classifier (`CharCat`)

r	0,1,2,...,9	EOF	**Other**
Register	*Digit*	*Other*	*Other*

Transition (δ)

	Register	***Digit***	***Other***
$\mathbf{s_0}$	s_1	s_e	s_e
$\mathbf{s_1}$	s_e	s_2	s_e
$\mathbf{s_2}$	s_e	s_2	s_e
$\mathbf{s_e}$	s_e	s_e	s_e

Token Type (`Type`)

$\mathbf{s_0}$	$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{s_e}$
invalid	*invalid*	*register*	*invalid*

Implementing DFA: Table-Driven Scanner (2)

The scanner then is implemented via a 4-stage skeleton:

```
NextWord()
  -- Stage 1:  Initialization
  state := $s_0$ ; word := $\epsilon$
  initialize an empty stack s ; s.push(bad)
  -- Stage 2:  Scanning Loop
  while (state $\neq$ $s_e$)
    NextChar(char) ; word := word + char
    if state $\in$ F then reset stack s end
    s.push(state)
    cat := CharCat[char]
    state := $\delta$[state, cat]
  -- Stage 3:  Rollback Loop
  while (state $\notin$ F $\wedge$ state $\neq$ bad)
    state := s.pop()
    truncate word
  -- Stage 4:  Interpret and Report
  if state $\in$ F then return Type[state]
  else return invalid
  end
```

Index (1)

Index (2)

Index (3)

Index (4)

Index (5)

Index (6)

Parser: Syntactic Analysis

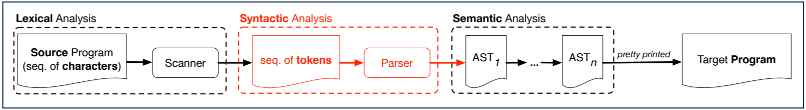
Readings: EAC2 Chapter 3

EECS4302 M:
Compilers and Interpreters
Winter 2020

CHEN-WEI WANG

Parser in Context

- Recall:



- Treats the input programas as a ***a sequence of classified tokens/words***
- Applies rules ***parsing*** token sequences as

  abstract syntax trees (ASTs) [***syntactic*** analysis]
- Upon termination:
  - Reports token sequences not derivable as ASTs
  - Produces an ***AST***
- No longer considers ***every character*** in input program.
- *Derivable* token sequences constitute a *context-free language (CFL)*.

Context-Free Languages: Introduction

- We have seen *regular languages*:
  - Can be described using *finite automata* or *regular expressions*.
  - Satisfy the *pumping lemma*.
- Languages with a *recursive* structure are provably *non-regular*.
  e.g., $\{0^n1^n \mid n \geq 0\}$
- *Context-free grammars (CFG's)* are used to describe strings that can be generated in a *recursive* fashion.
- *Context-free languages (CFL's)* are:
  - Languages that can be described using CFG's.
  - A proper superset of the set of regular languages.

CFG: Example (1.1)

- The language that we previously proved as *non-regular*

$$\{0^n\#1^n \mid n \geq 0\}$$

can be described using the following grammar:

$$\begin{array}{lcl} A & \rightarrow & 0A1 \\ A & \rightarrow & B \\ B & \rightarrow & \# \end{array}$$

- A grammar contains a collection of *substitution* or *production* rules, where:
  - A *terminal* is a word $w \in \Sigma^*$ (e.g., 0, 1, *etc.*).
  - A *variable* or *non-terminal* is a word $w \notin \Sigma^*$ (e.g., A, B, *etc.*).
  - A *start variable* occurs on the LHS of the topmost rule (e.g., A).

CFG: Example (1.2)

- Given a grammar, generate a string by:
  1. Write down the *start variable*.
  2. Choose a production rule where the *start variable* appears on the LHS of the arrow, and *substitute* it by the RHS.
  3. There are two cases of the re-written string:
     - **3.1** It contains *no* variables, then you are done.
     - **3.2** It contains *some* variables, then *substitute* each variable using the relevant *production rules*.
  4. Repeat Step 3.
- e.g., We can generate an *infinite* number of strings from

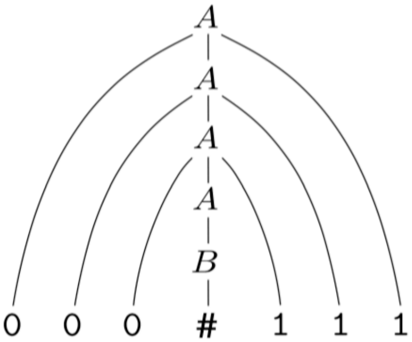
$$\begin{array}{lcl} A & \rightarrow & 0A1 \\ A & \rightarrow & B \\ B & \rightarrow & \# \end{array}$$

  - $A \Rightarrow B \Rightarrow \#$
  - $A \Rightarrow 0A1 \Rightarrow 0B1 \Rightarrow 0\#1$
  - $A \Rightarrow 0A1 \Rightarrow 00A11 \Rightarrow 00B11 \Rightarrow 00\#11$
  - ...

CFG: Example (1.2)

Given a CFG, the *derivation* of a string can be shown as a *parse tree*.

e.g., The derivation of 000#111 has the parse tree



CFG: Example (2)

Design a CFG for the following language:

$$\{w \mid w \in \{0,1\}^* \land w \texttt{ is a palidrome}\}$$

e.g., 00, 11, 0110, 1001, *etc.*

$$\begin{array}{lcl} P & \rightarrow & \epsilon \\ P & \rightarrow & 0 \\ P & \rightarrow & 1 \\ P & \rightarrow & 0P0 \\ P & \rightarrow & 1P1 \end{array}$$

CFG: Example (3)

Design a CFG for the following language:

$$\{ww^R \mid w \in \{0,1\}^*\}$$

e.g., 00, 11, 0110, *etc.*

$$\begin{array}{lcl} P & \rightarrow & \epsilon \\ P & \rightarrow & 0P0 \\ P & \rightarrow & 1P1 \end{array}$$

CFG: Example (4)

Design a CFG for the set of binary strings, where each block of 0's followed by at least as many 1's.

e.g., 000111, 0001111, *etc.*

- We use S to represent one such string, and A to represent each such block in S.

$$\begin{array}{lcll}
S & \rightarrow & \epsilon & \{\texttt{BC of } S\} \\
S & \rightarrow & AS & \{\texttt{RC of } S\} \\
A & \rightarrow & \epsilon & \{\texttt{BC of } A\} \\
A & \rightarrow & 01 & \{\texttt{BC of } A\} \\
A & \rightarrow & 0A1 & \{\texttt{RC of } A\texttt{: equal 0's and 1's}\} \\
A & \rightarrow & A1 & \{\texttt{RC of } A\texttt{: more 1's}\}
\end{array}$$

CFG: Example (5.1) Version 1

Design the grammar for the following small expression language, which supports:

- Arithmetic operations: `+`, `−`, `*`, `/`
- Relational operations: `>`, `<`, `>=`, `<=`, `==`, `/=`
- Logical operations: `true`, `false`, `!`, `&&`, `||`, `=>`

  Start with the variable *Expression*.
- There are two possible versions:
  1. **1.** All operations are mixed together. [e.g., $(1 + \mathit{true})/\mathit{false}$]
  2. **2.** Relevant operations are grouped together.
     Try both!

CFG: Example (5.2) Version 1

$$
\begin{array}{lcl}
\textit{Expression} & \rightarrow & \textit{IntegerConstant} \\
 & | & -\textit{IntegerConstant} \\
 & | & \textit{BooleanConstant} \\
 & | & \textit{BinaryOp} \\
 & | & \textit{UnaryOp} \\
 & | & \texttt{(}\ \textit{Expression}\ \texttt{)} \\
 & & \\
\textit{IntegerConstant} & \rightarrow & \textit{Digit} \\
 & | & \textit{Digit}\ \textit{IntegerConstant} \\
 & & \\
\textit{Digit} & \rightarrow & 0 \mid 1 \mid 2 \mid 3 \mid 4 \mid 5 \mid 6 \mid 7 \mid 8 \mid 9 \\
 & & \\
\textit{BooleanConstant} & \rightarrow & \texttt{TRUE} \\
 & | & \texttt{FALSE}
\end{array}
$$

CFG: Example (5.3) Version 1

$$
\begin{array}{lcl}
\mathit{BinaryOp} & \rightarrow & \mathit{Expression}\ \texttt{+}\ \mathit{Expression} \\
 & | & \mathit{Expression}\ \texttt{-}\ \mathit{Expression} \\
 & | & \mathit{Expression}\ \texttt{*}\ \mathit{Expression} \\
 & | & \mathit{Expression}\ \texttt{/}\ \mathit{Expression} \\
 & | & \mathit{Expression}\ \texttt{\&\&}\ \mathit{Expression} \\
 & | & \mathit{Expression}\ \texttt{||}\ \mathit{Expression} \\
 & | & \mathit{Expression}\ \texttt{=>}\ \mathit{Expression} \\
 & | & \mathit{Expression}\ \texttt{==}\ \mathit{Expression} \\
 & | & \mathit{Expression}\ \texttt{/=}\ \mathit{Expression} \\
 & | & \mathit{Expression}\ \texttt{>}\ \mathit{Expression} \\
 & | & \mathit{Expression}\ \texttt{<}\ \mathit{Expression} \\
 & & \\
\mathit{UnaryOp} & \rightarrow & \texttt{!}\ \mathit{Expression}
\end{array}
$$

CFG: Example (5.4) Version 1

However, Version 1 of CFG:

- ***Parses*** string that requires further *semantic analysis* (e.g., type checking):
  e.g., `3 => 6`
- Is *ambiguous*, meaning that a string may have more than one ways to interpret it.
  e.g., Draw the *parse tree(s)* for `3 * 5 + 4`

CFG: Example (5.5) Version 2

$$
\begin{array}{lcl}
\mathit{Expression} & \rightarrow & \mathit{ArithmeticOp} \\
 & \mid & \mathit{RelationalOp} \\
 & \mid & \mathit{LogicalOp} \\
 & \mid & \texttt{(}\ \mathit{Expression}\ \texttt{)} \\
 & & \\
\mathit{IntegerConstant} & \rightarrow & \mathit{Digit} \\
 & \mid & \mathit{Digit}\ \mathit{IntegerConstant} \\
 & & \\
\mathit{Digit} & \rightarrow & 0 \mid 1 \mid 2 \mid 3 \mid 4 \mid 5 \mid 6 \mid 7 \mid 8 \mid 9 \\
 & & \\
\mathit{BooleanConstant} & \rightarrow & \texttt{TRUE} \\
 & \mid & \texttt{FALSE}
\end{array}
$$

CFG: Example (5.6) Version 2

$$
\begin{array}{lcl}
\mathit{ArithmeticOp} & \rightarrow & \mathit{ArithmeticOp} + \mathit{ArithmeticOp} \\
 & | & \mathit{ArithmeticOp} - \mathit{ArithmeticOp} \\
 & | & \mathit{ArithmeticOp} \star \mathit{ArithmeticOp} \\
 & | & \mathit{ArithmeticOp} \; / \; \mathit{ArithmeticOp} \\
 & | & \texttt{(}\mathit{ArithmeticOp}\texttt{)} \\
 & | & \mathit{IntegerConstant} \\
 & | & -\mathit{IntegerConstant} \\
\mathit{RelationalOp} & \rightarrow & \mathit{ArithmeticOp} == \mathit{ArithmeticOp} \\
 & | & \mathit{ArithmeticOp} \; /= \mathit{ArithmeticOp} \\
 & | & \mathit{ArithmeticOp} > \mathit{ArithmeticOp} \\
 & | & \mathit{ArithmeticOp} < \mathit{ArithmeticOp} \\
\mathit{LogicalOp} & \rightarrow & \mathit{LogicalOp} \; \texttt{\&\&} \; \mathit{LogicalOp} \\
 & | & \mathit{LogicalOp} \; \texttt{||} \; \mathit{LogicalOp} \\
 & | & \mathit{LogicalOp} => \mathit{LogicalOp} \\
 & | & \texttt{!} \; \mathit{LogicalOp} \\
 & | & \texttt{(}\mathit{LogicalOp}\texttt{)} \\
 & | & \mathit{RelationalOp} \\
 & | & \mathit{BooleanConstant}
\end{array}
$$

CFG: Example (5.7) Version 2

However, Version 2 of CFG:

- Eliminates some cases for further semantic analysis:
  e.g., `(1 + 2) => (5 / 4)` [no parse tree]
- Still ***Parses*** string that might require further *semantic analysis*:
  e.g., `(1 + 2) / (5 - (2 + 3))`
- Is *ambiguous*, meaning that a string may have more than one ways to interpret it.
  e.g., Draw the *parse tree(s)* for `3 * 5 + 4`

CFG: Formal Definition (1)

- A *context-free grammar (CFG)* is a 4-tuple (V, Σ, R, S):
  - V is a finite set of *variables*.
  - Σ is a finite set of *terminals*. $[V \cap \Sigma = \varnothing]$
  - R is a finite set of *rules* s.t.

$$R \subseteq \{ v \to s \mid v \in V \land s \in (V \cup \Sigma)^* \}$$

  - $S \in V$ is is the *start variable*.
- Given strings $u, v, w \in (V \cup \Sigma)^*$, variable $A \in V$, and a rule $A \to w$:
  - $uAv \Rightarrow uwv$ menas that uAv *yields* uwv.
  - $u \overset{*}{\Rightarrow} v$ means that u *derives* v, if:
    - $u = v$; or
    - $u \Rightarrow u_1 \Rightarrow u_2 \Rightarrow \cdots \Rightarrow u_k \Rightarrow v$ [a yield sequence]
- Given a CFG $G = (V, \Sigma, R, S)$, the language of G

$$L(G) = \{ w \in \Sigma^* \mid S \overset{*}{\Rightarrow} w \}$$

CFG: Formal Definition (2): Example

- Design the CFG for strings of properly-nested parentheses.

  e.g., `()`, `()()`, `((()()))()`, *etc.*

  Present your answer in a formal manner.
- $G = (\{S\}, \{\texttt{(}, \texttt{)}\}, R, S)$, where R is

$$S \to \texttt{(}\ S\ \texttt{)} \mid SS \mid \epsilon$$

- Draw *parse trees* for the above three strings that G generates.

