Recursion

EECS2030 B: Advanced
Object Oriented Programming
Fall 2019

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Beyond this lecture . . .

- Fantastic resources for sharpening your recursive skills for the exam:

  `http://codingbat.com/java/Recursion-1`

  `http://codingbat.com/java/Recursion-2`

- The *best* approach to learning about recursion is via a functional programming language:

  Haskell Tutorial: `https://www.haskell.org/tutorial/`

Recursion: Principle

- *Recursion* is useful in expressing solutions to problems that can be *recursively* defined:
  - ***Base*** **Cases:** Small problem instances immediately solvable.
  - ***Recursive*** **Cases:**
    - Large problem instances *not immediately solvable*.
    - Solve by reusing *solution(s) to strictly smaller problem instances*.
- Similar idea learnt in high school: [*mathematical induction*]
- Recursion can be easily expressed programmatically in Java:

```
m (i) {
  if(i == ...) { /* base case: do something directly */ }
  else {
    m (j);/* recursive call with strictly smaller value */
  }
}
```

  - In the body of a method *m*, there might be *a call or calls to m itself*.
  - Each such self-call is said to be a *recursive call*.
  - Inside the execution of $m(i)$, a recursive call $m(j)$ must be that $j < i$.

Tracing Method Calls via a Stack

- When a method is called, it is ***activated*** (and becomes *active*) and *pushed* onto the stack.
- When the body of a method makes a (helper) method call, that (helper) method is ***activated*** (and becomes *active*) and *pushed* onto the stack.

  ⇒ The stack contains activation records of all *active* methods.
  - *Top* of stack denotes the current point of execution.
  - Remaining parts of stack are (temporarily) ***suspended***.
- When entire body of a method is executed, stack is *popped*.

  ⇒ The current point of execution is returned to the new *top* of stack (which was ***suspended*** and just became ***active***).
- Execution terminates when the stack becomes *empty*.

Recursion: Factorial (1)

- Recall the formal definition of calculating the *n* factorial:

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 3 \cdot 2 \cdot 1 & \text{if } n \geq 1 \end{cases}$$

- How do you define the same problem *recursively*?

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot (n-1)! & \text{if } n \geq 1 \end{cases}$$

- To solve *n*!, we combine *n* and the solution to (*n - 1*)!.

```
int factorial (int n) {
  int result;
  if(n == 0) { /* base case */ result = 1; }
  else { /* recursive case */
    result = n * factorial (n - 1);
  }
  return result;
}
```

Common Errors of Recursive Methods

- Missing Base Case(s).

```
int factorial (int n) {
  return n * factorial (n - 1);
}
```

Base case(s) are meant as points of stopping growing the runtime stack.

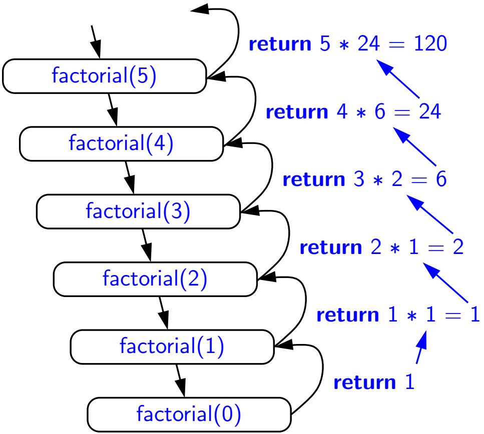
- Recursive Calls on Non-Smaller Problem Instances.

```
int factorial (int n) {
  if(n == 0) { /* base case */ return 1; }
  else { /* recursive case */ return n * factorial (n); }
}
```

Recursive calls on ***strictly smaller*** problem instances are meant for moving gradually towards the base case(s).

- In both cases, a `StackOverflowException` will be thrown.

Recursion: Factorial (2)



Recursion: Factorial (3)

- When running *factorial(5)*, a *recursive call factorial(4)* is made. Call to *factorial(5)* suspended until *factorial(4)* returns a value.
- When running *factorial(4)*, a *recursive call factorial(3)* is made. Call to *factorial(4)* suspended until *factorial(3)* returns a value.

  . . .
- *factorial(0)* returns 1 back to *suspended call factorial(1)*.
- *factorial(1)* receives 1 from *factorial(0)*, multiplies 1 to it, and returns 1 back to the *suspended call factorial(2)*.
- *factorial(2)* receives 1 from *factorial(1)*, multiplies 2 to it, and returns 2 back to the *suspended call factorial(3)*.
- *factorial(3)* receives 2 from *factorial(1)*, multiplies 3 to it, and returns 6 back to the *suspended call factorial(4)*.
- *factorial(4)* receives 6 from *factorial(3)*, multiplies 4 to it, and returns 24 back to the *suspended call factorial(5)*.
- *factorial(5)* receives 24 from *factorial(4)*, multiplies 5 to it, and returns 120 as the result.

