Program Correctness

OOSC2 Chapter 11

EECS3311 A: Software Design
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Weak vs. Strong Assertions

- Describe each assertion as ***a set of satisfying value***.

  $x > 3$ has satisfying values $\{ x \mid x > 3 \} = \{ 4, 5, 6, 7, \ldots \}$

  $x > 4$ has satisfying values $\{ x \mid x > 4 \} = \{ 5, 6, 7, \ldots \}$
- An assertion p is ***stronger*** than an assertion q $\boxed{\text{if}}$ p's set of satisfying values is a subset of q's set of satisfying values.
  - Logically speaking, p being stronger than q (or, q being weaker than p) means $p \Rightarrow q$.
  - e.g., $x > 4 \Rightarrow x > 3$
- What's the weakest assertion? [**TRUE**]
- What's the strongest assertion? [**FALSE**]
- In *Design by Contract* :
  - A weaker *invariant* has more acceptable object states

    e.g., $balance > 0$ vs. $balance > 100$ as an invariant for `ACCOUNT`
  - A weaker *precondition* has more acceptable input values
  - A weaker *postcondition* has more acceptable output values

Motivating Examples (1)

Is this feature correct?

```
class FOO
  i: INTEGER
  increment_by_9
    require
      i > 3
    do
      i := i + 9
    ensure
      i > 13
    end
end
```

Q: Is $i > 3$ is too weak or too strong?

A: Too weak

$\because$ assertion $i > 3$ allows value 4 which would fail postcondition.

Motivating Examples (2)

Is this feature correct?

```
class FOO
  i: INTEGER
  increment_by_9
    require
      i > 5
    do
      i := i + 9
    ensure
      i > 13
    end
end
```

Q: Is $i > 5$ too weak or too strong?

A: Maybe too strong

$\because$ assertion $i > 5$ disallows 5 which would not fail postcondition.

Whether 5 should be allowed depends on the requirements.

Software Correctness

- Correctness is a *relative* notion:

  consistency of *implementation* with respect to *specification*.

  ⇒ This assumes there is a specification!

- We introduce a formal and systematic way for formalizing a program **S** and its *specification* (pre-condition ***Q*** and post-condition ***R***) as a *Boolean predicate*: $\boxed{\{Q\}\ \mathbf{S}\ \{R\}}$
  - e.g., $\{i > 3\}$ `i := i + 9` $\{i > 13\}$
  - e.g., $\{i > 5\}$ `i := i + 9` $\{i > 13\}$
  - If $\boxed{\{Q\}\ \mathbf{S}\ \{R\}}$ **can** be proved **TRUE**, then the **S** is correct.

    e.g., $\{i > 5\}$ `i := i + 9` $\{i > 13\}$ can be proved TRUE.
  - If $\boxed{\{Q\}\ \mathbf{S}\ \{R\}}$ **cannot** be proved **TRUE**, then the **S** is incorrect.

    e.g., $\{i > 3\}$ `i := i + 9` $\{i > 13\}$ cannot be proved TRUE.

Hoare Logic

- Consider a program **S** with precondition ***Q*** and postcondition ***R***.
  - $\{Q\}$ S $\{R\}$ is a *correctness predicate* for program **S**
  - $\{Q\}$ S $\{R\}$ is TRUE if program **S** starts executing in a state satisfying the precondition ***Q***, and then:
    (a) The program **S** terminates.
    (b) Given that program **S** terminates, then it terminates in a state satisfying the postcondition ***R***.
- Separation of concerns

  (a) requires a proof of *termination*.

  (b) requires a proof of ***partial** correctness*.

  Proofs of (a) + (b) imply ***total** correctness*.

Hoare Logic and Software Correctness

Consider the *contract view* of a feature f (whose body of implementation is **S**) as a $\boxed{\text{Hoare Triple}}$:

$$\{\boldsymbol{Q}\}\ \mathtt{S}\ \{\boldsymbol{R}\}$$

Q is the *precondition* of f.
$\mathtt{S}$ is the implementation of f.
R is the *postcondition* of f.