CFG: Formal Definition (3): Example

- Consider the grammar $G = (V, \Sigma, R, S)$:
  - R is

$$
\begin{array}{lcll}
\mathit{Expr} & \rightarrow & \mathit{Expr} \; + & \mathit{Term} \\
 & \mid & \mathit{Term} & \\
\mathit{Term} & \rightarrow & \mathit{Term} \; \star & \mathit{Factor} \\
 & \mid & \mathit{Factor} & \\
\mathit{Factor} & \rightarrow & \texttt{(} \, \mathit{Expr} \, \texttt{)} & \\
 & \mid & \texttt{a} &
\end{array}
$$

  - $V = \{\mathit{Expr}, \mathit{Term}, \mathit{Factor}\}$
  - $\Sigma = \{\texttt{a}, +, \star, \texttt{(}, \texttt{)}\}$
  - $S = \mathit{Expr}$
- *Precedence* of operators $+$ and $\star$ is embedded in the grammar.
  - "Plus" is specified at a **higher** level (*Expr*) than is "times" (*Term*).
  - Both operands of a multiplication (*Factor*) may be **parenthesized**.

Regular Expressions to CFG's

- Recall the semantics of regular expressions (assuming that we do not consider $\varnothing$):

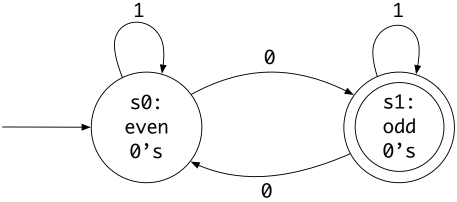
$$\begin{array}{lcl} L(\ \epsilon\ ) & = & \{\epsilon\} \\ L(\ a\ ) & = & \{a\} \\ L(\ E+F\ ) & = & L(E) \cup L(F) \\ L(\ EF\ ) & = & L(E)L(F) \\ L(\ E^*\ ) & = & (L(E))^* \\ L(\ (E)\ ) & = & L(E) \end{array}$$

- e.g., Grammar for $(00 + 1)^* + (11 + 0)^*$

$$\begin{array}{lcl} S & \rightarrow & A \mid B \\ A & \rightarrow & \epsilon \mid AC \\ C & \rightarrow & 00 \mid 1 \\ B & \rightarrow & \epsilon \mid BD \\ D & \rightarrow & 11 \mid 0 \end{array}$$

DFA to CFG's

- Given a DFA $M = (Q, \Sigma, \delta, q_0, F)$:
  - Make a variable R_i for each state $q_i \in Q$.
  - Make R_0 the start variable, where q_0 is the start state of M.
  - Add a rule $R_i \rightarrow aR_j$ to the grammar if $\delta(q_i, a) = q_j$.
  - Add a rule $R_i \rightarrow \epsilon$ if $q_i \in F$.
- e.g., Grammar for



$$\begin{array}{lcl} R_0 & \rightarrow & 1R_0 \mid 0R_1 \\ R_1 & \rightarrow & 0R_0 \mid 1R_1 \mid \epsilon \end{array}$$

CFG: Leftmost Derivations (1)

$$\begin{array}{lcl} \mathit{Expr} & \rightarrow & \mathit{Expr} + \mathit{Term} \mid \mathit{Term} \\ \mathit{Term} & \rightarrow & \mathit{Term} \star \mathit{Factor} \mid \mathit{Factor} \\ \mathit{Factor} & \rightarrow & (\mathit{Expr}) \mid a \end{array}$$

- *Unique leftmost derivation* for the string `a + a * a`:

$$\begin{array}{lcl} \mathit{Expr} & \Rightarrow & \mathit{Expr} + \mathit{Term} \\ & \Rightarrow & \mathit{Term} + \mathit{Term} \\ & \Rightarrow & \mathit{Factor} + \mathit{Term} \\ & \Rightarrow & a + \mathit{Term} \\ & \Rightarrow & a + \mathit{Term} \star \mathit{Factor} \\ & \Rightarrow & a + \mathit{Factor} \star \mathit{Factor} \\ & \Rightarrow & a + a \star \mathit{Factor} \\ & \Rightarrow & a + a \star a \end{array}$$

- This *leftmost derivation* suggests that `a * a` is the right operand of +.

CFG: Rightmost Derivations (1)

$$\begin{array}{lcl} \mathit{Expr} & \rightarrow & \mathit{Expr} \; + \; \mathit{Term} \mid \mathit{Term} \\ \mathit{Term} & \rightarrow & \mathit{Term} \; \star \; \mathit{Factor} \mid \mathit{Factor} \\ \mathit{Factor} & \rightarrow & (\, \mathit{Expr} \,) \mid a \end{array}$$

- *Unique rightmost derivation* for the string $\texttt{a} \; + \; \texttt{a} \; \star \; \texttt{a}$:

$$\begin{array}{lcl} \mathit{Expr} & \Rightarrow & \mathit{Expr} \; + \; \mathit{Term} \\ & \Rightarrow & \mathit{Expr} \; + \; \mathit{Term} \; \star \; \mathit{Factor} \\ & \Rightarrow & \mathit{Expr} \; + \; \mathit{Term} \; \star \; a \\ & \Rightarrow & \mathit{Expr} \; + \; \mathit{Factor} \; \star \; a \\ & \Rightarrow & \mathit{Expr} \; + \; a \; \star \; a \\ & \Rightarrow & \mathit{Term} \; + \; a \; \star \; a \\ & \Rightarrow & \mathit{Factor} \; + \; a \; \star \; a \\ & \Rightarrow & a \; + \; a \; \star \; a \end{array}$$

- This *rightmost derivation* suggests that $\texttt{a} \; \star \; \texttt{a}$ is the right operand of $+$.

CFG: Leftmost Derivations (2)

$$\begin{aligned}
\mathit{Expr} &\rightarrow \mathit{Expr} \; + \; \mathit{Term} \mid \mathit{Term} \\
\mathit{Term} &\rightarrow \mathit{Term} \; \star \; \mathit{Factor} \mid \mathit{Factor} \\
\mathit{Factor} &\rightarrow \texttt{(} \, \mathit{Expr} \, \texttt{)} \mid a
\end{aligned}$$

- *Unique leftmost derivation* for the string `(a + a) * a`:

$$\begin{aligned}
\mathit{Expr} &\Rightarrow \mathit{Term} \\
&\Rightarrow \mathit{Term} \; \star \; \mathit{Factor} \\
&\Rightarrow \mathit{Factor} \; \star \; \mathit{Factor} \\
&\Rightarrow (\; \mathit{Expr} \;) \; \star \; \mathit{Factor} \\
&\Rightarrow (\; \mathit{Expr} \; + \; \mathit{Term} \;) \; \star \; \mathit{Factor} \\
&\Rightarrow (\; \mathit{Term} \; + \; \mathit{Term} \;) \; \star \; \mathit{Factor} \\
&\Rightarrow (\; \mathit{Factor} \; + \; \mathit{Term} \;) \; \star \; \mathit{Factor} \\
&\Rightarrow (\; a \; + \; \mathit{Term} \;) \; \star \; \mathit{Factor} \\
&\Rightarrow (\; a \; + \; \mathit{Factor} \;) \; \star \; \mathit{Factor} \\
&\Rightarrow (\; a \; + \; a \;) \; \star \; \mathit{Factor} \\
&\Rightarrow (\; a \; + \; a \;) \; \star \; a
\end{aligned}$$

This *leftmost derivation* suggests that `(a + a)` is the left operand of $\star$.

CFG: Rightmost Derivations (2)

$$\begin{array}{lcl} \mathit{Expr} & \rightarrow & \mathit{Expr} \; + \; \mathit{Term} \mid \mathit{Term} \\ \mathit{Term} & \rightarrow & \mathit{Term} \; \star \; \mathit{Factor} \mid \mathit{Factor} \\ \mathit{Factor} & \rightarrow & (\,\mathit{Expr}\,) \mid a \end{array}$$

- *Unique rightmost derivation* for the string `(a + a) * a`:

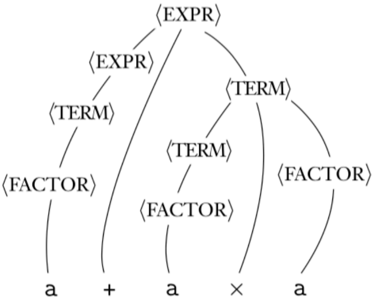
$$\begin{array}{lcl} \mathit{Expr} & \Rightarrow & \mathit{Term} \\ & \Rightarrow & \mathit{Term} \; \star \; \mathit{Factor} \\ & \Rightarrow & \mathit{Term} \; \star \; a \\ & \Rightarrow & \mathit{Factor} \; \star \; a \\ & \Rightarrow & (\; \mathit{Expr} \;) \; \star \; a \\ & \Rightarrow & (\; \mathit{Expr} \; + \; \mathit{Term} \;) \; \star \; a \\ & \Rightarrow & (\; \mathit{Expr} \; + \; \mathit{Factor} \;) \; \star \; a \\ & \Rightarrow & (\; \mathit{Expr} \; + \; a \;) \; \star \; a \\ & \Rightarrow & (\; \mathit{Term} \; + \; a \;) \; \star \; a \\ & \Rightarrow & (\; \mathit{Factor} \; + \; a \;) \; \star \; a \\ & \Rightarrow & (\; a \; + \; a \;) \; \star \; a \end{array}$$

This *rightmost derivation* suggests that `(a + a)` is the left operand of $\star$.

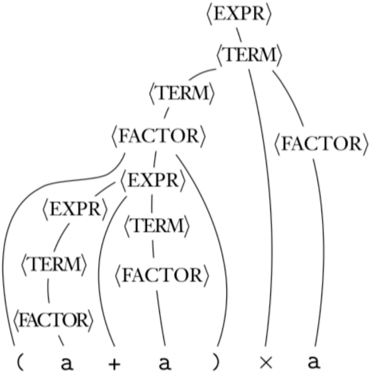
CFG: Parse Trees vs. Derivations (1)

- *Parse trees* for (leftmost & rightmost) *derivations* of expressions:

a + a * a



(a + a) * a



- Orders in which derivations are performed are ***not*** reflected on parse trees.

CFG: Parse Trees vs. Derivations (2)

- A string $w \in \Sigma^*$ may have more than one *derivations*.

  Q: distinct *derivations* for $w \in \Sigma^* \Rightarrow$ distinct *parse trees* for w?

  A: Not in general $\because$ Derivations with ***distinct orders*** of variable substitutions may still result in the ***same parse tree***.
- For example:

$$
\begin{array}{lcl}
Expr & \rightarrow & Expr \ \texttt{+} \ Term \mid Term \\
Term & \rightarrow & Term \ \star \ Factor \mid Factor \\
Factor & \rightarrow & \texttt{(} Expr \texttt{)} \mid a
\end{array}
$$

  For string `a + a * a`, the *leftmost* and *rightmost* derivations have ***distinct orders*** of variable substitutions, but their corresponding ***parse trees are the same***.

CFG: Ambiguity: Definition

Given a grammar $G = (V, \Sigma, R, S)$:

- A string $w \in \Sigma^*$ is derived *ambiguously* in G if there exist
  two or more ***distinct*** *parse trees* or, equally,
  two or more ***distinct*** *leftmost* derivations or, equally,
  two or more ***distinct*** *rightmost* derivations.

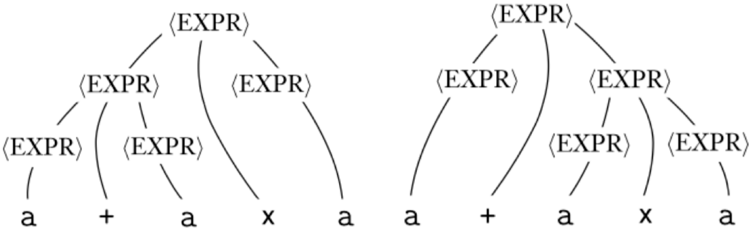
  Here we require that all such derivations have been completed by following a particular order (leftmost or rightmost) to avoid *false alarm*.
- G is *ambiguous* if it generates some string ambiguously.

CFG: Ambiguity: Exercise (1)

- Is the following grammar *ambiguous*?

$$Expr \rightarrow Expr \;\texttt{+}\; Expr \mid Expr \;\texttt{*}\; Expr \mid \texttt{(}\; Expr \;\texttt{)} \mid a$$

- Yes ∵ it generates the string `a + a * a` *ambiguously*:



- ***Distinct ASTs*** (for the ***same input***) mean ***distinct semantic interpretations***: e.g.,
  when a post-order traversal is used to implement evaluation
- **Exercise:** Show *leftmost* derivations for the two parse trees.