Recursion: Factorial (4)

- When the execution of a method (e.g., *factorial(5)*) leads to a nested method call (e.g., *factorial(4)*):
  - The execution of the current method (i.e., *factorial(5)*) is *suspended*, and a structure known as an *activation record* or *activation frame* is created to store information about the progress of that method (e.g., values of parameters and local variables).
  - The nested methods (e.g., *factorial(4)*) may call other nested methods (*factorial(3)*).
  - When all nested methods complete, the activation frame of the *latest suspended* method is re-activated, then continue its execution.
- What kind of data structure does this activation-suspension process correspond to? [LIFO Stack]

Recursion: Fibonacci (1)

Recall the formal definition of calculating the n_{th} number in a Fibonacci series (denoted as F_n), which is already itself recursive:

$$F_n = \begin{cases} 1 & \text{if } n = 1 \\ 1 & \text{if } n = 2 \\ F_{n-1} + F_{n-2} & \text{if } n > 2 \end{cases}$$

```
int fib (int n) {
  int result;
  if(n == 1) { /* base case */ result = 1; }
  else if(n == 2) { /* base case */ result = 1; }
  else { /* recursive case */
    result = fib (n - 1) + fib (n - 2);
  }
  return result;
}
```

Recursion: Fibonacci (2)

fib(5)

= {fib(5) = fib(4) + fib(3); push(fib(5)); *suspended*: ⟨fib(5)⟩; *active*: fib(4)}

fib(4) + fib(3)

= {fib(4) = fib(3) + fib(2); *suspended*: ⟨fib(4), fib(5)⟩; *active*: fib(3)}

(fib(3) + fib(2)) + fib(3)

= {fib(3) = fib(2) + fib(1); *suspended*: ⟨fib(3), fib(4), fib(5)⟩; *active*: fib(2)}

((fib(2) + fib(1)) + fib(2)) + fib(3)

= {fib(2) returns 1; *suspended*: ⟨fib(3), fib(4), fib(5)⟩; *active*: fib(1)}

((1 + fib(1)) + fib(2)) + fib(3)

= {fib(1) returns 1; *suspended*: ⟨fib(3), fib(4), fib(5)⟩; *active*: fib(3)}

((1 + 1) + fib(2)) + fib(3)

= {fib(3) returns 1 + 1; pop(); *suspended*: ⟨fib(4), fib(5)⟩; *active*: fib(2)}

(2 + fib(2)) + fib(3)

= {fib(2) returns 1; *suspended*: ⟨fib(4), fib(5)⟩; *active*: fib(4)}

(2 + 1) + fib(3)

= {fib(4) returns 2 + 1; pop(); *suspended*: ⟨fib(5)⟩; *active*: fib(3)}

3 + fib(3)

= {fib(3) = fib(2) + fib(1); *suspended*: ⟨fib(3),fib(5)⟩; *active*: fib(2)}

3 + (fib(2) + fib(1))

= {fib(2) returns 1; *suspended*: ⟨fib(3), fib(5)⟩; *active*: fib(1)}

3 + (1 + fib(1))

= {fib(1) returns 1; *suspended*: ⟨fib(3), fib(5)⟩; *active*: fib(3)}

3 + (1 + 1)

= {fib(3) returns 1 + 1; pop() ; *suspended*: ⟨fib(5)⟩; *active*: fib(5)}

3 + 2

= {fib(5) returns 3 + 2; *suspended*: ⟨⟩}

5

Java Library: String

```
public class StringTester {
  public static void main(String[] args) {
    String s = "abcd";
    System.out.println(s.isEmpty()); /* false */
    /* Characters in index range [0, 0) */
    String t0 = s.substring(0, 0);
    System.out.println(t0); /* "" */
    /* Characters in index range [0, 4) */
    String t1 = s.substring(0, 4);
    System.out.println(t1); /* "abcd" */
    /* Characters in index range [1, 3) */
    String t2 = s.substring(1, 3);
    System.out.println(t2); /* "bc" */
    String t3 = s.substring(0, 2) + s.substring(2, 4);
    System.out.println(s.equals(t3)); /* true */
    for(int i = 0; i < s.length(); i ++) {
      System.out.print(s.charAt(i));
    }
    System.out.println();
  }
}
```

Recursion: Palindrome (1)

Problem: A palindrome is a word that reads the same forwards and backwards. Write a method that takes a string and determines whether or not it is a palindrome.

```
System.out.println(isPalindrome(""));  true
System.out.println(isPalindrome("a"));  true
System.out.println(isPalindrome("madam"));  true
System.out.println(isPalindrome("racecar"));  true
System.out.println(isPalindrome("man"));  false
```

Base Case 1: Empty string $\longrightarrow$ Return *true* immediately.

Base Case 2: String of length 1 $\longrightarrow$ Return *true* immediately.

Recursive Case: String of length $\geq 2 \longrightarrow$

- 1st and last characters match, **and**
- *the rest (i.e., middle) of the string* *is a palindrome*.