- $\{\textbf{true}\}\ \mathtt{S}\ \{R\}$
  All input values are valid [Most-user friendly]
- $\{\textbf{false}\}\ \mathtt{S}\ \{R\}$
  All input values are invalid [Most useless for clients]
- $\{Q\}\ \mathtt{S}\ \{\textbf{true}\}$
  All output values are valid [Most risky for clients; Easiest for suppliers]
- $\{Q\}\ \mathtt{S}\ \{\textbf{false}\}$
  All output values are invalid [Most challenging coding task]
- $\{\textbf{true}\}\ \mathtt{S}\ \{\textbf{true}\}$
  All inputs/outputs are valid (No contracts) [Least informative]

Proof of Hoare Triple using *wp*

$$\{Q\}\ \texttt{S}\ \{R\} \equiv Q \Rightarrow wp(S, R)$$

- $wp(S, R)$ is the *weakest precondition for S to establish R*.
- *S* can be:
  - Assignments (`x := y`)
  - Alternations (**`if`** ... **`then`** ... **`else`** ... **`end`**)
  - Sequential compositions (S_1 ; S_2)
  - Loops (**`from`** ... **`until`** ... **`loop`** ... **`end`**)
- We will learn how to calculate the *wp* for the above programming constructs.

Hoare Logic A Simple Example

Given $\{??\}n := n + 9\{n > 13\}$:

- $\boxed{n > 4}$ is the *weakest precondition (wp)* for the given implementation (n := n + 9) to start and establish the postcondition ($n > 13$).
- Any precondition that is ***equal to or stronger than*** the *wp* ($n > 4$) will result in a correct program.

  e.g., $\{n > 5\}n := n + 9\{n > 13\}$ can be proved **TRUE**.
- Any precondition that is ***weaker than*** the *wp* ($n > 4$) will result in an incorrect program.

  e.g., $\{n > 3\}n := n + 9\{n > 13\}$ cannot be proved **TRUE**.

  Counterexample: $n = 4$ satisfies precondition $n > 3$ but the output $n = 13$ fails postcondition $n > 13$.

Denoting New and Old Values

In the *postcondition*, for a program variable x:

- We write $\boxed{x_0}$ to denote its ***pre-state (old)*** value.
- We write $\boxed{x}$ to denote its ***post-state (new)*** value.

  Implicitly, in the *precondition*, all program variables have their ***pre-state*** values.

e.g., $\{b_0 > a\}$ `b := b - a` $\{b = b_0 - a\}$

- Notice that:
  - We may choose to write "b" rather than "b_0" in preconditions
    $\because$ All variables are pre-state values in preconditions
  - We don't write "b_0" in program
    $\because$ there might be *multiple intermediate values* of a variable due to sequential composition

wp Rule: Assignments (1)

$$wp(\texttt{x := e}, \boldsymbol{R}) = \boldsymbol{R}[x := e]$$

$\boldsymbol{R}[x := e]$ means to substitute all *free occurrences* of variable *x* in postcondition ***R*** by expression *e*.

wp Rule: Assignments (2)

Recall:

$$\{Q\}\ \texttt{S}\ \{R\} \equiv Q \Rightarrow wp(S, R)$$

How do we prove $\{Q\}\ \texttt{x := e}\ \{R\}$?

$$\{Q\}\ \texttt{x := e}\ \{R\} \iff Q \Rightarrow \underbrace{R[x := e]}_{wp(\texttt{x := e}, R)}$$

wp Rule: Assignments (3) Exercise

What is the weakest precondition for a program `x := x + 1` to establish the postcondition $x > x_0$?

$$\{??\}\ \texttt{x := x + 1}\ \{x > x_0\}$$

For the above Hoare triple to be ***TRUE***, it must be that $?? \Rightarrow wp(\texttt{x := x + 1}, x > x_0)$.

$$
\begin{array}{ll}
 & wp(\texttt{x := x + 1}, x > x_0) \\
= & \{\textit{Rule of } \textbf{\textit{wp}}\textit{: Assignments}\} \\
 & x > x_0[x := x_0 + 1] \\
= & \{\textit{Replacing } x \textit{ by } x_0 + 1\} \\
 & x_0 + 1 > x_0 \\
= & \{1 > 0 \textit{ always true}\} \\
 & \textit{True}
\end{array}
$$