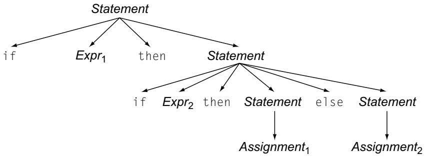
CFG: Ambiguity: Exercise (2.1)

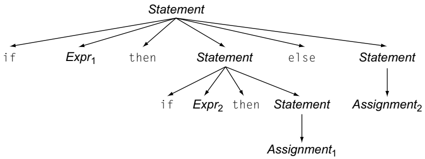
- Is the following grammar *ambiguous*?

$$\begin{array}{lcl} \mathit{Statement} & \rightarrow & \texttt{if}\ \mathit{Expr}\ \texttt{then}\ \mathit{Statement} \\ & | & \texttt{if}\ \mathit{Expr}\ \texttt{then}\ \mathit{Statement}\ \texttt{else}\ \mathit{Statement} \\ & | & \mathit{Assignment} \\ & \ldots & \end{array}$$

- Yes ∵ it generates the following string *ambiguously*:

$$\texttt{if}\ \mathit{Expr}_1\ \texttt{then}\ \texttt{if}\ \mathit{Expr}_2\ \texttt{then}\ \mathit{Assignment}_1\ \texttt{else}\ \mathit{Assignment}_2$$

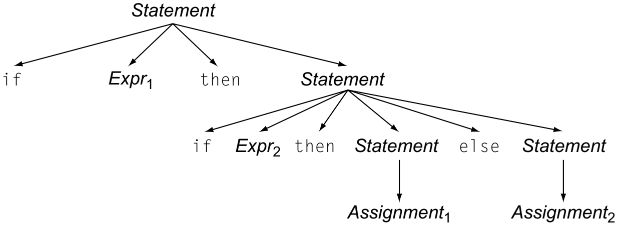




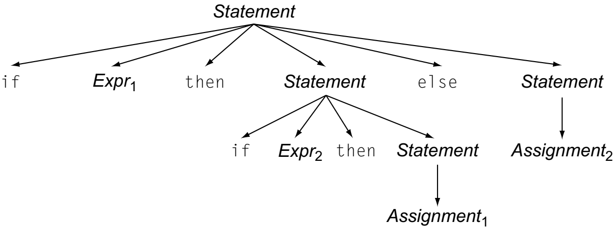
- This is called the *dangling else* problem.
- **Exercise:** Show *leftmost* derivations for the two parse trees.

CFG: Ambiguity: Exercise (2.2)

(***Meaning 1***) *Assignment*$_2$ may be associated with the inner `if`:



(***Meaning 2***) *Assignment*$_2$ may be associated with the outer `if`:



CFG: Ambiguity: Exercise (2.3)

- We may remove the ***ambiguity*** by specifying that the *dangling* `else` is associated with the *nearest* `if`:

$$\begin{array}{lcl}
\mathit{Statement} & \rightarrow & \texttt{if}\ \mathit{Expr}\ \texttt{then}\ \mathit{Statement} \\
 & | & \texttt{if}\ \mathit{Expr}\ \texttt{then}\ \mathit{WithElse}\ \texttt{else}\ \mathit{Statement} \\
 & | & \mathit{Assignment} \\
\mathit{WithElse} & \rightarrow & \texttt{if}\ \mathit{Expr}\ \texttt{then}\ \mathit{WithElse}\ \texttt{else}\ \mathit{WithElse} \\
 & | & \mathit{Assignment}
\end{array}$$

- When applying $\boxed{\texttt{if ... then}\ \mathit{WithElse}\ \texttt{else}\ \mathit{Statement}}$:
  - The ***true*** branch will be produced via *WithElse*.
  - The ***false*** branch will be produced via *Statement*.

  There is ***no circularity*** between the two non-terminals.

Discovering Derivations

- Given a CFG $G = (V, \Sigma, R, S)$ and an input program $p \in \Sigma^*$:
  - So far we ***manually*** come up a valid derivation $S \overset{*}{\Rightarrow} p$.
  - A parser is supposed to *automate* this derivation process.

    Given an input sequence of (t, c) pairs, where *token* t (e.g., `r241`) belongs to some *syntactic category* c (e.g., `register`):

    Either output a ***valid derivation*** (as an ***AST***), or signal an ***error***.
- In the process of building an *AST* for the input program:
  - **Root** of AST: **start symbol** S of G
  - **Internal nodes**: A subset of variables V of G
  - **Leaves** of AST: **token sequence** input by the scanner

  ⇒ Discovering the *grammatical connections* (according to R) between the root, internal nodes, and leaves is the hard part!
- Approaches to Parsing: $[\, w \in (V \cup \Sigma)^*,\ A \in V,\ \boxed{\boldsymbol{A} \rightarrow \boldsymbol{w}} \in R \,]$
  - **Top-down** parsing

    For a node representing ***A***, extend it with a subtree representing ***w***.
  - **Bottom-up** parsing

    For a substring matching ***w***, build a node representing ***A*** accordingly.

TDP: Discovering Leftmost Derivation

```
ALGORITHM: TDParse
  INPUT: CFG $G = (V, \Sigma, R, S)$
  OUTPUT: Root of a Parse Tree or Syntax Error
PROCEDURE:
  $root$ := a new node for the start symbol $S$
  $focus$ := $root$
  initialize an empty stack $trace$
  $trace$.push(null)
  $word$ := NextWord()
  while (true):
     if $focus \in V$ then
        if $\exists$ unvisited rule $focus \rightarrow \beta_1 \beta_2 \ldots \beta_n \in R$ then
           create $\beta_1, \beta_2 \ldots \beta_n$ as children of $focus$
           $trace$.push($\beta_n \beta_{n-1} \ldots \beta_2$)
           $focus$ := $\beta_1$
        else
           if $focus$ = $S$ then report syntax error
           else backtrack
     elseif $word$ matches $focus$ then
        $word$ := NextWord()
        $focus$ := $trace$.pop()
     elseif $word = EOF \land focus =$ null then  return root
     else backtrack
```

backtrack $\triangleq$ pop *focus*.siblings; *focus* := *focus*.parent; *focus*.resetChildren

TDP: Exercise (1)

- Given the following CFG **G**:

$$\begin{array}{lcl} \mathit{Expr} & \rightarrow & \mathit{Expr} \ + \ \mathit{Term} \\ & | & \mathit{Term} \\ \mathit{Term} & \rightarrow & \mathit{Term} \ \star \ \mathit{Factor} \\ & | & \mathit{Factor} \\ \mathit{Factor} & \rightarrow & (\,\mathit{Expr}\,) \\ & | & \mathtt{a} \end{array}$$

  Trace *TDParse* on how to build an AST for input `a + a * a`.
- Running *TDParse* with **G** results an ***infinite loop*** !!!
  - *TDParse* focuses on the ***leftmost*** non-terminal.
  - The grammar **G** contains ***left-recursions***.
- We must first convert left-recursions in **G** to ***right-recursions***.

TDP: Exercise (2)

- Given the following CFG **G**:

$$\begin{array}{lcl} \mathit{Expr} & \rightarrow & \mathit{Term} \quad \mathit{Expr}' \\ \mathit{Expr}' & \rightarrow & + \; \mathit{Term} \quad \mathit{Expr}' \\ & | & \epsilon \\ \mathit{Term} & \rightarrow & \mathit{Factor} \quad \mathit{Term}' \\ \mathit{Term}' & \rightarrow & \star \; \mathit{Factor} \quad \mathit{Term}' \\ & | & \epsilon \\ \mathit{Factor} & \rightarrow & (\,\mathit{Expr}\,) \\ & | & \mathtt{a} \end{array}$$

  Exercise. Trace *TDParse* on building AST for `a + a * a`.

  Exercise. Trace *TDParse* on building AST for `(a + a) * a`.

  Q: How to handle ϵ-productions (e.g., $\mathit{Expr} \rightarrow \epsilon$)?

  A: Execute $\mathit{focus} := \mathit{trace}.\mathtt{pop()}$ to advance to next node.

- Running *TDParse* will ***terminate*** $\because$ **G** is ***right-recursive***.
- We will learn about a systematic approach to converting left-recursions in a given grammar to ***right-recursions***.

Left-Recursions (LR): Direct vs. Indirect

Given CFG $G = (V, \Sigma, R, S)$, $\alpha, \beta, \gamma \in (V \cup \Sigma)^*$, G contains:

- A *cycle* if $\exists A \in V \bullet A \overset{*}{\Rightarrow} A$
- A ***direct*** LR if $A \rightarrow A\alpha \in R$ for non-terminal $A \in V$

  e.g.,

$$\begin{array}{lcl} \mathit{Expr} & \rightarrow & \mathit{Expr} \; + \; \mathit{Term} \\ & | & \mathit{Term} \\ \mathit{Term} & \rightarrow & \mathit{Term} \; * \; \mathit{Factor} \\ & | & \mathit{Factor} \\ \mathit{Factor} & \rightarrow & (\; \mathit{Expr} \;) \\ & | & \mathtt{a} \end{array}$$

  e.g.,

$$\begin{array}{lcl} \mathit{Expr} & \rightarrow & \mathit{Expr} \; + \; \mathit{Term} \\ & | & \mathit{Expr} \; - \; \mathit{Term} \\ & | & \mathit{Term} \\ \mathit{Term} & \rightarrow & \mathit{Term} \; * \; \mathit{Factor} \\ & | & \mathit{Term} \; / \; \mathit{Factor} \\ & | & \mathit{Factor} \end{array}$$

- An ***indirect*** LR if $A \rightarrow B\beta \in R$ for non-terminals $A, B \in V$, $B \overset{*}{\Rightarrow} A\gamma$

$$\begin{array}{lcl} A & \rightarrow & Br \\ B & \rightarrow & Cd \\ C & \rightarrow & At \end{array}$$

$$A \rightarrow Br, B \overset{*}{\Rightarrow} Atd$$

$$\begin{array}{lclclcl} A & \rightarrow & Ba & | & b & & \\ B & \rightarrow & Cd & | & e & & \\ C & \rightarrow & Df & | & g & & \\ D & \rightarrow & f & | & Aa & | & Cg \end{array}$$

$$A \rightarrow Ba, B \overset{*}{\Rightarrow} Aafd$$

TDP: (Preventively) Eliminating LRs

```
1   ALGORITHM: RemoveLR
2     INPUT: CFG $G = (V, \Sigma, R, S)$
3     ASSUME: $G$ acyclic $\wedge$ with no $\epsilon$-productions
4     OUTPUT: $G'$ s.t. $G' \equiv G$, $G'$ has no
5             indirect & direct left-recursions
6   PROCEDURE:
7     impose an order on $V$: $\langle\langle A_1, A_2, \ldots, A_n \rangle\rangle$
8     for $i$: 1 .. $n$:
9       for $j$: 1 .. $i-1$:
10        if $\exists\ A_i \rightarrow A_j\gamma \in R \ \wedge\ A_j \rightarrow \delta_1 \mid \delta_2 \mid \ldots \mid \delta_m \in R$ then
11          replace $A_i \rightarrow A_j\gamma$ with $A_i \rightarrow \delta_1\gamma \mid \delta_2\gamma \mid \ldots \mid \delta_m\gamma$
12        end
13      for $A_i \rightarrow A_i\alpha \mid \beta \in R$:
14        replace it with: $A_i \rightarrow \beta A'$, $A' \rightarrow \alpha A' \mid \epsilon$
```

L9 to **L11**: Remove ***indirect*** left-recursions from A_1 to A_{i-1}.

L12 to **L13**: Remove ***direct*** left-recursions from A_1 to A_{i-1}.

Loop Invariant (outer for-loop)? At the start of i^{th} iteration:

- No direct or indirect left-recursions for $A_1, A_2, \ldots, A_{i-1}$.
- More precisely: $\forall k : k < i \bullet \neg(\exists l \bullet l \leq k \wedge A_k \rightarrow A_l \cdots \in R)$

CFG: Eliminating ϵ-Productions (1)

- Motivations:
  - *TDParse* requires CFG with no ϵ-productions.
  - *RemoveLR* produces CFG which may contain ϵ-productions.
- $\epsilon \notin L \Rightarrow \exists$ CFG $G = (V, \Sigma, R, S)$ s.t. G has no ϵ-productions.

  An ***ϵ-production*** has the form $A \rightarrow \epsilon$.
- A variable A is ***nullable*** if $A \overset{*}{\Rightarrow} \epsilon$.
  - Each terminal symbol is *not nullable*.
  - Variable A is *nullable* if either:
    - $A \rightarrow \epsilon \in R$; or
    - $A \rightarrow B_1 B_2 \dots B_k \in R$, where each variable B_i $(1 \le i \le k)$ is a *nullable*.
- Given a production $B \rightarrow CAD$, if only variable A is nullable, then there are 2 versions of B: $B \rightarrow CAD \mid CD$
- In general, given a production $A \rightarrow X_1 X_2 \dots X_k$ with k symbols, if m of the k symbols are *nullable*:
  - $m < k$: There are 2^m versions of A.
  - $m = k$: There are $2^m - 1$ versions of A. [excluding $A \rightarrow \epsilon$]

CFG: Eliminating ϵ-Productions (2)

- Eliminate ϵ-productions from the following grammar:

$$\begin{array}{rcl} S & \rightarrow & AB \\ A & \rightarrow & aAA \mid \epsilon \\ B & \rightarrow & bBB \mid \epsilon \end{array}$$

- Which are the *nullable* variables? [S, A, B]

$$\begin{array}{rcll} S & \rightarrow & A \mid B \mid AB & \{S \rightarrow \epsilon \texttt{ not included}\} \\ A & \rightarrow & aAA \mid aA \mid a & \{A \rightarrow aA \texttt{ duplicated}\} \\ B & \rightarrow & bBB \mid bB \mid b & \{B \rightarrow bB \texttt{ duplicated}\} \end{array}$$

Backtrack-Free Parsing (1)

- `TDParse` automates the ***top-down***, ***leftmost*** derivation process by consistently choosing production rules (e.g., in order of their appearance in CFG).
  - This ***inflexibility*** may lead to ***inefficient*** runtime performance due to the need to *backtrack*.
  - e.g., It may take the ***construction of a giant subtree*** to find out a ***mismatch*** with the input tokens, which end up requiring it to *backtrack* all the way back to the ***root*** (start symbol).
- We may avoid backtracking with a modification to the parser:
  - When deciding which production rule to choose, consider:
    (1) the *current* input symbol
    (2) the **consequential** ***first*** symbol if a rule was applied for ***focus***
    [*lookahead* symbol]
  - Using a ***one symbol lookhead***, w.r.t. a ***right-recursive*** CFG, each alternative for the ***leftmost nonterminal*** leads to a ***unique terminal***, allowing the parser to decide on a choice that prevents *backtracking*.
  - Such CFG is ***backtrack free*** with a *lookhead* of one symbol.
  - We also call such backtrack-free CFG a ***predictive grammar***.