Recursion: Palindrome (2)

```
boolean isPalindrome (String word) {
  if(word.length() == 0 || word.length() == 1) {
    /* base case */
    return true;
  }
  else {
    /* recursive case */
    char firstChar = word.charAt(0);
    char lastChar = word.charAt(word.length() - 1);
    String middle = word.substring(1, word.length() - 1);
    return
        firstChar == lastChar
        /* See the API of java.lang.String.substring. */
        && isPalindrome (middle);
  }
}
```

Recursion: Reverse of String (1)

Problem: The reverse of a string is written backwards. Write a method that takes a string and returns its reverse.

```
System.out.println(reverseOf(""));  /* "" */
System.out.println(reverseOf("a"));  "a"
System.out.println(reverseOf("ab"));  "ba"
System.out.println(reverseOf("abc"));  "cba"
System.out.println(reverseof("abcd"));  "dcba"
```

Base Case 1: Empty string ⟶ Return *empty string*.

Base Case 2: String of length 1 ⟶ Return *that string*.

Recursive Case: String of length ≥ 2 ⟶

1) Head of string (i.e., first character)

2) Reverse of the tail of string (i.e., all but the first character)

Return the concatenation of **2)** and **1)**.

Recursion: Reverse of a String (2)

```
String reverseOf (String s) {
  if(s.isEmpty()) { /* base case 1 */
    return "";
  }
  else if(s.length() == 1) { /* base case 2 */
    return s;
  }
  else { /* recursive case */
    String tail = s.substring(1, s.length());
    String reverseOfTail = reverseOf (tail);
    char head = s.charAt(0);
    return reverseOfTail + head;
  }
}
```

Recursion: Number of Occurrences (1)

Problem: Write a method that takes a string `s` and a character `c`, then count the number of occurrences of `c` in `s`.

```
System.out.println(occurrencesOf("", 'a'));  /* 0 */
System.out.println(occurrencesOf("a", 'a'));  /* 1 */
System.out.println(occurrencesOf("b", 'a'));  /* 0 */
System.out.println(occurrencesOf("baaba", 'a'));  /* 3 */
System.out.println(occurrencesOf("baaba", 'b'));  /* 2 */
System.out.println(occurrencesOf("baaba", 'c'));  /* 0 */
```

Base Case: Empty string ⟶ Return *0*.

Recursive Case: String of length ≥ 1 ⟶

1) Head of `s` (i.e., first character)
2) Number of occurrences of `c` in the tail of `s` (i.e., all but the first character)

If head is equal to `c`, return 1 + **2)**.
If head is not equal to `c`, return 0 + **2)**.

Recursion: Number of Occurrences (2)

```
int occurrencesOf (String s, char c) {
  if(s.isEmpty()) {
    /* Base Case */
    return 0;
  }
  else {
    /* Recursive Case */
    char head = s.charAt(0);
    String tail = s.substring(1, s.length());
    if(head == c) {
      return 1 + occurrencesOf (tail, c);
    }
    else {
      return 0 + occurrencesOf (tail, c);
    }
  }
}
```

Making Recursive Calls on an Array

- Recursive calls denote solutions to *smaller* sub-problems.
- *Naively*, explicitly create a new, smaller array:

```
void m(int[] a) {
  if(a.length == 0) { /* base case */ }
  else if(a.length == 1) { /* base case */ }
  else {
    int[] sub = new int[a.length - 1];
    for(int i = 1; i < a.length; i ++) { sub[0] = a[i - 1]; }
    m(sub) } }
```

- For *efficiency*, we pass the ***reference*** of the same array and specify the *range of indices* to be considered:

```
void m(int[] a, int from, int to) {
  if(from > to) { /* base case */ }
  else if(from == to) { /* base case */ }
  else { m(a, from + 1, to) } }
```

  - `m(a, 0, a.length - 1)` [Initial call; entire array]
  - `m(a, 1, a.length - 1)` [1st r.c. on array of size *a.length* – 1]
  - `m(a, a.length-1, a.length-1)` [Last r.c. on array of size 1]

Recursion: All Positive (1)

Problem: Determine if an array of integers are all positive.

```
System.out.println(allPositive({}));  /* true */
System.out.println(allPositive({1, 2, 3, 4, 5}));  /* true */
System.out.println(allPositive({1, 2, -3, 4, 5}));  /* false */
```

Base Case: Empty array ⟶ Return *true* immediately.

The base case is *true* ∵ we can *not* find a counter-example (i.e., a number *not* positive) from an empty array.

Recursive Case: Non-Empty array ⟶

- 1st element positive, **and**
- *the rest of the array* *is all positive*.

Exercise: Write a method `boolean somePostive(int[] a)` which *recursively* returns *true* if there is some positive number in `a`, and *false* if there are no positive numbers in `a`.