Any precondition is OK. ***False*** is valid but not useful.

wp Rule: Assignments (4) Exercise

What is the weakest precondition for a program `x := x + 1` to establish the postcondition $x > x_0$?

$$\{??\}\ \texttt{x := x + 1}\ \{x = 23\}$$

For the above Hoare triple to be ***TRUE***, it must be that $?? \Rightarrow wp(\texttt{x := x + 1}, x = 23)$.

$$\begin{array}{ll}
 & wp(\texttt{x := x + 1}, x = 23) \\
= & \{\textit{Rule of } \textbf{\textit{wp:}} \textit{ Assignments}\} \\
 & x = 23[x := x_0 + 1] \\
= & \{\textit{Replacing } x \textit{ by } x_0 + 1\} \\
 & x_0 + 1 = 23 \\
= & \{\textit{arithmetic}\} \\
 & x_0 = 22
\end{array}$$

Any precondition weaker than $x = 22$ is not OK.

wp Rule: Alternations (1)

$$wp(\texttt{if } B \texttt{ then } S_1 \texttt{ else } S_2 \texttt{ end}, \boldsymbol{R}) = \left(\begin{array}{l} B \Rightarrow wp(S_1, \boldsymbol{R}) \\ \wedge \\ \neg B \Rightarrow wp(S_2, \boldsymbol{R}) \end{array} \right)$$

The *wp* of an alternation is such that ***all branches*** are able to establish the postcondition ***R***.

wp Rule: Alternations (2)

Recall: $\{Q\}\ S\ \{R\} \equiv Q \Rightarrow wp(S, R)$

How do we prove that $\{Q\}$ **if** B **then** S_1 **else** S_2 **end** $\{R\}$?

```
{Q}
if B then
  {Q ∧ B} S1 {R}
else
  {Q ∧ ¬B} S2 {R}
end
{R}
```

$$\{Q\}\ \textbf{if}\ B\ \textbf{then}\ S_1\ \textbf{else}\ S_2\ \textbf{end}\ \{R\}$$

$$\iff \left(\begin{array}{l} \{\,Q \land B\,\}\ S_1\ \{\,R\,\} \\ \land \\ \{\,Q \land \neg B\,\}\ S_2\ \{\,R\,\} \end{array}\right) \iff \left(\begin{array}{l} (Q \land B) \Rightarrow wp(S_1, R) \\ \land \\ (Q \land \neg B) \Rightarrow wp(S_2, R) \end{array}\right)$$

wp Rule: Alternations (3) Exercise

Is this program correct?

```
{x > 0 ∧ y > 0}
if x > y then
 bigger := x ; smaller := y
else
 bigger := y ; smaller := x
end
{bigger ≥ smaller}
```

$$\left(\begin{array}{l}\{(x > 0 \land y > 0) \land (x > y)\} \\ \qquad \texttt{bigger := x ; smaller := y} \\ \{\mathit{bigger} \geq \mathit{smaller}\}\end{array}\right)$$
$$\land$$
$$\left(\begin{array}{l}\{(x > 0 \land y > 0) \land \neg(x > y)\} \\ \qquad \texttt{bigger := y ; smaller := x} \\ \{\mathit{bigger} \geq \mathit{smaller}\}\end{array}\right)$$

wp Rule: Sequential Composition (1)

$$wp(S_1 \; ; \; S_2, \boldsymbol{R}) = wp(S_1, wp(S_2, \boldsymbol{R}))$$

The *wp* of a sequential composition is such that the $\boxed{\text{first phase}}$ establishes the *wp* for the $\boxed{\text{second phase}}$ to establish the postcondition ***R***.

wp Rule: Sequential Composition (2)

Recall:

$$\{\boldsymbol{Q}\}\ \texttt{S}\ \{\boldsymbol{R}\}\ \equiv\ \boldsymbol{Q} \Rightarrow wp(S, \boldsymbol{R})$$

How do we prove $\{\boldsymbol{Q}\}\ S_1\ \ ;\ \ S_2\ \{\boldsymbol{R}\}$?

$$\{\boldsymbol{Q}\}\ S_1\ \ ;\ \ S_2\ \{\boldsymbol{R}\}\ \iff\ \boldsymbol{Q} \Rightarrow \underbrace{wp(S_1,\ wp(S_2,\ \boldsymbol{R}))}_{wp(S_1\ \ ;\ \ S_2, \boldsymbol{R})}$$

wp Rule: Sequential Composition (3) Exercise

Is $\{\ \textit{True}\ \}$ `tmp := x; x := y; y := tmp` $\{\ x > y\ \}$ correct?