The FIRST Set: Definition

- Say we write $T \subset \mathbb{P}(\Sigma^*)$ to denote the set of valid tokens recognizable by the scanner.
- **FIRST** $(\alpha) \triangleq$ set of symbols that can appear as the ***first word*** in some string derived from α.
- More precisely:

$$\textbf{FIRST}(\alpha) = \begin{cases} \{\alpha\} & \text{if } \alpha \in T \\ \{w \mid w \in \Sigma^* \land \alpha \overset{*}{\Rightarrow} w\beta \land \beta \in (V \cup \Sigma)^*\} & \text{if } \alpha \in V \end{cases}$$

The FIRST Set: Examples

- Consider this ***right***-recursive CFG:

0	$Goal$	→	$Expr$	6	$Term'$	→	`x` $Factor\ Term'$
1	$Expr$	→	$Term\ Expr'$	7		|	÷ $Factor\ Term'$
2	$Expr'$	→	+ $Term\ Expr'$	8		|	ϵ
3		|	- $Term\ Expr'$	9	$Factor$	→	$\underline{(}\ Expr\ \underline{)}$
4		|	ϵ	10		|	`num`
5	$Term$	→	$Factor\ Term'$	11		|	`name`

- Compute **FIRST** for each terminal (e.g., `num`, +, $\underline{(}$):

	`num`	`name`	+	-	×	÷	$\underline{(}$	$\underline{)}$	`eof`	ϵ
FIRST	`num`	`name`	+	-	x	÷	$\underline{(}$	$\underline{)}$	`eof`	ϵ

- Compute **FIRST** for each non-terminal (e.g., $Expr$, $Term'$):

	Expr	***Expr′***	***Term***	***Term′***	***Factor***
FIRST	$\underline{(}$, `name`, `num`	+, -, ϵ	$\underline{(}$, `name`, `num`	x, ÷, ϵ	$\underline{(}$, `name`, `num`

Computing the FIRST Set

$$\text{FIRST}(\alpha) = \begin{cases} \{\alpha\} & \text{if } \alpha \in T \\ \{w \mid w \in \Sigma^* \wedge \alpha \overset{*}{\Rightarrow} w\beta \wedge \beta \in (V \cup \Sigma)^*\} & \text{if } \alpha \in V \end{cases}$$

```
ALGORITHM: GetFirst
  INPUT: CFG $G = (V, \Sigma, R, S)$
  $T \subset \Sigma^*$ denotes valid terminals
  OUTPUT: FIRST : $V \cup T \cup \{\epsilon, eof\} \longrightarrow \mathbb{P}(T \cup \{\epsilon, eof\})$
PROCEDURE:
  for $\alpha \in (T \cup \{eof, \epsilon\})$: FIRST($\alpha$) := $\{\alpha\}$
  for $A \in V$: FIRST($A$) := $\varnothing$
  $lastFirst$ := $\varnothing$
  while ($lastFirst \neq$ FIRST):
    $lastFirst$ := FIRST
    for $A \rightarrow \beta_1\beta_2 \ldots \beta_k \in R$ s.t. $\forall \beta_j : \beta_j \in (T \cup V)$:
      rhs := FIRST($\beta_1$) $- \{\epsilon\}$
      for ($i$ := 1; $\epsilon \in$ FIRST($\beta_i$) $\wedge\ i < k$; $i$++):
        rhs := rhs $\cup$ (FIRST($\beta_{i+1}$) $- \{\epsilon\}$)
      if $i = k \wedge \epsilon \in$ FIRST($\beta_k$) then
        rhs := rhs $\cup \{\epsilon\}$
      end
      FIRST($A$) := FIRST($A$) $\cup$ rhs
```

Computing the FIRST Set: Extension

- Recall: **FIRST** takes as input a token or a variable.

$$\boxed{\textbf{FIRST} : V \cup T \cup \{\epsilon, eof\} \longrightarrow \mathbb{P}(T \cup \{\epsilon, eof\})}$$

- The computation of variable ***rhs*** in algoritm `GetFirst` actually suggests an extended, overloaded version:

$$\boxed{\textbf{FIRST} : (V \cup T \cup \{\epsilon, eof\})^* \longrightarrow \mathbb{P}(T \cup \{\epsilon, eof\})}$$

  FIRST may also take as input a string $\beta_1 \beta_2 \dots \beta_n$ (RHS of rules).

- More precisely:

$$\textbf{FIRST}(\beta_1 \beta_2 \dots \beta_n) = \left\{ \textbf{FIRST}(\beta_1) \cup \textbf{FIRST}(\beta_2) \cup \dots \beta_k \;\middle|\; \begin{array}{l} \forall i : 1 \leq i < k \bullet \epsilon \in \textbf{FIRST}(\beta_i) \\ \wedge \\ \epsilon \notin \textbf{FIRST}(\beta_k) \end{array} \right\}$$

Note. β_k is the first symbol whose **FIRST** set does not contain ϵ.

Extended FIRST Set: Examples

Consider this ***right***-recursive CFG:

0	$Goal$	$\rightarrow$	$Expr$	6	$Term'$	$\rightarrow$	$\texttt{x}\ Factor\ Term'$
1	$Expr$	$\rightarrow$	$Term\ Expr'$	7		$\mid$	$\div\ Factor\ Term'$
2	$Expr'$	$\rightarrow$	$+\ Term\ Expr'$	8		$\mid$	ϵ
3		$\mid$	$\texttt{-}\ Term\ Expr'$	9	$Factor$	$\rightarrow$	$\underline{(}\ Expr\ \underline{)}$
4		$\mid$	ϵ	10		$\mid$	$\texttt{num}$
5	$Term$	$\rightarrow$	$Factor\ Term'$	11		$\mid$	$\texttt{name}$

e.g., **FIRST**$(Term\ Expr')$ = **FIRST**$(Term)$ $=\{\underline{(}, \texttt{name}, \texttt{num}\}$

e.g., **FIRST**$(+\ Term\ Expr')$ = **FIRST**$(+)$ = $\{+\}$

e.g., **FIRST**$(-\ Term\ Expr')$ = **FIRST**$(-)$ = $\{-\}$

e.g., **FIRST**(ϵ) = $\{\epsilon\}$

Is the FIRST Set Sufficient

- Consider the following three productions:

$$\begin{array}{llllll} Expr' & \rightarrow & + & Term & Term' & \texttt{(1)} \\ & \mid & - & Term & Term' & \texttt{(2)} \\ & \mid & \epsilon & & & \texttt{(3)} \end{array}$$

  In TDP, when the parser attempts to expand an *Expr′* node, it ***looks ahead with one symbol*** to decide on the choice of rule: **FIRST**$(+) = \{+\}$, **FIRST**$(-) = \{-\}$, and **FIRST**$(\epsilon) = \{\epsilon\}$.

  Q. When to choose rule `(3)` (causing ***focus := trace.pop()***)?

  A?. Choose rule `(3)` when $focus \neq$ **FIRST**$(+) \wedge focus \neq$ **FIRST**$(-)$?

  - ***Correct*** but ***inefficient*** in case of illegal input string: syntax error is only reported after possibly a long series of ***backtrack***.
  - Useful if parser knows which words can appear, after an application of the ϵ-production (rule `(3)`), as leadling symbols.

- **FOLLOW** $(v : V) \triangleq$ set of symbols that can appear to the immediate right of a string derived from α.

$$\text{FOLLOW}(v) = \{w \mid w, x, y \in \Sigma^* \wedge v \overset{*}{\Rightarrow} x \wedge S \overset{*}{\Rightarrow} xwy\}$$

The FOLLOW Set: Examples

- Consider this ***right***-recursive CFG:

0	$Goal$	→	$Expr$	6	$Term'$	→	x $Factor\ Term'$
1	$Expr$	→	$Term\ Expr'$	7		|	÷ $Factor\ Term'$
2	$Expr'$	→	+ $Term\ Expr'$	8		|	ϵ
3		|	- $Term\ Expr'$	9	$Factor$	→	$\underline{(}$ $Expr$ $\underline{)}$
4		|	ϵ	10		|	num
5	$Term$	→	$Factor\ Term'$	11		|	name

- Compute **FOLLOW** for each non-terminal (e.g., *Expr*, *Term′*):

	Expr	***Expr′***	***Term***	***Term′***	***Factor***
FOLLOW	eof, $\underline{)}$	eof, $\underline{)}$	eof, +, -, $\underline{)}$	eof, +, -, $\underline{)}$	eof, +, -, x, ÷, $\underline{)}$

Computing the FOLLOW Set

$$\text{FOLLOW}(v) = \{w \mid w, x, y \in \Sigma^* \land v \overset{*}{\Rightarrow} x \land S \overset{*}{\Rightarrow} xwy\}$$

```
ALGORITHM: GetFollow
  INPUT: CFG $G = (V,\ \Sigma,\ R,\ S)$
  OUTPUT: FOLLOW : $V \longrightarrow \mathbb{P}(T \cup \{eof\})$
PROCEDURE:
  for $A \in V$: FOLLOW($A$) := $\varnothing$
  FOLLOW($S$) := $\{eof\}$
  $lastFollow$ := $\varnothing$
  while ($lastFollow \neq$ FOLLOW):
    $lastFollow$ := FOLLOW
    for $A \rightarrow \beta_1\beta_2 \dots \beta_k \in R$:
      trailer := FOLLOW($A$)
      for i: k .. 1:
        if $\beta_i \in V$ then
          FOLLOW($\beta_i$) := FOLLOW($\beta_i$) $\cup$ trailer
          if $\epsilon \in$ FIRST($\beta_i$)
            then trailer := trailer $\cup$ (FIRST($\beta_i$) $- \epsilon$)
            else trailer := FIRST($\beta_i$)
        else
          trailer := FIRST($\beta_i$)
```

Backtrack-Free Grammar

- A *backtrack-free grammar* (for a ***top-down parser***), when expanding the `focus` ***internal node***, is always able to choose a ***unique*** rule with the ***one-symbol lookahead*** (or report a ***syntax error*** when no rule applies).
- To formulate this, we first define:

$$\textbf{FIRST}^{+}(A \rightarrow \beta) = \begin{cases} \textbf{FIRST}(\beta) & \text{if } \epsilon \notin \textbf{FIRST}(\beta) \\ \textbf{FIRST}(\beta) \cup \textbf{FOLLOW}(A) & \text{otherwise} \end{cases}$$

  FIRST(β) is the extended version where β may be $\beta_1 \beta_2 \dots \beta_n$
- Now, a *backtrack-free grammar* has each of its productions $A \rightarrow \gamma_1 \mid \gamma_2 \mid \dots \mid \gamma_n$ satisfying:

$$\forall i, j : 1 \leq i, j \leq n \wedge i \neq j \bullet \textbf{FIRST}^{+}(\gamma_i) \cap \textbf{FIRST}^{+}(\gamma_j) = \varnothing$$

TDP: Lookahead with One Symbol

```
ALGORITHM: TDParse
  INPUT: CFG $G = (V, \Sigma, R, S)$
  OUTPUT: Root of a Parse Tree or Syntax Error
PROCEDURE:
  root := a new node for the start symbol $S$
  focus := root
  initialize an empty stack trace
  trace.push(null)
  word := NextWord()
  while (true):
    if $focus \in V$ then % use FOLLOW set as well?
      if $\exists$ unvisited rule $focus \rightarrow \beta_1 \beta_2 \dots \beta_n \in R \land$ $word \in \textsc{First}^{+}(\beta)$ then
        create $\beta_1, \beta_2 \dots \beta_n$ as children of focus
        trace.push($\beta_n \beta_{n-1} \dots \beta_2$)
        focus := $\beta_1$
      else
        if focus = $S$ then report syntax error
        else backtrack
    elseif word matches focus then
      word := NextWord()
      focus := trace.pop()
    elseif word = EOF $\land$ focus = null then return root
    else backtrack
```

backtrack ≜ pop *focus*.siblings; *focus* := *focus*.parent; *focus*.resetChildren

Backtrack-Free Grammar: Exercise

Is the following CFG ***backtrack free***?

11	*Factor*	$\rightarrow$	`name`
12		$\mid$	`name` `[` *ArgList* `]`
13		$\mid$	`name` `(` *ArgList* `)`
15	*ArgList*	$\rightarrow$	*Expr MoreArgs*
16	*MoreArgs*	$\rightarrow$	`,` *Expr MoreArgs*
17		$\mid$	ϵ

- $\epsilon \notin \textbf{FIRST}(\textit{Factor}) \Rightarrow \textbf{FIRST}^{+}(\textit{Factor}) = \textbf{FIRST}(\textit{Factor})$
- $\textbf{FIRST}(\textit{Factor} \rightarrow \texttt{name}) = \{\texttt{name}\}$
- $\textbf{FIRST}(\textit{Factor} \rightarrow \texttt{name}\ \texttt{[}\textit{ArgList}\texttt{]}) = \{\texttt{name}\}$
- $\textbf{FIRST}(\textit{Factor} \rightarrow \texttt{name}\ \texttt{(}\textit{ArgList}\texttt{)}) = \{\texttt{name}\}$

$\therefore$ The above grammar is ***not*** backtrack free.
$\Rightarrow$ To expand an AST node of *Factor*, with a ***lookahead*** of `name`, the parser has no basis to choose among rules 11, 12, and 13.