Hint: What to return in the base case of an empty array? [*false*]

∵ No witness (i.e., a positive number) from an empty array

Recursion: All Positive (2)

```
boolean allPositive(int[] a) {
  return allPositiveHelper (a, 0, a.length - 1);
}

boolean allPositiveHelper (int[] a, int from, int to) {
  if (from > to) { /* base case 1: empty range */
    return true;
  }
  else if(from == to) { /* base case 2: range of one element */
    return a[from] > 0;
  }
  else { /* recursive case */
    return a[from] > 0 && allPositiveHelper (a, from + 1, to);
  }
}
```

Recursion: Is an Array Sorted? (1)

Problem: Determine if an array of integers are sorted in a non-descending order.

```
System.out.println(isSorted({}));  true

System.out.println(isSorted({1, 2, 2, 3, 4}));  true

System.out.println(isSorted({1, 2, 2, 1, 3}));  false
```

Base Case: Empty array $\longrightarrow$ Return *true* immediately.
The base case is *true* $\because$ we can *not* find a counter-example (i.e., a pair of adjacent numbers that are *not* sorted in a non-descending order) from an empty array.
Recursive Case: Non-Empty array $\longrightarrow$

- 1st and 2nd elements are sorted in a non-descending order, **and**
- *the rest of the array*, starting from the 2nd element, *are sorted in a non-descending positive* .

Recursion: Is an Array Sorted? (2)

```
boolean isSorted(int[] a) {
  return isSortedHelper (a, 0, a.length - 1);
}

boolean isSortedHelper (int[] a, int from, int to) {
  if (from > to) { /* base case 1: empty range */
    return true;
  }
  else if(from == to) { /* base case 2: range of one element */
    return true;
  }
  else {
    return a[from] <= a[from + 1]
      && isSortedHelper (a, from + 1, to);
  }
}
```

Recursive Methods: Correctness Proofs

```
1 boolean allPositive(int[] a) { return allPosH (a, 0, a.length - 1); }
2 boolean allPosH (int[] a, int from, int to) {
3   if (from > to) { return true; }
4   else if(from == to) { return a[from] > 0; }
5   else { return a[from] > 0 && allPosH (a, from + 1, to); } }
```

- Via mathematical induction, prove that `allPosH` is correct:

  Base Cases

  - In an empty array, there is no non-positive number $\therefore$ result is ***true***. **[L3]**
  - In an array of size 1, the only one elements determines the result. **[L4]**

  Inductive Cases

  - **Inductive Hypothesis**: `allPosH(a, from + 1, to)` returns ***true*** if a[from + 1], a[from + 2], . . . , a[to] are all positive; ***false*** otherwise.
  - `allPosH(a, from, to)` should return ***true*** if: **1)** a[from] is positive; and **2)** a[from + 1], a[from + 2], . . . , a[to] are all positive.
  - By *I.H.*, result is $a[from] > 0 \wedge$ `allPosH(a, from + 1, to)`. **[L5]**

- `allPositive(a)` is correct by invoking `allPosH(a, 0, a.length - 1)`, examining the entire array. **[L1]**

Recursion: Binary Search (1)

- **Searching Problem**

  Input: A number a and a *sorted* list of n numbers $\langle a_1, a_2, \ldots, a_n \rangle$ such that $a'_1 \leq a'_2 \leq \ldots \leq a'_n$

  Output: Whether or not a exists in the input list

- **An Efficient Recursive Solution**

  Base Case: Empty list $\longrightarrow$ *False*.

  Recursive Case: List of size $\geq 1 \longrightarrow$

  - *Compare* the *middle* element against a.
    - All elements to the left of *middle* are $\leq a$
    - All elements to the right of *middle* are $\geq a$
  - If the *middle* element *is* equal to $a \longrightarrow$ *True*.
  - If the *middle* element *is not* equal to a:
    - If $a <$ *middle*, recursively find a on the left half.
    - If $a >$ *middle*, recursively find a on the right half.

Recursion: Binary Search (2)

```
boolean binarySearch(int[] sorted, int key) {
  return binarySearchHelper (sorted, 0, sorted.length - 1, key);
}
boolean binarySearchHelper (int[] sorted, int from, int to, int key) {
  if (from > to) { /* base case 1: empty range */
    return false; }
  else if(from == to) { /* base case 2: range of one element */
    return sorted[from] == key; }
  else {
    int middle = (from + to) / 2;
    int middleValue = sorted[middle];
    if(key < middleValue) {
      return binarySearchHelper (sorted, from, middle - 1, key);
    }
    else if (key > middleValue) {
      return binarySearchHelper (sorted, middle + 1, to, key);
    }
    else  { return true; }
  }
}
```

Running Time: Binary Search (1)

We use *T(n)* to denote the running time function of a binary search, where *n* is the size of the input array.

$$\begin{cases} T(0) & = & 1 \\ T(1) & = & 1 \\ T(n) & = & T(\frac{n}{2}) + 1 \quad \texttt{where} \quad n \geq 2 \end{cases}$$

To solve this recurrence relation, we study the pattern of *T(n)* and observe how it reaches the *base case(s)*.