If and only if $\textit{True} \Rightarrow wp(\texttt{tmp := x ; x := y ; y := tmp}, x > y)$

$$
\begin{array}{ll}
 & wp(\texttt{tmp := x ; } \boxed{\texttt{x := y ; y := tmp}}, x > y) \\
= & \{wp\ \textit{rule for seq. comp.}\} \\
 & wp(\texttt{tmp := x}, wp(\texttt{x := y ; } \boxed{\texttt{y := tmp}}, x > y)) \\
= & \{wp\ \textit{rule for seq. comp.}\} \\
 & wp(\texttt{tmp := x}, wp(\texttt{x := y}, wp(\texttt{y := tmp}, x > \boxed{\mathsf{y}}))) \\
= & \{wp\ \textit{rule for assignment}\} \\
 & wp(\texttt{tmp := x}, wp(\texttt{x := y}, \boxed{\mathsf{x}} > tmp)) \\
= & \{wp\ \textit{rule for assignment}\} \\
 & wp(\texttt{tmp := x}, y > \boxed{\mathsf{tmp}}) \\
= & \{wp\ \textit{rule for assignment}\} \\
 & y > x
\end{array}
$$

$\because$ ***True*** $\Rightarrow y > x$ does not hold in general.

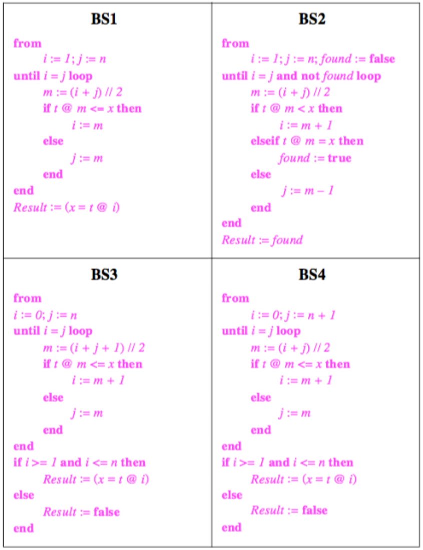
$\therefore$ The above program is not correct.

Loops

- A loop is a way to compute a certain result by *successive approximations*.

  e.g. computing the maximum value of an array of integers
- Loops are needed and powerful
- But loops *very hard* to get right:
  - Infinite loops [termination]
  - “off-by-one” error [partial correctness]
  - Improper handling of borderline cases [partial correctness]
  - Not establishing the desired condition [partial correctness]

Loops: Binary Search



BS1

```
from
    i := 1; j := n
until i = j loop
    m := (i + j) // 2
    if t @ m <= x then
        i := m
    else
        j := m
    end
end
Result := (x = t @ i)
```

BS2

```
from
    i := 1; j := n; found := false
until i = j and not found loop
    m := (i + j) // 2
    if t @ m < x then
        i := m + 1
    elseif t @ m = x then
        found := true
    else
        j := m – 1
    end
end
Result := found
```

BS3

```
from
i := 0; j := n
until i = j loop
    m := (i + j + 1) // 2
    if t @ m <= x then
        i := m + 1
    else
        j := m
    end
end
if i >= 1 and i <= n then
    Result := (x = t @ i)
else
    Result := false
end
```

BS4

```
from
    i := 0; j := n + 1
until i = j loop
    m := (i + j) // 2
    if t @ m <= x then
        i := m + 1
    else
        j := m
    end
end
if i >= 1 and i <= n then
    Result := (x = t @ i)
else
    Result := false
end
```

4 implementations for binary search: published, but *wrong*!