Backtrack-Free Grammar: Left-Factoring

- A CFG is not backtrack free if there exists a ***common prefix*** (`name`) among the RHS of ***multiple*** production rules.
- To make such a CFG ***backtrack-free***, we may transform it using *left factoring* : a process of extracting and isolating ***common prefixes*** in a set of production rules.
  - Identify a common prefix α:

$$A \rightarrow \alpha\beta_1 \mid \alpha\beta_2 \mid \dots \mid \alpha\beta_n \mid \gamma_1 \mid \gamma_2 \mid \dots \mid \gamma_j$$

[each of $\gamma_1, \gamma_2, \dots, \gamma_j$ does not begin with α]

  - Rewrite that production rule as:

$$\begin{array}{lcl} A & \rightarrow & \alpha B \mid \gamma_1 \mid \gamma_2 \mid \dots \mid \gamma_j \\ B & \rightarrow & \beta_1 \mid \beta_2 \mid \dots \mid \beta_n \end{array}$$

  - New rule $B \rightarrow \beta_1 \mid \beta_2 \mid \dots \mid \beta_n$ may also contain ***common prefixes***.
  - Rewriting continues until no common prefixes are identified.

Left-Factoring: Exercise

- Use *left-factoring* to remove all ***common prefixes*** from the following grammar.

$$\begin{array}{llcl}
11 & \mathit{Factor} & \rightarrow & \texttt{name} \\
12 & & \mid & \texttt{name}\ \underline{\texttt{[}}\ \mathit{ArgList}\ \underline{\texttt{]}} \\
13 & & \mid & \texttt{name}\ \underline{\texttt{(}}\ \mathit{ArgList}\ \underline{\texttt{)}} \\
15 & \mathit{ArgList} & \rightarrow & \mathit{Expr}\ \mathit{MoreArgs} \\
16 & \mathit{MoreArgs} & \rightarrow & ,\ \mathit{Expr}\ \mathit{MoreArgs} \\
17 & & \mid & \epsilon
\end{array}$$

- Identify common prefix `name` and rewrite rules 11, 12, and 13:

$$\begin{array}{lcllll}
\mathit{Factor} & \rightarrow & \texttt{name} & \mathit{Arguments} & \\
\mathit{Arguments} & \rightarrow & \texttt{[} & \mathit{ArgList} & \texttt{]} \\
 & \mid & \texttt{(} & \mathit{ArgList} & \texttt{)} \\
 & \mid & \epsilon & &
\end{array}$$

Any more ***common prefixes***? [No]

TDP: Terminating and Backtrack-Free

- Given an arbitrary CFG as input to a ***top-down parser***:
  - **Q.** How do we avoid a ***non-terminating*** parsing process?
    A. Convert left-recursions to right-recursion.
  - **Q.** How do we minimize the need of ***backtracking***?
    A. left-factoring & one-symbol lookahead using $\textbf{FIRST}^{+}$
- ***Not*** every context-free ***language*** has a corresponding ***backtrack***-free context-free ***grammar***.

  Given a CFL l, the following is *undecidable*:

  $$\boxed{\exists cfg \mid L(cfg) = l \wedge isBacktrackFree(cfg)}$$

- Given a CFG $g = (V, \Sigma, R, S)$, whether or not g is ***backtrack-free*** is *decidable*:

  For each $A \rightarrow \gamma_1 \mid \gamma_2 \mid \ldots \mid \gamma_n \in R$:

  $$\boxed{\forall i, j : 1 \leq i, j \leq n \wedge i \neq j \bullet \textbf{FIRST}^{+}(\gamma_i) \cap \textbf{FIRST}^{+}(\gamma_j) = \varnothing}$$

Backtrack-Free Parsing (2.1)

- A *recursive-descent* parser is:
  - A top-down parser
  - Structured as a set of *mutually recursive* procedures

    Each procedure corresponds to a ***non-terminal*** in the grammar.

  See an example.
- Given a ***backtrack-free*** grammar, a tool (a.k.a. *parser generator*) can automatically generate:
  - **FIRST**, **FOLLOW**, and **FIRST**$^+$ sets
  - An efficient ***recursive-descent*** parser

    This generated parser is called an *LL(1) parser*, which:

    - Processes input from **L**eft to right
    - Constructs a **L**eftmost derivation
    - Uses a lookahead of **1** symbol
- *LL(1) grammars* are those working in an *LL(1)* scheme.

  LL(1) grammars are ***backtrack-free*** by definition.

Backtrack-Free Parsing (2.2)

- Consider this CFG with **FIRST**$^+$ sets of the RHSs:

	Production	**FIRST**$^+$
2	$Expr' \rightarrow + \; Term \; Expr'$	$\{+\}$
3	$\mid \; - \; Term \; Expr'$	$\{-\}$
4	$\mid \; \epsilon$	$\{\epsilon, \mathtt{eof}, \underline{)}\}$

- The corresponding ***recursive-descent*** parser is structured as:

```
ExprPrim()
  if word = + ∨ word = − then /* Rules 2, 3 */
    word := NextWord()
    if(Term())
      then return ExprPrim()
      else return false
  elseif word = ) ∨ word = eof then /* Rule 4 */
    return true
  else
    report a syntax error
    return false
  end

Term()
  ...
```

See: parser generator

LL(1) Parser: Exercise

Consider the following grammar:

$$\begin{array}{lcl} L & \rightarrow & R\ \mathtt{a} \\ & | & Q\ \mathtt{ba} \end{array} \qquad \begin{array}{lcl} R & \rightarrow & \mathtt{aba} \\ & | & \mathtt{caba} \\ & | & R\ \mathtt{bc} \end{array} \qquad \begin{array}{lcl} Q & \rightarrow & \mathtt{bbc} \\ & | & \mathtt{bc} \end{array}$$

Q. Is it suitable for a ***top-down predictive*** parser?

- If so, show that it satisfies the *LL(1)* condition.
- If not, identify the problem(s) and correct it (them). Also show that the revised grammar satisfies the *LL(1)* condition.

BUP: Discovering Rightmost Derivation

- In TDP, we build the start variable as the ***root node***, and then work towards the ***leaves***. [**leftmost** derivation]
- In Bottom-Up Parsing (BUP):
  - Words (terminals) are still returned from **left** to **right** by the scanner.
  - As terminals, or a mix of terminals and variables, are identified as *reducible* to some variable A (i.e., matching the RHS of some production rule for A), then a layer is added.
  - Eventually:
    - ***accept***: The ***start variable*** is reduced and **all** words have been consumed.
    - ***reject***: The next word is not `eof`, but no further *reduction* can be identified.

Q. Why can BUP find the ***rightmost*** derivation (RMD), if any?

A. BUP discovers steps in a ***RMD*** in its *reverse* order.

BUP: Discovering Rightmost Derivation (1)

- ***table***-driven *LR(1)* parser: an implementation for BUP, which
  - Processes input from **L**eft to right
  - Constructs a **R**ightmost derivation
  - Uses a lookahead of **1** symbol
- A language has the *LR(1)* property if it:
  - Can be parsed in a single **L**eft to right scan,
  - To build a *reversed* **R**ightmost derivation,
  - Using a lookahead of **1** symbol to determine parsing actions.
- Critical step in a bottom-up parser is to find the ***next*** *handle*.

BUP: Discovering Rightmost Derivation (2)

```
ALGORITHM: BUParse
  INPUT: CFG $G = (V, \Sigma, R, S)$, Action & Goto Tables
  OUTPUT: Report Parse Success or Syntax Error
PROCEDURE:
  initialize an empty stack $trace$
  $trace$.push($S$) /* start state */
  $word$ := NextWord()
  while(true)
    $state$ := $trace$.top()
    $act$ := Action[$state$, $word$]
    if $act$ = ``accept'' then
      succeed()
    elseif $act$ = ``reduce $A \rightarrow \beta$'' then
      $trace$.pop() $2 \times |\beta|$ times /* word + state */
      $state$ := $trace$.top()
      $trace$.push($A$)
      $next$ := Goto[state, A]
      $trace$.push($next$)
    elseif $act$ = ``shift $s_i$'' then
      $trace$.push($word$)
      $trace$.push($s_i$)
      $word$ := NextWord()
    else
      fail()
```

BUP: Example Tracing (1)

- Consider the following grammar for parentheses:

1 $Goal \rightarrow List$
2 $List \rightarrow List\ Pair$
3 $\quad\quad\ \mid\ Pair$
4 $Pair \rightarrow \underline{(}\ Pair\ \underline{)}$
5 $\quad\quad\ \mid\ \underline{(}\ \underline{)}$

- Assume: tables ***Action*** and ***Goto*** constructed accordingly:

	Action **Table**			Goto **Table**	
State	eof	(	)	***List***	***Pair***
0		s 3		1	2
1	acc	s 3			4
2	r 3	r 3			
3		s 6	s 7		5
4	r 2	r 2			
5			s 8		
6		s 6	s 10		9
7	r 5	r 5			
8	r 4	r 4			
9			s 11		
10			r 5		
11			r 4		

In **Action** table:

- s_i: shift to state i
- r_j: reduce to the LHS of production #j

BUP: Example Tracing (2.1)

Consider the steps of performing BUP on input $\boxed{\texttt{()}}$:

Iteration	**State**	**word**	**Stack**	**Handle**	**Action**
initial	—	$\underline{(}$	$ 0	— *none* —	—
1	0	$\underline{(}$	$ 0	— *none* —	*shift 3*
2	3	$\underline{)}$	$ 0 $\underline{(}$ 3	— *none* —	*shift 7*
3	7	`eof`	$ 0 $\underline{(}$ 3 $\underline{)}$ 7	$\underline{(}$ $\underline{)}$	*reduce 5*
4	2	`eof`	$ 0 *Pair* 2	*Pair*	*reduce 3*
5	1	`eof`	$ 0 *List* 1	*List*	*accept*

BUP: Example Tracing (2.2)

Consider the steps of performing BUP on input (( )) ( ) :

Iteration	State	word	Stack	Handle	Action
initial	—	(	$ 0	— *none* —	—
1	0	(	$ 0	— *none* —	*shift 3*
2	3	(	$ 0 (3	— *none* —	*shift 6*
3	6	)	$ 0 (3 (6	— *none* —	*shift 10*
4	10	)	$ 0 (3 (6) 10	( )	*reduce 5*
5	5	)	$ 0 (3 *Pair* 5	— *none* —	*shift 8*
6	8	(	$ 0 (3 *Pair* 5) 8	(*Pair*)	*reduce 4*
7	2	(	$ 0 *Pair* 2	*Pair*	*reduce 3*
8	1	(	$ 0 *List* 1	— *none* —	*shift 3*
9	3	)	$ 0 *List* 1 (3	— *none* —	*shift 7*
10	7	eof	$ 0 *List* 1 (3) 7	( )	*reduce 5*
11	4	eof	$ 0 *List* 1 *Pair* 4	*List Pair*	*reduce 2*
12	1	eof	$ 0 *List* 1	*List*	*accept*

BUP: Example Tracing (2.3)

Consider the steps of performing BUP on input `( ) )`:

Iteration	**State**	`word`	**Stack**	**Handle**	**Action**
initial	—	(	$ 0	— *none* —	—
1	0	(	$ 0	— *none* —	*shift 3*
2	3	)	$ 0 (3	— *none* —	*shift 7*
3	7	)	$ 0 (3) 7	— *none* —	*error*

LR(1) Items: Definition

- In *LR(1)* parsing, **Action** and **Goto** tabeles encode legitimate ways (w.r.t. a grammar) for finding ***handles*** (for ***reductions***).
- In a ***table***-driven *LR(1)* parser, the table-construction algorithm represents each potential ***handle*** (for a ***reduction***) with an *LR(1) item* e.g.,

$$[A \to \beta \bullet \gamma,\ \mathtt{a}]$$

where:
  - A production rule $A \to \beta\gamma$ is currently being applied.
  - A placeholder, $\bullet$, indicates the position of the parser's ***stack top***.
    - ✓ The parser's stack contains β ("left context").
    - ✓ γ is yet to be matched.

      Remark. Upon matching $\beta\gamma$, if $\mathtt{a}$ matches the current word, then we "replace" $\beta\gamma$ (and their corresponding states) with A (and its corresponding state).
  - A terminal symbol $\mathtt{a}$ servers as a ***lookahead symbol***.

LR(1) Items: Scenarios

An *LR(1) item* can be:

1. **POSSIBILITY** $[A \rightarrow \bullet\beta\gamma,\ \mathtt{a}]$
   - In the current parsing context, an A would be valid.
   - $\bullet$ represents the position of the parser's ***stack top***
   - Recognizing a β next would be one step towards discovering an A.
2. **PARTIALLY COMPLETION** $[A \rightarrow \beta\bullet\gamma,\ \mathtt{a}]$
   - The parser has progressed from $[A \rightarrow \bullet\beta\gamma,\ \mathtt{a}]$ by recognizing β.
   - Recognizing a γ next would be one step towards discovering an A.
3. **COMPLETION** $[A \rightarrow \beta\gamma\bullet,\ \mathtt{a}]$
   - Parser has progressed from $[A \rightarrow \bullet\beta\gamma,\ \mathtt{a}]$ by recognizing $\beta\gamma$.
   - $\beta\gamma$ found in a context where an A followed by $\mathtt{a}$ would be valid.
   - If the current input word matches $\mathtt{a}$, then:
     - Current ***complet item*** is a *handle*.
     - Parser can ***reduce*** $\beta\gamma$ to A (and replace $\beta\gamma$ with A in its stack).