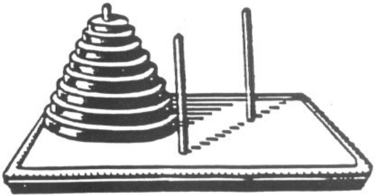
Running Time: Binary Search (2)

Without loss of generality, assume $n = 2^i$ for some non-negative i.

$$
\begin{aligned}
T(n) &= T(\tfrac{n}{2}) + 1 \\
&= \underbrace{(T(\frac{n}{4}) + 1)}_{T(\frac{n}{2})} + \underbrace{1}_{1 \texttt{ time}} \\
&= \underbrace{((T(\frac{n}{8}) + 1)}_{T(\frac{n}{4})} + \underbrace{1) + 1}_{2 \texttt{ times}} \\
&= \dots \\
&= (\ ((\ \underbrace{1}_{T(\frac{n}{2^{log\ n}})=T(1)}\ ) + \underbrace{1)\ \dots\ ) + 1}_{log\ n \texttt{ times}}
\end{aligned}
$$

$\therefore T(n)$ is $O(log\ n)$

Tower of Hanoi: Specification



- *Given*: A tower of 8 disks, initially stacked in decreasing size on one of 3 pegs
- *Rules*:
  - Move only one disk at a time
  - Never move a larger disk onto a smaller one
- *Problem*: Transfer the entire tower to one of the other pegs.

Tower of Hanoi: A Recursive Solution

The general, recursive solution requires 3 steps:

1. Transfer the *n - 1* smallest disks to a different peg.
2. Move the largest to the remaining free peg.
3. Transfer the *n - 1* disks back onto the largest disk.

Tower of Hanoi in Java (1)

```
void towerOfHanoi(String[] disks) {
  tohHelper (disks, 0, disks.length - 1, 1, 3);
}
void tohHelper(String[] disks, int from, int to, int ori, int des){
  if(from > to) { }
  else if(from == to) {
    print("move " + disks[to] + " from " + ori + " to " + des);
  }
  else {
    int intermediate = 6 - ori - des;
    tohHelper (disks, from, to - 1, ori, intermediate);
    print("move " + disks[to] + " from " + ori + " to " + des);
    tohHelper (disks, from, to - 1, intermediate, des);
  }
}
```

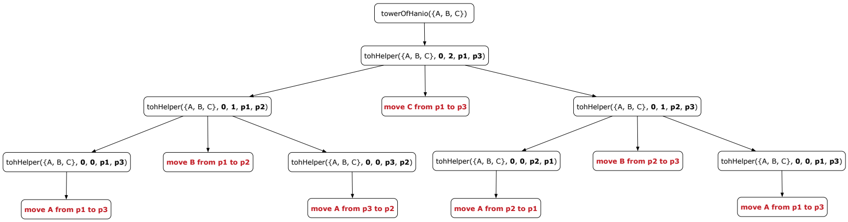
- `tohHelper(disks, from, to, ori, des)` moves disks $\{disks[from], disks[from+1], \ldots, disks[to]\}$ from peg *ori* to peg *des*.
- Peg id's are 1, 2, and 3 $\Rightarrow$ The intermediate one is 6 − *ori* − *des*.

Tower of Hanoi in Java (2)

Say *ds* (disks) is $\{A, B, C\}$, where $A < B < C$.

$$
tohH(ds, \underbrace{0,2}_{\{A,B,C\}}, p1, p3) = \begin{cases}
tohH(ds, \underbrace{0,1}_{\{A,B\}}, p1, p2) = \begin{cases}
tohH(ds, \underbrace{0,0}_{\{A\}}, p1, p3) = \begin{cases} \texttt{Move A: p1 to p3} \end{cases} \\
\texttt{Move B: p1 to p2} \\
tohH(ds, \underbrace{0,0}_{\{A\}}, p3, p2) = \begin{cases} \texttt{Move A: p3 to p2} \end{cases}
\end{cases} \\
\texttt{Move C: p1 to p3} \\
tohH(ds, \underbrace{0,1}_{\{A,B\}}, p2, p3) = \begin{cases}
tohH(ds, \underbrace{0,0}_{\{A\}}, p2, p1) = \begin{cases} \texttt{Move A: p2 to p1} \end{cases} \\
\texttt{Move B: p2 to p3} \\
tohH(ds, \underbrace{0,0}_{\{A\}}, p1, p3) = \begin{cases} \texttt{Move A: p1 to p3} \end{cases}
\end{cases}
\end{cases}
$$

Tower of Hanoi in Java (3)



Running Time: Tower of Hanoi (1)

- Generalize the problem by considering *n* disks.
- Let $T(n)$ denote the number of moves required to to transfer *n* disks from one to another under the rules.
- Recall the general solution pattern:
  1. Transfer the *n - 1* smallest disks to a different peg.
  2. Move the largest to the remaining free peg.
  3. Transfer the *n - 1* disks back onto the largest disk.
- We end up with the following recurrence relation that allows us to compute T_n for any *n* we like:

$$\begin{cases} T(1) & = & 1 \\ T(n) & = & 2 \times T(n-1) + 1 \quad \text{where } n > 0 \end{cases}$$

- To solve this recurrence relation, we study the pattern of T(n) and observe how it reaches the base case(s).