See page 381 in *Object Oriented Software Construction*

Correctness of Loops

How do we prove that the following loops are correct?

```
{Q}
from
  $S_{init}$
until
  $B$
loop
  $S_{body}$
end
{R}
```

```
{Q}
$S_{init}$
while (¬ $B$) {
  $S_{body}$
}
{R}
```

- In case of C/Java, $\boxed{\neg B}$ denotes the ***stay condition***.
- In case of Eiffel, $\boxed{B}$ denotes the ***exit condition***.

  There is native, syntactic support for checking/proving the *total correctness* of loops.

Contracts for Loops: Syntax

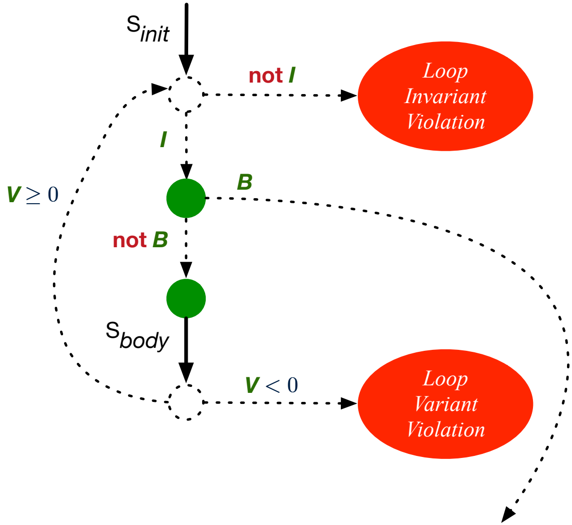
```
from
  $S_{init}$
invariant
  invariant_tag: $I$ -- Boolean expression for partial correctness
until
  $B$
loop
  $S_{body}$
variant
  variant_tag: $V$ -- Integer expression for termination
end
```

Contracts for Loops

- Use of ***loop invariants (LI)*** and ***loop variants (LV)***.
  - ***Invariants***: $\boxed{\text{Boolean}}$ expressions for ***partial*** *correctness*.
    - Typically a special case of the postcondition.
      e.g., Given postcondition " *Result is maximum of the array* ":
      LI can be " *Result is maximum of the* ***part of array scanned so far*** ".
    - Established before the very first iteration.
    - Maintained TRUE after each iteration.
  - ***Variants***: $\boxed{\text{Integer}}$ expressions for ***termination***
    - Denotes the *number of iterations remaining*
    - *Decreased* at the end of each subsequent iteration
    - Maintained *non-negative* at the end of each iteration.
    - As soon as value of ***LV*** reaches *zero*, meaning that no more iterations remaining, the loop must exit.
- Remember:

  total *correctness* = ***partial*** *correctness* + ***termination***

Contracts for Loops: Runtime Checks (1)



Contracts for Loops: Runtime Checks (2)

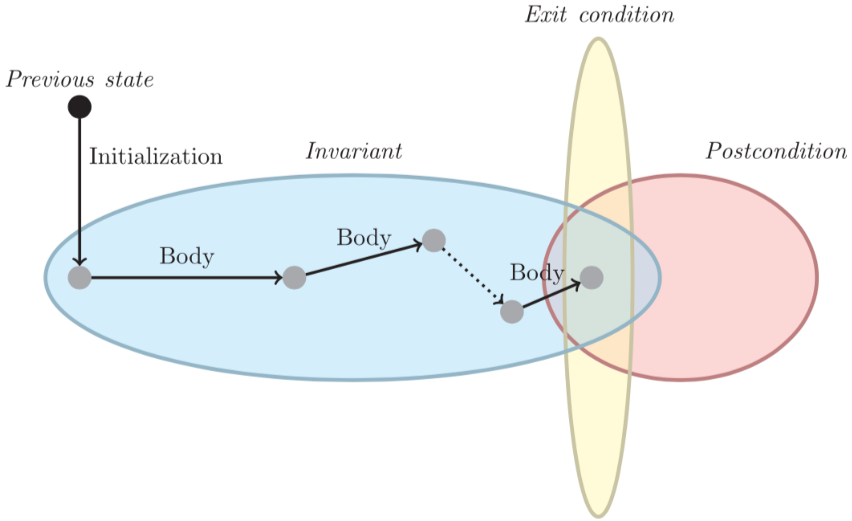
```
1  test
2    local
3      i: INTEGER
4    do
5      from
6        i := 1
7      invariant
8        1 <= i and i <= 6
9      until
10       i > 5
11     loop
12       io.put_string ("iteration " + i.out + "%N")
13       i := i + 1
14     variant
15       6 - i
16     end
17 end
```