LR(1) Items: Example (1.1)

Consider the following grammar for parentheses:

1 $\mathit{Goal} \rightarrow \mathit{List}$
2 $\mathit{List} \rightarrow \mathit{List}\ \mathit{Pair}$
3 $\quad\quad\ \mid\ \mathit{Pair}$
4 $\mathit{Pair} \rightarrow \underline{(}\ \mathit{Pair}\ \underline{)}$
5 $\quad\quad\ \mid\ \underline{(}\ \underline{)}$

Initial State: $[\mathit{Goal} \rightarrow \bullet \mathit{List},\ \texttt{eof}]$

Desired Final State: $[\mathit{Goal} \rightarrow \mathit{List}\bullet,\ \texttt{eof}]$

Intermediate States: Subset Construction

Q. Derive all *LR(1) items* for the above grammar.

- **Follow**(*List*) = $\{\texttt{eof}, \texttt{(}\}$ **Follow**(*Pair*) = $\{\texttt{eof}, \texttt{(}, \texttt{)}\}$
- For each production $A \rightarrow \beta$, given **Follow**(A), *LR(1) items* are:

$$\{\ [A \rightarrow \bullet\beta\gamma,\ \texttt{a}] \mid a \in \textsc{Follow}(A)\ \}$$
$$\cup$$
$$\{\ [A \rightarrow \beta\bullet\gamma,\ \texttt{a}] \mid a \in \textsc{Follow}(A)\ \}$$
$$\cup$$
$$\{\ [A \rightarrow \beta\gamma\bullet,\ \texttt{a}] \mid a \in \textsc{Follow}(A)\ \}$$

LR(1) Items: Example (1.2)

Q. Given production $A \to \beta$ (e.g., $\mathit{Pair} \to \texttt{(}\ \mathit{Pair}\ \texttt{)}$), how many *LR(1) items* can be generated?

- The current parsing progress (on matching the RHS) can be:
  1. $\bullet\texttt{(}\ \mathit{Pair}\ \texttt{)}$
  2. $\texttt{(}\ \bullet\mathit{Pair}\ \texttt{)}$
  3. $\texttt{(}\ \mathit{Pair}\bullet\ \texttt{)}$
  4. $\texttt{(}\ \mathit{Pair}\ \texttt{)}\bullet$
- Lookahead symbol following *Pair*? $\textsc{Follow}(\mathit{Pair}) = \{\texttt{eof}, \texttt{(}, \texttt{)}\}$
- All possible *LR(1) items* related to $\mathit{Pair} \to \texttt{(}\ \mathit{Pair}\ \texttt{)}$?

✓ $[\bullet\texttt{(}\ \mathit{Pair}\ \texttt{)}, \texttt{eof}]$	$[\bullet\texttt{(}\ \mathit{Pair}\ \texttt{)}, \texttt{(}]$	$[\bullet\texttt{(}\ \mathit{Pair}\ \texttt{)}, \texttt{)}]$
✓ $[\texttt{(}\ \bullet\mathit{Pair}\ \texttt{)}, \texttt{eof}]$	$[\texttt{(}\ \bullet\mathit{Pair}\ \texttt{)}, \texttt{(}]$	$[\texttt{(}\ \bullet\mathit{Pair}\ \texttt{)}, \texttt{)}]$
✓ $[\texttt{(}\ \mathit{Pair}\bullet\ \texttt{)}, \texttt{eof}]$	$[\texttt{(}\ \mathit{Pair}\bullet\ \texttt{)}, \texttt{(}]$	$[\texttt{(}\ \mathit{Pair}\bullet\ \texttt{)}, \texttt{)}]$
✓ $[\texttt{(}\ \mathit{Pair}\ \texttt{)}\bullet, \texttt{eof}]$	$[\texttt{(}\ \mathit{Pair}\ \texttt{)}\bullet, \texttt{(}]$	$[\texttt{(}\ \mathit{Pair}\ \texttt{)}\bullet, \texttt{)}]$

A. How many in general (in terms of A and β)?

$$\underbrace{|\beta| + 1}_{\texttt{possible positions of } \bullet} \times \underbrace{|\textsc{Follow}(A)|}_{\texttt{possible lookahead symbols}}$$

LR(1) Items: Example (1.3)

A. There are 33 *LR(1) items* in the parentheses grammar.

$[\mathit{Goal} \rightarrow \bullet\, \mathit{List}, \texttt{eof}]$

$[\mathit{Goal} \rightarrow \mathit{List}\, \bullet, \texttt{eof}]$

$[\mathit{List} \rightarrow \bullet\, \mathit{List}\ \mathit{Pair}, \texttt{eof}]$ $[\mathit{List} \rightarrow \bullet\, \mathit{List}\ \mathit{Pair}, \underline{(}\,]$
$[\mathit{List} \rightarrow \mathit{List}\, \bullet\, \mathit{Pair}, \texttt{eof}]$ $[\mathit{List} \rightarrow \mathit{List}\, \bullet\, \mathit{Pair}, \underline{(}\,]$
$[\mathit{List} \rightarrow \mathit{List}\ \mathit{Pair}\, \bullet, \texttt{eof}]$ $[\mathit{List} \rightarrow \mathit{List}\ \mathit{Pair}\, \bullet, \underline{(}\,]$

$[\mathit{List} \rightarrow \bullet\, \mathit{Pair}, \texttt{eof}\,]$ $[\mathit{List} \rightarrow \bullet\, \mathit{Pair}, \underline{(}\,]$
$[\mathit{List} \rightarrow \mathit{Pair}\, \bullet, \texttt{eof}\,]$ $[\mathit{List} \rightarrow \mathit{Pair}\, \bullet, \underline{(}\,]$

$[\mathit{Pair} \rightarrow \bullet\, \underline{(}\ \mathit{Pair}\ \underline{)}, \texttt{eof}\,]$ $[\mathit{Pair} \rightarrow \bullet\, \underline{(}\ \mathit{Pair}\ \underline{)}, \underline{)}]$ $[\mathit{Pair} \rightarrow \bullet\, \underline{(}\ \mathit{Pair}\ \underline{)}, \underline{(}]$
$[\mathit{Pair} \rightarrow \underline{(}\ \bullet\, \mathit{Pair}\ \underline{)}, \texttt{eof}\,]$ $[\mathit{Pair} \rightarrow \underline{(}\ \bullet\, \mathit{Pair}\ \underline{)}, \underline{)}]$ $[\mathit{Pair} \rightarrow \underline{(}\ \bullet\, \mathit{Pair}\ \underline{)}, \underline{(}]$
$[\mathit{Pair} \rightarrow \underline{(}\ \mathit{Pair}\, \bullet\ \underline{)}, \texttt{eof}\,]$ $[\mathit{Pair} \rightarrow \underline{(}\ \mathit{Pair}\, \bullet\ \underline{)}, \underline{)}]$ $[\mathit{Pair} \rightarrow \underline{(}\ \mathit{Pair}\, \bullet\ \underline{)}, \underline{(}]$
$[\mathit{Pair} \rightarrow \underline{(}\ \mathit{Pair}\ \underline{)}\, \bullet, \texttt{eof}\,]$ $[\mathit{Pair} \rightarrow \underline{(}\ \mathit{Pair}\ \underline{)}\, \bullet, \underline{)}]$ $[\mathit{Pair} \rightarrow \underline{(}\ \mathit{Pair}\ \underline{)}\, \bullet, \underline{(}]$

$[\mathit{Pair} \rightarrow \bullet\, \underline{(}\ \ \underline{)}, \texttt{eof}]$ $[\mathit{Pair} \rightarrow \bullet\, \underline{(}\ \ \underline{)}, \underline{(}]$ $[\mathit{Pair} \rightarrow \bullet\, \underline{(}\ \ \underline{)}, \underline{)}]$
$[\mathit{Pair} \rightarrow \underline{(}\ \bullet\ \underline{)}, \texttt{eof}]$ $[\mathit{Pair} \rightarrow \underline{(}\ \bullet\ \underline{)}, \underline{(}]$ $[\mathit{Pair} \rightarrow \underline{(}\ \bullet\ \underline{)}, \underline{)}]$
$[\mathit{Pair} \rightarrow \underline{(}\ \ \underline{)}\, \bullet, \texttt{eof}]$ $[\mathit{Pair} \rightarrow \underline{(}\ \ \underline{)}\, \bullet, \underline{(}]$ $[\mathit{Pair} \rightarrow \underline{(}\ \ \underline{)}\, \bullet, \underline{)}]$

LR(1) Items: Example (2)

Consider the following grammar for expressions:

0	*Goal*	→	*Expr*	6	*Term′*	→	x *Factor Term′*
1	*Expr*	→	*Term Expr′*	7		|	÷ *Factor Term′*
2	*Expr′*	→	+ *Term Expr′*	8		|	ϵ
3		|	- *Term Expr′*	9	*Factor*	→	(*Expr*)
4		|	ϵ	10		|	num
5	*Term*	→	*Factor Term′*	11		|	name

Q. Derive all *LR(1) items* for the above grammar.

Hints. First compute **FOLLOW** for each non-terminal:

	Expr	***Expr′***	***Term***	***Term′***	***Factor***
FOLLOW	eof,)	eof,)	eof, +, -,)	eof, +, -,)	eof, +, -, x, ÷,)

Tips. Ignore ϵ ***production*** such as *Expr′* → ϵ
since the **FOLLOW** sets already take them into consideration.

Canonical Collection ($\mathcal{CC}$) vs. LR(1) items

1	$Goal$	$\rightarrow$	$List$
2	$List$	$\rightarrow$	$List\ Pair$
3		$\mid$	$Pair$
4	$Pair$	$\rightarrow$	$\underline{(}\ Pair\ \underline{)}$
5		$\mid$	$\underline{(}\ \underline{)}$

Recall:

LR(1) Items: 33 items

Initial State: $[Goal \rightarrow \bullet List, \texttt{eof}]$

Desired Final State: $[Goal \rightarrow List \bullet, \texttt{eof}]$

- The *canonical collection*

$$\mathcal{CC} = \{cc_0, cc_1, cc_2, \ldots, cc_n\}$$

  denotes the set of ***valid states*** of a ***LR(1)*** parser.
  - Each $cc_i \in \mathcal{CC}$ ($0 \leq i \leq n$) is a set of **LR(1) items**.
  - $\mathcal{CC} \subseteq \mathbb{P}(\textbf{LR(1) items})$ $|\mathcal{CC}|$? $[\ |\mathcal{CC}| \leq 2^{|\textbf{LR(1) items}|}\ ]$
- To model a ***LR(1)*** parser, we use techniques similar to how we construct a DFA from an NFA (subset construction and ϵ-closure).
- **Analogies.**
  - ✓ **LR(1) items** ≈ states of source *NFA*
  - ✓ $\mathcal{CC}$ ≈ states of target *DFA*

Constructing $\mathcal{CC}$: The *closure* Procedure (1)

```
1  ALGORITHM: closure
2    INPUT: CFG $G = (V, \Sigma, R, S)$, a set $s$ of LR(1) items
3    OUTPUT: a set of LR(1) items
4  PROCEDURE:
5    $lastS := \varnothing$
6    while ($lastS \neq s$):
7      $lastS := s$
8      for $[A \rightarrow \cdots \bullet C\,\delta,\ a] \in s$:
9        for $C \rightarrow \gamma \in R$:
10         for $b \in \text{FIRST}(\delta a)$:
11           $s := s \cup \{\ [C \rightarrow \bullet\gamma,\ b]\ \}$
12   return $s$
```

- **Line 8**: $[A \rightarrow \cdots \bullet C\,\delta,\ a] \in s$ indicates that the parser's next task is to match $C\,\delta$ with a lookahead symbol a.
- **Line 9**: Given: matching γ can reduce to C
- **Line 10**: Given: $b \in \text{FIRST}(\delta a)$ is a valid lookahead symbol after reducing γ to C
- **Line 11**: Add a new item $[C \rightarrow \bullet\gamma,\ b]$ into s.
- **Line 6**: Termination is guaranteed.

  ∵ Each iteration adds ≥ 1 item to s (otherwise $lastS \neq s$ is *false*).

Constructing $\mathcal{CC}$: The *closure* Procedure (2.1)

$$\begin{array}{llcl}
1 & Goal & \rightarrow & List \\
2 & List & \rightarrow & List\ Pair \\
3 & & | & Pair \\
4 & Pair & \rightarrow & \underline{(}\ \ Pair\ \ \underline{)} \\
5 & & | & \underline{(}\ \ \underline{)}
\end{array}$$

Initial State: $[Goal \rightarrow \bullet List,\ \texttt{eof}]$

Calculate $cc_0 = closure([Goal \rightarrow \bullet List,\ \texttt{eof}])$.

Constructing $\mathcal{CC}$: The *goto* Procedure (1)

```
1  ALGORITHM: goto
2    INPUT: a set s of LR(1) items, a symbol x
3    OUTPUT: a set of LR(1) items
4  PROCEDURE:
5    moved := $\varnothing$
6    for item $\in$ s:
7      if item = $[\alpha \to \beta \bullet x\delta,\ \mathtt{a}]$ then
8        moved := moved $\cup$ { $[\alpha \to \beta x \bullet \delta,\ \mathtt{a}]$ }
9      end
10   return closure(moved)
```

Line 7: Given: item $[\alpha \to \beta \bullet x\delta,\ \mathtt{a}]$ (where *x* is the next to match)
Line 8: Add $[\alpha \to \beta x \bullet \delta,\ \mathtt{a}]$ (indicating $\mathtt{x}$ is matched) to *moved*
Line 10: Calculate and return *closure*(*moved*) as the "next state" from *s* with a "transition" $\mathtt{x}$.