Running Time: Tower of Hanoi (2)

$$
\begin{aligned}
T(n) &= 2 \times T(n-1) + 1 \\
&= 2 \times (\underbrace{2 \times T(n-2) + 1}_{T(n-1)}) + 1 \\
&= 2 \times (2 \times (\underbrace{2 \times T(n-3) + 1}_{T(n-2)}) + 1) + 1 \\
&= \dots \\
&= 2 \times (2 \times (2 \times (\underbrace{\cdots \times (\overbrace{2 \times T(1) + 1}^{T(2)}) + \dots}_{T(n-3)}) + 1) + 1) + 1 \\
&= 2^{n-1} + (n-1)
\end{aligned}
$$

$\therefore T(n)$ is $O(2^n)$

Recursion: Merge Sort

- **Sorting Problem**

  Input: A list of n numbers $\langle a_1,\ a_2, \ldots,\ a_n \rangle$

  Output: A permutation (reordering) $\langle a'_1,\ a'_2, \ldots,\ a'_n \rangle$ of the input list such that $a'_1 \leq a'_2 \leq \ldots \leq a'_n$

- **Recursive Solution**

  Base Case 1: Empty list $\longrightarrow$ Automatically sorted.

  Base Case 2: List of size 1 $\longrightarrow$ Automatically sorted.

  Recursive Case: List of size ≥ 2 $\longrightarrow$

  - Split the list into two (unsorted) halves: *L* and *R*;
  - **Recursively** sort *L* and *R*: *sortedL* and *sortedR*;
  - Return the *merge* of *sortedL* and *sortedR*.

Recursion: Merge Sort in Java (1)

```
/* Assumption: L and R are both already sorted. */
private List<Integer> merge(List<Integer> L, List<Integer> R) {
  List<Integer> merge = new ArrayList<>();
  if(L.isEmpty()||R.isEmpty()) { merge.addAll(L); merge.addAll(R); }
  else {
    int i = 0;
    int j = 0;
    while(i < L.size() && j < R.size()) {
      if( L.get(i) <= R.get(j) ) { merge.add(L.get(i)); i ++; }
      else { merge.add(R.get(j)); j ++; }
    }
    /* If i >= L.size(), then this for loop is skipped. */
    for(int k = i; k < L.size(); k ++) { merge.add(L.get(k)); }
    /* If j >= R.size(), then this for loop is skipped. */
    for(int k = j; k < R.size(); k ++) { merge.add(R.get(k)); }
  }
  return merge;
}
```

RT(merge)? [O(n)]

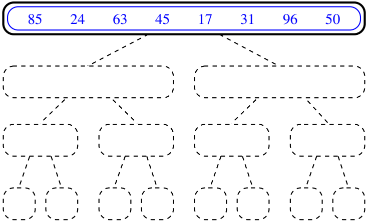
Recursion: Merge Sort in Java (2)

```
public List<Integer> sort (List<Integer> list) {
  List<Integer> sortedList;
  if(list.size() == 0) { sortedList = new ArrayList<>(); }
  else if(list.size() == 1) {
    sortedList = new ArrayList<>();
    sortedList.add(list.get(0));
  }
  else {
    int middle = list.size() / 2;
    List<Integer> left = list.subList(0, middle);
    List<Integer> right = list.subList(middle, list.size());
    List<Integer> sortedLeft = sort (left);
    List<Integer> sortedRight = sort (right);
    sortedList = merge (sortedLeft, sortedRight);
  }
  return sortedList;
}
```

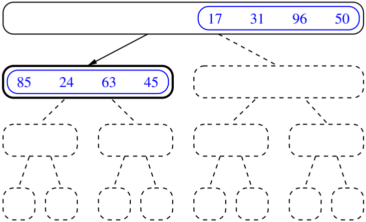
$$\text{RT(sort)} = \underbrace{RT(merge)}_{O(n)} \times \underbrace{\texttt{\# splits until size 0 or 1}}_{O(log\ n)}$$

Recursion: Merge Sort Example (1)