L8: Change to `1 <= i and i <= 5` for a ***Loop Invariant Violation***.

L10: Change to `i > 0` to bypass the body of loop.

L15: Change to `5 - i` for a ***Loop Variant Violation***.

Contracts for Loops: Visualization



Digram Source: page 5 in *Loop Invariants: Analysis, Classification, and Examples*

Contracts for Loops: Example 1.1

```
find_max (a: ARRAY [INTEGER]): INTEGER
 local i: INTEGER
 do
  from
   i := a.lower ; Result := a[i]
  invariant
   loop_invariant: -- $\forall j \mid a.lower \leq j \leq i \bullet Result \geq a[j]$
    across a.lower |..| i as j all Result >= a [j.item] end
  until
   i > a.upper
  loop
   if a [i] > Result then Result := a [i] end
   i := i + 1
  variant
   loop_variant: a.upper - i + 1
  end
 ensure
  correct_result: -- $\forall j \mid a.lower \leq j \leq a.upper \bullet Result \geq a[j]$
   across a.lower |..| a.upper as j all Result >= a [j.item]
 end
end
```

Contracts for Loops: Example 1.2

Consider the feature call $\boxed{\text{find\_max}(\ \langle\langle 20, 10, 40, 30 \rangle\rangle\ )}$, given:

- ***Loop Invariant***: $\forall j \mid a.lower \leq j \leq i \bullet Result \geq a[j]$
- ***Loop Variant***: $a.upper - i + 1$

AFTER ITERATION	i	Result	***LI***	EXIT ($i > a.upper$)?	***LV***
Initialization	1	20	✓	×	–
1st	2	20	✓	×	3
2nd	***3***	20	×	–	–

Loop invariant violation at the end of the 2nd iteration:

$$\forall j \mid a.lower \leq j \leq \boxed{3} \bullet \boxed{20} \geq a[j]$$

evaluates to ***false*** $\because 20 \not\geq a[3] = 40$

Contracts for Loops: Example 2.1

```
find_max (a: ARRAY [INTEGER]): INTEGER
 local i: INTEGER
 do
   from
     i := a.lower ; Result := a[i]
   invariant
     loop_invariant: -- ∀j | a.lower ≤ j < i • Result ≥ a[j]
       across a.lower |..| (i - 1) as j all Result >= a [j.item] end
   until
     i > a.upper
   loop
     if a [i] > Result then Result := a [i] end
     i := i + 1
   variant
     loop_variant: a.upper - i
   end
 ensure
   correct_result: -- ∀j | a.lower ≤ j ≤ a.upper • Result ≥ a[j]
     across a.lower |..| a.upper as j all Result >= a [j.item]
 end
end
```

Contracts for Loops: Example 2.2

Consider the feature call $\boxed{\text{find\_max}(\langle\langle 20, 10, 40, 30\rangle\rangle)}$, given:

- ***Loop Invariant***: $\forall j \mid a.lower \leq j < i \bullet Result \geq a[j]$
- ***Loop Variant***: $a.upper - i$

AFTER ITERATION	`i`	`Result`	***LI***	EXIT ($i > a.upper$)?	***LV***
Initialization	1	20	✓	×	–
1st	2	20	✓	×	2
2nd	3	20	✓	×	1
3rd	4	40	✓	×	0
4th	5	40	✓	✓	***-1***

Loop variant violation at the end of the 2nd iteration
$\because a.upper - i = 4 - 5$ evaluates to ***non-zero***.