Constructing $\mathcal{CC}$: The *goto* Procedure (2)

$$\begin{array}{llcl} 1 & Goal & \rightarrow & List \\ 2 & List & \rightarrow & List\ Pair \\ 3 & & \mid & Pair \\ 4 & Pair & \rightarrow & \underline{(}\ \ Pair\ \ \underline{)} \\ 5 & & \mid & \underline{(}\ \ \underline{)} \end{array}$$

$$cc_0 = \left\{ \begin{array}{ccc} {[}Goal \rightarrow \bullet\, List, \texttt{eof}] & {[}List \rightarrow \bullet\, List\ Pair, \texttt{eof}] & {[}List \rightarrow \bullet\, List\ Pair, \underline{(}] \\ {[}List \rightarrow \bullet\, Pair, \texttt{eof}] & {[}List \rightarrow \bullet\, Pair, \underline{(}] & {[}Pair \rightarrow \bullet\, \underline{(}\ Pair\ \underline{)}, \texttt{eof}] \\ {[}Pair \rightarrow \bullet\, \underline{(}\ Pair\ \underline{)}, \underline{(}] & {[}Pair \rightarrow \bullet\, \underline{(}\ \underline{)}, \texttt{eof}] & {[}Pair \rightarrow \bullet\, \underline{(}\ \underline{)}, \underline{(}] \end{array} \right\}$$

Calculate $goto(cc_0,\ \texttt{(})$. [“next state” from cc_0 taking `(`]

Constructing $\mathcal{CC}$: The Algorithm (1)

```
ALGORITHM: $Build\mathcal{CC}$
  INPUT: a grammar $G = (V, \Sigma, R, S)$, goal production $S \to S'$
  OUTPUT:
    (1) a set $\mathcal{CC} = \{cc_0, cc_1, \dots, cc_n\}$ where $cc_i \subseteq G'$s $LR(1)$ items
    (2) a transition function
PROCEDURE:
  $cc_0 := closure(\{[S' \to \bullet S, \texttt{eof}]\})$
  $\mathcal{CC} := \{cc_0\}$
  $processed := \{cc_0\}$
  $last\mathcal{CC} := \varnothing$
  while($last\mathcal{CC} \neq \mathcal{CC}$):
    $last\mathcal{CC} := \mathcal{CC}$
    for $cc_i$ s.t. $cc_i \in \mathcal{CC} \land cc_i \notin processed$:
      $processed := processed \cup \{cc_i\}$
      for x s.t. $[\cdots \to \cdots \bullet \texttt{x} \dots] \in cc_i$
        $temp := goto(cc_i, \texttt{x})$
        if $temp \notin \mathcal{CC}$ then
          $\mathcal{CC} := \mathcal{CC} \cup \{temp\}$
        end
        $\delta := \delta \cup (cc_i, \texttt{x}, temp)$
```

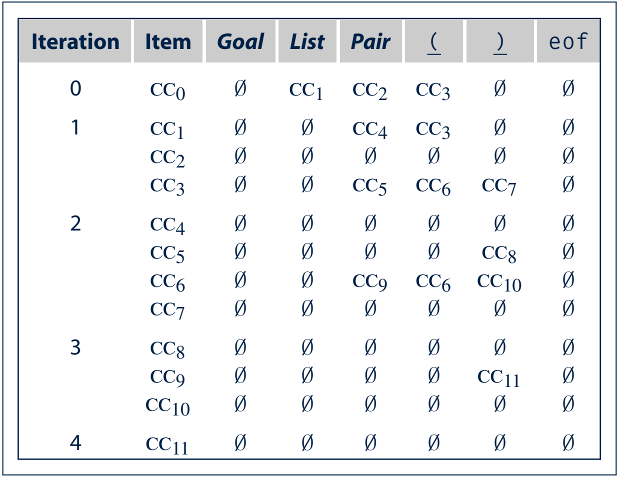
Constructing $\mathcal{CC}$: The Algorithm (2.1)

$$\begin{array}{rlcl}
1 & \mathit{Goal} & \rightarrow & \mathit{List} \\
2 & \mathit{List} & \rightarrow & \mathit{List}\ \mathit{Pair} \\
3 & & | & \mathit{Pair} \\
4 & \mathit{Pair} & \rightarrow & \underline{(}\ \ \mathit{Pair}\ \ \underline{)} \\
5 & & | & \underline{(}\ \ \underline{)}
\end{array}$$

- Calculate $\mathcal{CC} = \{cc_0, cc_1, \ldots, cc_{11}\}$
- Calculate the transition function $\delta : \mathcal{CC} \times \Sigma \rightarrow \mathcal{CC}$

Constructing $\mathcal{CC}$: The Algorithm (2.2)

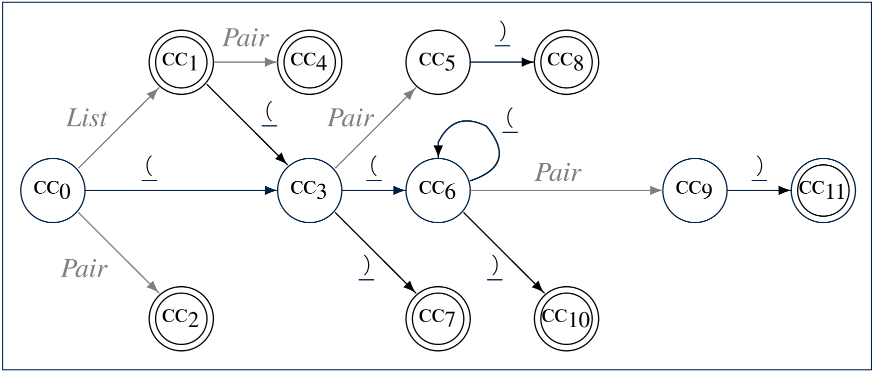
Resulting transition table:



Iteration	Item	*Goal*	*List*	*Pair*	(	)	`eof`
0	CC_0	∅	CC_1	CC_2	CC_3	∅	∅
1	CC_1	∅	∅	CC_4	CC_3	∅	∅
	CC_2	∅	∅	∅	∅	∅	∅
	CC_3	∅	∅	CC_5	CC_6	CC_7	∅
2	CC_4	∅	∅	∅	∅	∅	∅
	CC_5	∅	∅	∅	∅	CC_8	∅
	CC_6	∅	∅	CC_9	CC_6	CC_{10}	∅
	CC_7	∅	∅	∅	∅	∅	∅
3	CC_8	∅	∅	∅	∅	∅	∅
	CC_9	∅	∅	∅	∅	CC_{11}	∅
	CC_{10}	∅	∅	∅	∅	∅	∅
4	CC_{11}	∅	∅	∅	∅	∅	∅

Constructing $\mathcal{CC}$: The Algorithm (2.3)

Resulting DFA for the parser:



Constructing $\mathcal{CC}$: The Algorithm (2.4.1)

Resulting canonical collection $\mathcal{CC}$:

$$\mathrm{CC}_0 = \left\{ \begin{array}{ccc} [\mathit{Goal} \rightarrow \bullet\, \mathit{List}, \texttt{eof}] & [\mathit{List} \rightarrow \bullet\, \mathit{List}\ \mathit{Pair}, \texttt{eof}] & [\mathit{List} \rightarrow \bullet\, \mathit{List}\ \mathit{Pair}, \underline{\texttt{(}}] \\ [\mathit{List} \rightarrow \bullet\, \mathit{Pair}, \texttt{eof}] & [\mathit{List} \rightarrow \bullet\, \mathit{Pair}, \underline{\texttt{(}}] & [\mathit{Pair} \rightarrow \bullet\, \underline{\texttt{(}}\ \mathit{Pair}\ \underline{\texttt{)}}, \texttt{eof}] \\ [\mathit{Pair} \rightarrow \bullet\, \underline{\texttt{(}}\ \mathit{Pair}\ \underline{\texttt{)}}, \underline{\texttt{(}}] & [\mathit{Pair} \rightarrow \bullet\, \underline{\texttt{(}}\ \underline{\texttt{)}}, \texttt{eof}] & [\mathit{Pair} \rightarrow \bullet\, \underline{\texttt{(}}\ \underline{\texttt{)}}, \underline{\texttt{(}}] \end{array} \right\}$$

$$\mathrm{CC}_1 = \left\{ \begin{array}{ccc} [\mathit{Goal} \rightarrow \mathit{List}\, \bullet, \texttt{eof}] & [\mathit{List} \rightarrow \mathit{List} \bullet \mathit{Pair}, \texttt{eof}] & [\mathit{List} \rightarrow \mathit{List} \bullet \mathit{Pair}, \underline{\texttt{(}}] \\ [\mathit{Pair} \rightarrow \bullet\, \underline{\texttt{(}}\ \mathit{Pair}\ \underline{\texttt{)}}, \texttt{eof}] & [\mathit{Pair} \rightarrow \bullet\, \underline{\texttt{(}}\ \mathit{Pair}\ \underline{\texttt{)}}, \underline{\texttt{(}}] & [\mathit{Pair} \rightarrow \bullet\, \underline{\texttt{(}}\ \underline{\texttt{)}}, \texttt{eof}] \\ & [\mathit{Pair} \rightarrow \bullet\, \underline{\texttt{(}}\ \underline{\texttt{)}}, \underline{\texttt{(}}] & \end{array} \right\}$$

$$\mathrm{CC}_2 = \left\{ [\mathit{List} \rightarrow \mathit{Pair}\, \bullet, \texttt{eof}] \quad [\mathit{List} \rightarrow \mathit{Pair}\, \bullet, \underline{\texttt{(}}] \right\}$$

$$\mathrm{CC}_3 = \left\{ \begin{array}{ccc} [\mathit{Pair} \rightarrow \bullet\, \underline{\texttt{(}}\ \mathit{Pair}\ \underline{\texttt{)}}, \underline{\texttt{)}}] & [\mathit{Pair} \rightarrow \underline{\texttt{(}} \bullet \mathit{Pair}\ \underline{\texttt{)}}, \texttt{eof}] & [\mathit{Pair} \rightarrow \underline{\texttt{(}} \bullet \mathit{Pair}\ \underline{\texttt{)}}, \underline{\texttt{(}}] \\ [\mathit{Pair} \rightarrow \bullet\, \underline{\texttt{(}}\ \underline{\texttt{)}}, \underline{\texttt{)}}] & [\mathit{Pair} \rightarrow \underline{\texttt{(}} \bullet \underline{\texttt{)}}, \texttt{eof}] & [\mathit{Pair} \rightarrow \underline{\texttt{(}} \bullet \underline{\texttt{)}}, \underline{\texttt{(}}] \end{array} \right\}$$

$$\mathrm{CC}_4 = \left\{ [\mathit{List} \rightarrow \mathit{List}\ \mathit{Pair}\, \bullet, \texttt{eof}] \quad [\mathit{List} \rightarrow \mathit{List}\ \mathit{Pair}\, \bullet, \underline{\texttt{(}}] \right\}$$

$$\mathrm{CC}_5 = \left\{ [\mathit{Pair} \rightarrow \underline{\texttt{(}}\ \mathit{Pair} \bullet \underline{\texttt{)}}, \texttt{eof}] \quad [\mathit{Pair} \rightarrow \underline{\texttt{(}}\ \mathit{Pair} \bullet \underline{\texttt{)}}, \underline{\texttt{(}}] \right\}$$

$$\mathrm{CC}_6 = \left\{ \begin{array}{cc} [\mathit{Pair} \rightarrow \bullet\, \underline{\texttt{(}}\ \mathit{Pair}\ \underline{\texttt{)}}, \underline{\texttt{)}}] & [\mathit{Pair} \rightarrow \underline{\texttt{(}} \bullet \mathit{Pair}\ \underline{\texttt{)}}, \underline{\texttt{)}}] \\ [\mathit{Pair} \rightarrow \bullet\, \underline{\texttt{(}}\ \underline{\texttt{)}}, \underline{\texttt{)}}] & [\mathit{Pair} \rightarrow \underline{\texttt{(}} \bullet \underline{\texttt{)}}, \underline{\texttt{)}}] \end{array} \right\}$$

$$\mathrm{CC}_7 = \left\{ [\mathit{Pair} \rightarrow \underline{\texttt{(}}\ \underline{\texttt{)}}\, \bullet, \texttt{eof}] \quad [\mathit{Pair} \rightarrow \underline{\texttt{(}}\ \underline{\texttt{)}}\, \bullet, \underline{\texttt{(}}] \right\}$$

$$\mathrm{CC}_8 = \left\{ [\mathit{Pair} \rightarrow \underline{\texttt{(}}\ \mathit{Pair}\ \underline{\texttt{)}}\, \bullet, \texttt{eof}] \quad [\mathit{Pair} \rightarrow \underline{\texttt{(}}\ \mathit{Pair}\ \underline{\texttt{)}}\, \bullet, \underline{\texttt{(}}] \right\}$$

$$\mathrm{CC}_9 = \left\{ [\mathit{Pair} \rightarrow \underline{\texttt{(}}\ \mathit{Pair} \bullet \underline{\texttt{)}}, \underline{\texttt{)}}] \right\}$$

$$\mathrm{CC}_{10} = \left\{ [\mathit{Pair} \rightarrow \underline{\texttt{(}}\ \underline{\texttt{)}}\, \bullet, \underline{\texttt{)}}] \right\}$$

$$\mathrm{CC}_{11} = \left\{ [\mathit{Pair} \rightarrow \underline{\texttt{(}}\ \mathit{Pair}\ \underline{\texttt{)}}\, \bullet, \underline{\texttt{)}}] \right\}$$