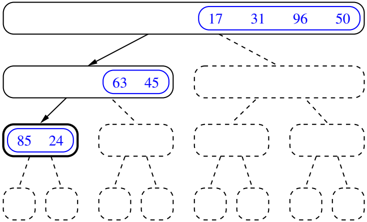
(1) Start with input list of size 8



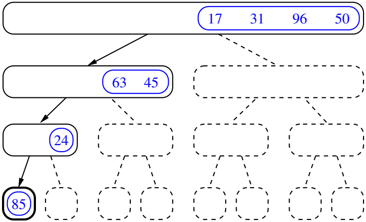
(2) Split and recur on L of size 4



(3) Split and recur on L of size 2

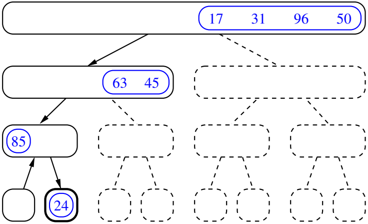


(4) Split and recur on L of size 1, *return*

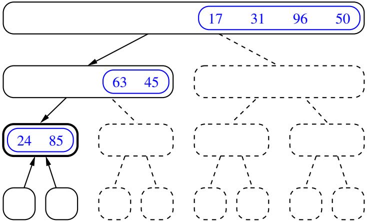


Recursion: Merge Sort Example (2)

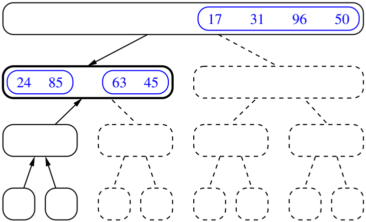
(5) Recur on R of size 1 and *return*



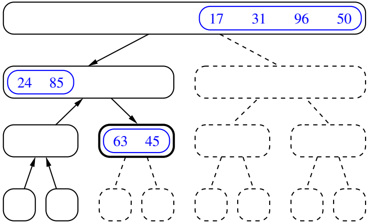
(6) Merge sorted L and R of sizes 1



(7) Return merged list of size 2

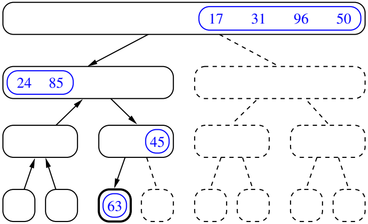


(8) Recur on R of size 2

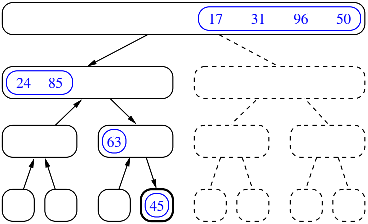


Recursion: Merge Sort Example (3)

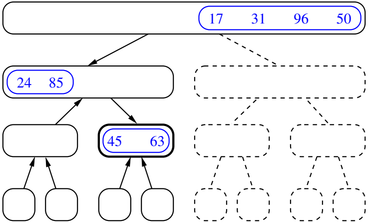
(9) Split and recur on L of size 1, *return*



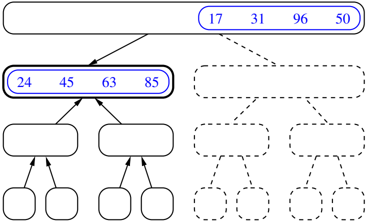
(10) Recur on R of size 1, *return*



(11) Merge sorted L and R of sizes 1, *return*

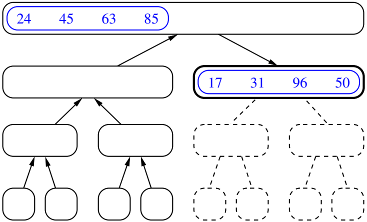


(12) Merge sorted L and R of sizes 2

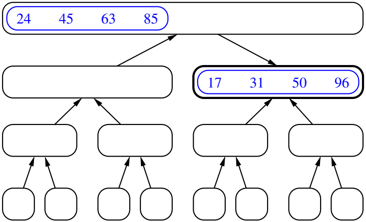


Recursion: Merge Sort Example (4)

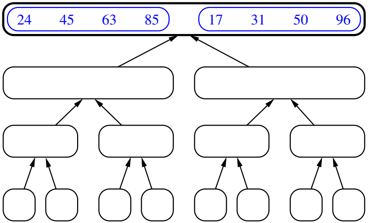
(13) Recur on R of size 4



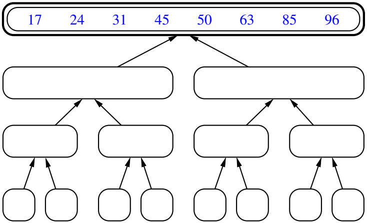
(14) *Return* a sorted list of size 4



(15) Merge sorted L and R of sizes 4

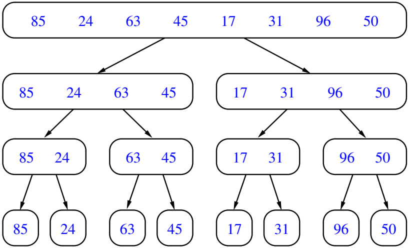


(16) *Return* a sorted list of size 8

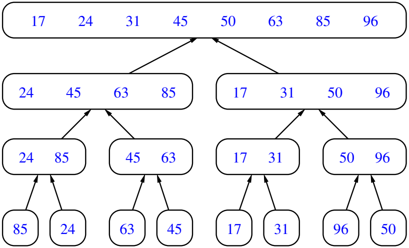


Recursion: Merge Sort Example (5)