Contracts for Loops: Example 3.1

```
find_max (a: ARRAY [INTEGER]): INTEGER
 local i: INTEGER
 do
   from
     i := a.lower ; Result := a[i]
   invariant
     loop_invariant: -- $\forall j \mid a.lower \leq j < i \bullet Result \geq a[j]$
       across a.lower |..| (i - 1) as j all Result >= a [j.item] end
   until
     i > a.upper
   loop
     if a [i] > Result then Result := a [i] end
     i := i + 1
   variant
     loop_variant: a.upper - i + 1
   end
 ensure
   correct_result: -- $\forall j \mid a.lower \leq j \leq a.upper \bullet Result \geq a[j]$
     across a.lower |..| a.upper as j all Result >= a [j.item]
 end
end
```

Contracts for Loops: Example 3.2

Consider the feature call $\boxed{\text{find\_max}(\ \langle\langle 20, 10, 40, 30 \rangle\rangle\ )}$, given:

- ***Loop Invariant***: $\forall j \mid a.lower \leq j < i \bullet Result \geq a[j]$
- ***Loop Variant***: $a.upper - i + 1$
- *Postcondition*: $\forall j \mid a.lower \leq j \leq a.upper \bullet Result \geq a[j]$

After Iteration	i	Result	*LI*	Exit ($i > a.upper$)?	*LV*
Initialization	1	20	✓	×	–
1st	2	20	✓	×	3
2nd	3	20	✓	×	2
3rd	4	40	✓	×	1
4th	5	40	✓	✓	0

Contracts for Loops: Exercise

```
class DICTIONARY[V, K]
feature {NONE} -- Implementations
 values: ARRAY[K]
 keys: ARRAY[K]
feature -- Abstraction Function
 model: FUN[K, V]
feature -- Queries
 get_keys(v: V): ITERABLE[K]
   local i: INTEGER; ks: LINKED_LIST[K]
   do
     from i := keys.lower ; create ks.make_empty
     invariant ??
     until i > keys.upper
     do if values[i] ~ v then ks.extend(keys[i]) end
     end
     Result := ks.new_cursor
   ensure
     result_valid: ∀k | k ∈ Result • model.item(k) ~ v
     no_missing_keys: ∀k | k ∈ model.domain • model.item(k) ~ v ⇒ k ∈ Result
   end
```

Proving Correctness of Loops (1)

```
{Q}     from
           $S_{init}$
        invariant
           $I$
        until
           $B$
        loop
           $S_{body}$
        variant
           $V$
        end     {R}
```

- A loop is ***partially** correct* if:
  - Given precondition ***Q***, the initialization step S_{init} establishes ***LI*** I.
  - At the end of S_{body}, if not yet to exit, ***LI*** I is maintained.
  - If ready to exit and ***LI*** I maintained, postcondition ***R*** is established.
- A loop *terminates* if:
  - Given ***LI*** I, and not yet to exit, S_{body} maintains ***LV*** V as non-negative.
  - Given ***LI*** I, and not yet to exit, S_{body} decrements ***LV*** V.

Proving Correctness of Loops (2)

$\{\boldsymbol{Q}\}$ **from** S_{init} **invariant** I **until** B **loop** S_{body} **variant** V **end** $\{\boldsymbol{R}\}$

- A loop is ***partially** correct* if:
  - Given precondition $\boldsymbol{Q}$, the initialization step S_{init} establishes ***LI*** I.

    $$\boxed{\{\boldsymbol{Q}\}\ S_{init}\ \{I\}}$$

  - At the end of S_{body}, if not yet to exit, ***LI*** I is maintained.

    $$\boxed{\{I \land \neg B\}\ S_{body}\ \{I\}}$$

  - If ready to exit and ***LI*** I maintained, postcondition $\boldsymbol{R}$ is established.

    $$\boxed{I \land B \Rightarrow \boldsymbol{R}}$$

- A loop *terminates* if:
  - Given ***LI*** I, and not yet to exit, S_{body} maintains ***LV*** V as non-negative.

    $$\boxed{\{I \land \neg B\}\ S_{body}\ \{V \geq 0\}}$$

  - Given ***LI*** I, and not yet to exit, S_{body} decrements ***LV*** V.