Constructing *Action* and *Goto* Tables (1)

```
1   ALGORITHM: BuildActionGotoTables
2     INPUT:
3       (1) a grammar $G = (V, \Sigma, R, S)$
4       (2) goal production $S \to S'$
5       (3) a canonical collection $\mathcal{CC} = \{cc_0, cc_1, \dots, cc_n\}$
6       (4) a transition function $\delta : \mathcal{CC} \times \Sigma \to \mathcal{CC}$
7     OUTPUT: Action Table & Goto Table
8   PROCEDURE:
9     for $cc_i \in \mathcal{CC}$:
10      for $item \in cc_i$:
11        if $item = [A \to \beta \bullet \mathtt{x}\gamma,\ \mathtt{a}]\backslash pause \wedge \delta(cc_i, \mathtt{x}) = cc_j$ then
12          Action[$i$, x] := shift $j$
13        elseif $item = [A \to \beta\bullet,\ \mathtt{a}]$ then
14          Action[$i$, a] := reduce $A \to \beta$
15        elseif $item = [S \to S'\bullet,\ \mathtt{eof}]$ then
16          Action[$i$, eof] := accept
17        end
18      for $v \in V$:
19        if $\delta(cc_i, v) = cc_j$ then
20          Goto[$i$, $v$] = $j$
21        end
```

- **L12, 13**: Next valid step in discovering A is to match terminal symbol $\mathtt{x}$.
- **L14, 15**: Having recognized β, if current word matches lookahead $\mathtt{a}$, reduce β to A.
- **L16, 17**: Accept if input exhausted and what's recognized reducible to start var. S.
- **L20, 21**: Record consequence of a reduction to non-terminal v from state i

Constructing *Action* and *Goto* Tables (2)

Resulting **Action** and **Goto** tables:

	`Action` **Table**			`Goto` **Table**	
State	`eof`	(	)	***List***	***Pair***
0		s 3		1	2
1	acc	s 3			4
2	r 3	r 3			
3		s 6	s 7		5
4	r 2	r 2			
5			s 8		
6		s 6	s 10		9
7	r 5	r 5			
8	r 4	r 4			
9			s 11		
10			r 5		
11			r 4		

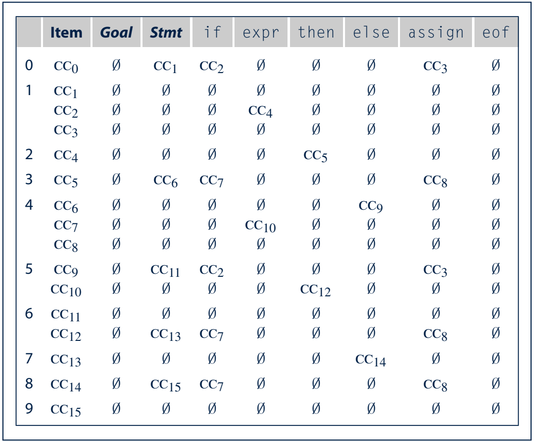
BUP: Discovering Ambiguity (1)

1	Goal	$\rightarrow$	Stmt
2	Stmt	$\rightarrow$	`if expr then` Stmt
3		|	`if expr then` Stmt `else` Stmt
4		|	`assign`

- Calculate $\mathcal{CC} = \{cc_0, cc_1, \dots,\}$
- Calculate the transition function $\delta : \mathcal{CC} \times \Sigma \rightarrow \mathcal{CC}$

BUP: Discovering Ambiguity (2.1)

Resulting transition table:



	Item	***Goal***	***Stmt***	`if`	`expr`	`then`	`else`	`assign`	`eof`
0	CC_0	∅	CC_1	CC_2	∅	∅	∅	CC_3	∅
1	CC_1	∅	∅	∅	∅	∅	∅	∅	∅
	CC_2	∅	∅	∅	CC_4	∅	∅	∅	∅
	CC_3	∅	∅	∅	∅	∅	∅	∅	∅
2	CC_4	∅	∅	∅	∅	CC_5	∅	∅	∅
3	CC_5	∅	CC_6	CC_7	∅	∅	∅	CC_8	∅
4	CC_6	∅	∅	∅	∅	∅	CC_9	∅	∅
	CC_7	∅	∅	∅	CC_{10}	∅	∅	∅	∅
	CC_8	∅	∅	∅	∅	∅	∅	∅	∅
5	CC_9	∅	CC_{11}	CC_2	∅	∅	∅	CC_3	∅
	CC_{10}	∅	∅	∅	∅	CC_{12}	∅	∅	∅
6	CC_{11}	∅	∅	∅	∅	∅	∅	∅	∅
	CC_{12}	∅	CC_{13}	CC_7	∅	∅	∅	CC_8	∅
7	CC_{13}	∅	∅	∅	∅	∅	CC_{14}	∅	∅
8	CC_{14}	∅	CC_{15}	CC_7	∅	∅	∅	CC_8	∅
9	CC_{15}	∅	∅	∅	∅	∅	∅	∅	∅

BUP: Discovering Ambiguity (2.2.1)

Resulting canonical collection $\mathcal{CC}$:

$$cc_0 = \left\{ \begin{array}{ll} [\mathit{Goal} \rightarrow \bullet\ \mathit{Stmt}, \texttt{eof}] & [\mathit{Stmt} \rightarrow \bullet\ \texttt{if expr then } \mathit{Stmt}, \texttt{eof}] \\ [\mathit{Stmt} \rightarrow \bullet\ \texttt{assign}, \texttt{eof}] & [\mathit{Stmt} \rightarrow \bullet\ \texttt{if expr then } \mathit{Stmt}\ \texttt{else}\ \mathit{Stmt}, \texttt{eof}] \end{array} \right\}$$

$$cc_1 = \left\{ [\mathit{Goal} \rightarrow \mathit{Stmt}\ \bullet, \texttt{eof}] \right\}$$

$$cc_2 = \left\{ \begin{array}{l} [\mathit{Stmt} \rightarrow \texttt{if}\ \bullet\ \texttt{expr then } \mathit{Stmt}, \texttt{eof}], \\ [\mathit{Stmt} \rightarrow \texttt{if}\ \bullet\ \texttt{expr then } \mathit{Stmt}\ \texttt{else}\ \mathit{Stmt}, \texttt{eof}] \end{array} \right\}$$

$$cc_3 = \left\{ [\mathit{Stmt} \rightarrow \texttt{assign}\ \bullet, \texttt{eof}] \right\}$$

$$cc_4 = \left\{ \begin{array}{l} [\mathit{Stmt} \rightarrow \texttt{if expr}\ \bullet\ \texttt{then } \mathit{Stmt}, \texttt{eof}], \\ [\mathit{Stmt} \rightarrow \texttt{if expr}\ \bullet\ \texttt{then } \mathit{Stmt}\ \texttt{else}\ \mathit{Stmt}, \texttt{eof}] \end{array} \right\}$$

$$cc_5 = \left\{ \begin{array}{l} [\mathit{Stmt} \rightarrow \texttt{if expr then}\ \bullet\ \mathit{Stmt}, \texttt{eof}], \\ [\mathit{Stmt} \rightarrow \texttt{if expr then}\ \bullet\ \mathit{Stmt}\ \texttt{else}\ \mathit{Stmt}, \texttt{eof}], \\ [\mathit{Stmt} \rightarrow\ \bullet\ \texttt{if expr then } \mathit{Stmt}, \{\texttt{eof}, \texttt{else}\}], \\ [\mathit{Stmt} \rightarrow\ \bullet\ \texttt{assign}, \{\texttt{eof}, \texttt{else}\}], \\ [\mathit{Stmt} \rightarrow\ \bullet\ \texttt{if expr then } \mathit{Stmt}\ \texttt{else}\ \mathit{Stmt}, \{\texttt{eof}, \texttt{else}\}] \end{array} \right\}$$

$$cc_6 = \left\{ \begin{array}{l} [\mathit{Stmt} \rightarrow \texttt{if expr then } \mathit{Stmt}\ \bullet, \texttt{eof}], \\ [\mathit{Stmt} \rightarrow \texttt{if expr then } \mathit{Stmt}\ \bullet\ \texttt{else}\ \mathit{Stmt}, \texttt{eof}] \end{array} \right\}$$

$$cc_7 = \left\{ \begin{array}{l} [\mathit{Stmt} \rightarrow \texttt{if}\ \bullet\ \texttt{expr then } \mathit{Stmt}, \{\texttt{eof}, \texttt{else}\}], \\ [\mathit{Stmt} \rightarrow \texttt{if}\ \bullet\ \texttt{expr then } \mathit{Stmt}\ \texttt{else}\ \mathit{Stmt}, \{\texttt{eof}, \texttt{else}\}] \end{array} \right\}$$

BUP: Discovering Ambiguity (2.2.2)

Resulting canonical collection $\mathcal{CC}$:

$$cc_8 = \{[\mathit{Stmt} \rightarrow \texttt{assign} \bullet, \{\texttt{eof}, \texttt{else}\}]\}$$

$$cc_9 = \left\{ \begin{array}{l} [\mathit{Stmt} \rightarrow \texttt{if expr then } \mathit{Stmt} \texttt{ else } \bullet\ \mathit{Stmt}, \texttt{eof}], \\ [\mathit{Stmt} \rightarrow \bullet \texttt{ if expr then } \mathit{Stmt}, \texttt{eof}], \\ [\mathit{Stmt} \rightarrow \bullet \texttt{ if expr then } \mathit{Stmt} \texttt{ else } \mathit{Stmt}, \texttt{eof}], \\ [\mathit{Stmt} \rightarrow \bullet \texttt{ assign}, \texttt{eof}] \end{array} \right\}$$

$$cc_{10} = \left\{ \begin{array}{l} [\mathit{Stmt} \rightarrow \texttt{if expr} \bullet \texttt{ then } \mathit{Stmt}, \{\texttt{eof}, \texttt{else}\}], \\ [\mathit{Stmt} \rightarrow \texttt{if expr} \bullet \texttt{ then } \mathit{Stmt} \texttt{ else } \mathit{Stmt}, \{\texttt{eof}, \texttt{else}\}] \end{array} \right\}$$

$$cc_{11} = \{[\mathit{Stmt} \rightarrow \texttt{if expr then } \mathit{Stmt} \texttt{ else } \mathit{Stmt} \bullet, \texttt{eof}]\}$$

$$cc_{12} = \left\{ \begin{array}{l} [\mathit{Stmt} \rightarrow \texttt{if expr then} \bullet \mathit{Stmt}, \{\texttt{eof}, \texttt{else}\}], \\ [\mathit{Stmt} \rightarrow \texttt{if expr then} \bullet \mathit{Stmt} \texttt{ else } \mathit{Stmt}, \{\texttt{eof}, \texttt{else}\}], \\ [\mathit{Stmt} \rightarrow \bullet \texttt{if expr then } \mathit{Stmt}, \{\texttt{eof}, \texttt{else}\}], \\ [\mathit{Stmt} \rightarrow \bullet \texttt{if expr then } \mathit{Stmt} \texttt{ else } \mathit{Stmt}, \{\texttt{eof}, \texttt{else}\}], \\ [\mathit{Stmt} \rightarrow \bullet \texttt{assign}, \{\texttt{eof}, \texttt{else}\}] \end{array} \right\}$$

$$cc_{13} = \left\{ \begin{array}{l} [\mathit{Stmt} \rightarrow \texttt{if expr then } \mathit{Stmt} \bullet, \{\texttt{eof}, \texttt{else}\}], \\ [\mathit{Stmt} \rightarrow \texttt{if expr then } \mathit{Stmt} \bullet \texttt{ else } \mathit{Stmt}, \{\texttt{eof}, \texttt{else}\}] \end{array} \right\}$$

$$cc_{14} = \left\{ \begin{array}{l} [\mathit{Stmt} \rightarrow \texttt{if expr then } \mathit{Stmt} \texttt{ else } \bullet\ \mathit{Stmt}, \{\texttt{eof}, \texttt{else}\}], \\ [\mathit{Stmt} \rightarrow \bullet \texttt{ if expr then } \mathit{Stmt}, \{\texttt{eof}, \texttt{else}\}], \\ [\mathit{Stmt} \rightarrow \bullet \texttt{ if expr then } \mathit{Stmt} \texttt{ else } \mathit{Stmt}, \{\texttt{eof}, \texttt{else}\}], \\ [\mathit{Stmt} \rightarrow \bullet \texttt{ assign}, \{\texttt{eof}, \texttt{else}\}] \end{array} \right\}$$

BUP: Discovering Ambiguity (3)

- Consider cc_{13}

$$\text{cc}_{13} = \left\{ \begin{array}{l} [Stmt \rightarrow \texttt{if expr then}\ Stmt\ \bullet\ , \{\texttt{eof}, \texttt{else}\}], \\ [Stmt \rightarrow \texttt{if expr then}\ Stmt\ \bullet\ \texttt{else}\ Stmt, \{\texttt{eof}, \texttt{else}\}] \end{array} \right\}$$

  Q. What does it mean if the current word to consume is `else`?

  A. We can either ***shift*** (then expecting to match another *Stmt*) or reduce to a *Stmt*.

  A single *Action* table entry cannot hold these two alternatives.

  This is known as the *shift-reduce conflict*.

- Consider another scenario, say:

  $[A \rightarrow \gamma\delta\bullet,\ \texttt{a}]$

  $[B \rightarrow \gamma\delta\bullet,\ \texttt{a}]$

  Q. What does it mean if the current word to consume is `a`?

  A. We can either ***reduce*** to *A* or ***reduce*** to *B*.

  A single *Action* table entry cannot hold these two alternatives.

  This is known as the *reduce-reduce conflict*.

Index (1)

Index (2)

Index (3)

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Index (6)

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