(1) Recursion trees of *unsorted* lists



(2) Recursion trees of *sorted* lists



Recursion: Merge Sort Running Time (1)

Base Case 1: Empty list ⟶ Automatically sorted. [O(1)]

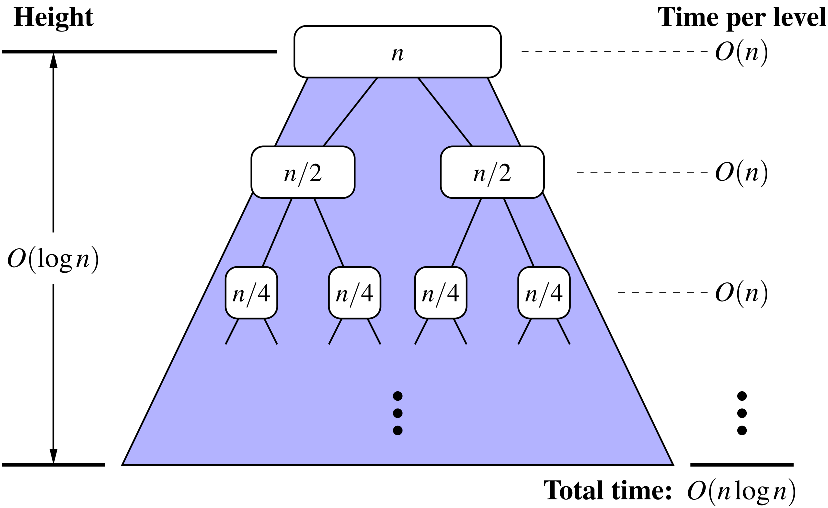
Base Case 2: List of size 1 ⟶ Automatically sorted. [O(1)]

Recursive Case: List of size ≥ 2 ⟶

- Split the list into two (unsorted) halves: *L* and *R*; [O(1)]
- **Recursively** sort *L* and *R*: *sortedL* and *sortedR*;
  How many times to split until *L* and *R* have size 0 or 1? [*O(log n)*]
- Return the *merge* of *sortedL* and *sortedR*. [*O(n)*]

RT
= (**RT** each RC) × (# **RC**s)
= (**RT** merging *sortedL* and *sortedR*) × (# splits until bases)
= $n \cdot \log n$

Recursion: Merge Sort Running Time (2)



Recursion: Merge Sort Running Time (3)

We use *T(n)* to denote the running time function of a merge sort, where *n* is the size of the input list.

$$\begin{cases} T(0) & = & 1 \\ T(1) & = & 1 \\ T(n) & = & 2 \cdot T(\frac{n}{2}) + n \quad \texttt{where} \quad n \geq 2 \end{cases}$$

To solve this recurrence relation, we study the pattern of *T(n)* and observe how it reaches the *base case(s)*.

Recursion: Merge Sort Running Time (4)

Without loss of generality, assume $n = 2^i$ for some non-negative i.

$$
\begin{aligned}
T(n) &= 2 \times T(\tfrac{n}{2}) + n \\
&= \underbrace{2 \times (2}_{2 \texttt{ terms}} \times T(\tfrac{n}{4}) + \underbrace{\frac{n}{2}) + n}_{2 \texttt{ terms}} \\
&= \underbrace{2 \times (2 \times (2}_{3 \texttt{ terms}} \times T(\tfrac{n}{8}) + \underbrace{\frac{n}{4}) + \frac{n}{2}) + n}_{3 \texttt{ terms}} \\
&= \dots \\
&= \underbrace{2 \times (2 \times (2 \times \cdots \times (2}_{log n \texttt{ terms}} \times T(\tfrac{n}{2^{log n}}) + \underbrace{\frac{n}{2^{log n-1}}) + \cdots + \frac{n}{4}) + \frac{n}{2}) + n}_{log n \texttt{ terms}} \\
&= 2^{log n} + \underbrace{(2 \cdot \frac{n}{2} + 2^2 \cdot \frac{n}{4} + \cdots + 2^{log n-1} \cdot \frac{n}{2^{log n-1}} + n)}_{log n \texttt{ terms}}
\end{aligned}
$$

$\therefore T(n)$ is $O(n \cdot log\ n)$

Beyond this lecture ...

- Notes on Recursion:
  `http://www.eecs.yorku.ca/~jackie/teaching/lectures/2019/F/EECS2030/slides/EECS2030_F19_Notes_Recursion.pdf`
- API for `String`:
  `https://docs.oracle.com/javase/8/docs/api/java/lang/String.html`
- Fantastic resources for sharpening your recursive skills for the exam:
  `http://codingbat.com/java/Recursion-1`
  `http://codingbat.com/java/Recursion-2`
- The *best* approach to learning about recursion is via a functional programming language:
  Haskell Tutorial: `https://www.haskell.org/tutorial/`

Index (1)

Index (2)

Index (3)

Index (4)