    $$\boxed{\{I \land \neg B\}\ S_{body}\ \{V < V_0\}}$$

Proving Correctness of Loops: Exercise (1.1)

Prove that the following program is correct:

```
find_max (a: ARRAY [INTEGER]): INTEGER
 local i: INTEGER
 do
   from
     i := a.lower ; Result := a[i]
   invariant
     loop_invariant: $\forall j \mid a.lower \le j < i \bullet Result \ge a[j]$
   until
     i > a.upper
   loop
     if a [i] > Result then Result := a [i] end
     i := i + 1
   variant
     loop_variant: a.upper - i + 1
   end
 ensure
   correct_result: $\forall j \mid a.lower \le j \le a.upper \bullet Result \ge a[j]$
 end
end
```

Proving Correctness of Loops: Exercise (1.2)

Prove that each of the following *Hoare Triples* is TRUE.

1. Establishment of Loop Invariant:

```
{ $True$ }
  i := a.lower
  Result := a[i]
{ $\forall j \mid a.lower \leq j < i \bullet Result \geq a[j]$ }
```

2. Maintenance of Loop Invariant:

```
{ $(\forall j \mid a.lower \leq j < i \bullet Result \geq a[j]) \land \neg(i > a.upper)$ }
  if a [i] > Result then Result := a [i] end
  i := i + 1
{ $(\forall j \mid a.lower \leq j < i \bullet Result \geq a[j])$ }
```

3. Establishment of Postcondition upon Termination:

$$(\forall j \mid a.lower \leq j < i \bullet Result \geq a[j]) \land i > a.upper$$
$$\Rightarrow \forall j \mid a.lower \leq j \leq a.upper \bullet Result \geq a[j]$$

Proving Correctness of Loops: Exercise (1.3)

Prove that each of the following *Hoare Triples* is TRUE.

4. Loop Variant Stays Non-Negative Before Exit:

```
{ $(\forall j \mid a.lower \leq j < i \bullet Result \geq a[j]) \wedge \neg(i > a.upper)$ }
  if a [i] > Result then Result := a [i] end
  i := i + 1
{ $a.upper - i + 1 \geq 0$ }
```

5. Loop Variant Keeps Decrementing before Exit:

```
{ $(\forall j \mid a.lower \leq j < i \bullet Result \geq a[j]) \wedge \neg(i > a.upper)$ }
  if a [i] > Result then Result := a [i] end
  i := i + 1
{ $a.upper - i + 1 < (a.upper - i + 1)_0$ }
```

where $(a.upper - i + 1)_0 \equiv a.upper_0 - i_0 + 1$

Proof Tips (1)

$$\{Q\}\ \mathtt{S}\ \{R\} \Rightarrow \{Q \land P\}\ \mathtt{S}\ \{R\}$$

In order to prove $\{Q \land P\}\ \mathtt{S}\ \{R\}$, it is sufficient to prove a version with a *weaker* precondition: $\{Q\}\ \mathtt{S}\ \{R\}$.

Proof:

- Assume: $\{Q\}\ \mathtt{S}\ \{R\}$

  It's equivalent to assuming: $\boxed{Q} \Rightarrow wp(\mathtt{S}, R)$ **(A1)**
- To prove: $\{Q \land P\}\ \mathtt{S}\ \{R\}$
  - It's equivalent to proving: $Q \land P \Rightarrow wp(\mathtt{S}, R)$
  - Assume: $Q \land P$, which implies $\boxed{Q}$
  - According to **(A1)**, we have $wp(\mathtt{S}, R)$. ■

Proof Tips (2)

When calculating $wp(\texttt{S}, R)$, if either program $\texttt{S}$ or postcondition R involves array indexing, then R should be augmented accordingly.

e.g., Before calculating $wp(\texttt{S}, a[i] > 0)$, augment it as

$$wp(\texttt{S}, a.lower \leq i \leq a.upper \wedge a[i] > 0)$$

e.g., Before calculating $wp(\texttt{x := a[i]}, R)$, augment it as

$$wp(\texttt{x := a[i]}, a.lower \leq i \leq a.upper \wedge R)$$

Index (1)

Index (2)

Index (3)