Chapter 2
Binary Numbers and Number Systems

Number Categories – Ch. 2.1

Definition of
- number
- natural numbers
- negative numbers
- integers
- rational numbers

The Idea of a Positional Number System

4357	four thousand, three hundred and fifty seven	
	four units of a thousand (4 x 1000)	4000
	three units of a hundred (3 x 100)	300
	five units of ten (5 x 10)	50
	seven units of one (7 x 1)	7

$$4 \times 10^3 + 3 \times 10^2 + 5 \times 10^1 + 7 \times 10^0$$

5743 same digits, different positions, different number

The ***position*** of each digit determines that digit's contribution to the number.

The beauty of the idea of a positional number system:

no matter what column you are in, the column to the left is the base times bigger.

Example: in decimal, if you are in the hundreds column, the thousands column is ten times bigger than the hundreds column

$$4 \times 10^3 + 3 \times 10^2 = (4 \times 10^1 + 3) \times 10^2$$

Positional notation – Ch. 2.2

base: b any integer > 1

digits: `0`, `1`, ..., b–`1`

number $d_{n-1}d_{n-2}\ldots d_2 d_1 d_0$

its definition

$$d_{n-1}\times b^{n-1}+d_{n-2}\times b^{n-2}+\cdots+d_2\times b^2+d_1\times b^1+d_0\times b^0$$

Examples:

base	digits
`2`	`0,1`
`5`	`0,1,2,3,4`
`8`	`0,1,2,3,4,5,6,7`
`10`	`0,1,2,3,4,5,6,7,8,9`
`16`	`0,1,2,3,4,5,6,7,8,9,A,B,C,D,E,F`

Binary, Octal and Hexadecimal

Side-by-side comparison of the first twenty-three natural numbers in four different bases.

Decimal	Binary	Octal	Hexadecimal	
0	0	0	0	
1	1	1	1	
2	10	2	2	
3	11	3	3	
4	100	4	4	
5	101	5	5	
6	110	6	6	
7	111	7	7	
8	1000	10	8	
9	1001	11	9	
10	1010	12	A	
11	1011	13	B	
12	1100	14	C	
13	1101	15	D	
14	1110	16	E	
15	1111	17	F	
16	10000	20	10	
17	10001	21	11	
18	10010	22	12	
19	10011	23	13	
20	10100	24	14	
21	10101	25	15	
22	10110	26	16	
23	10111	27	17	*etc.*

Arithmetic in Other Bases

All the familiar rules and techniques of pencil-and-paper decimal arithmetic carry over to any other base.

Addition

```
  46      101110
+ 27    +  11011
```

Subtraction

```
 5037     57     111001
-  95    - 6    -   110
```

Decimal Grid:

	0	1	2	3	4	5	6	7	8	9
0	00	01	02	03	04	05	06	07	08	09
1	01	02	03	04	05	06	07	08	09	10
2	02	03	04	05	06	07	08	09	10	11
3	03	04	05	06	07	08	09	10	11	12
4	04	05	06	07	08	09	10	11	12	13
5	05	06	07	08	09	10	11	12	13	14
6	06	07	08	09	10	11	12	13	14	15
7	07	08	09	10	11	12	13	14	15	16
8	08	09	10	11	12	13	14	15	16	17
9	09	10	11	12	13	14	15	16	17	18

Binary Grid:

	0	1
0	00	01
1	01	10

Octal Grid:

	0	1	2	3	4	5	6	7
0	00	01	02	03	04	05	06	07
1	01	02	03	04	05	06	07	10
2	02	03	04	05	06	07	10	11
3	03	04	05	06	07	10	11	12
4	04	05	06	07	10	11	12	13
5	05	06	07	10	11	12	13	14
6	06	07	10	11	12	13	14	15
7	07	10	11	12	13	14	15	16

Hexadecimal Grid:

	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
0	00	01	02	03	04	05	06	07	08	09	0A	0B	0C	0D	0E	0F
1	01	02	03	04	05	06	07	08	09	0A	0B	0C	0D	0E	0F	10
2	02	03	04	05	06	07	08	09	0A	0B	0C	0D	0E	0F	10	11
3	03	04	05	06	07	08	09	0A	0B	0C	0D	0E	0F	10	11	12
4	04	05	06	07	08	09	0A	0B	0C	0D	0E	0F	10	11	12	13
5	05	06	07	08	09	0A	0B	0C	0D	0E	0F	10	11	12	13	14
6	06	07	08	09	0A	0B	0C	0D	0E	0F	10	11	12	13	14	15
7	07	08	09	0A	0B	0C	0D	0E	0F	10	11	12	13	14	15	16
8	08	09	0A	0B	0C	0D	0E	0F	10	11	12	13	14	15	16	17
9	09	0A	0B	0C	0D	0E	0F	10	11	12	13	14	15	16	17	18
A	0A	0B	0C	0D	0E	0F	10	11	12	13	14	15	16	17	18	19
B	0B	0C	0D	0E	0F	10	11	12	13	14	15	16	17	18	19	1A
C	0C	0D	0E	0F	10	11	12	13	14	15	16	17	18	19	1A	1B
D	0D	0E	0F	10	11	12	13	14	15	16	17	18	19	1A	1B	1C
E	0E	0F	10	11	12	13	14	15	16	17	18	19	1A	1B	1C	1D
F	0F	10	11	12	13	14	15	16	17	18	19	1A	1B	1C	1D	1E

Examples of Arithmetic:

In Octal:

```
  3754
+ 6317
------
```

In Hexadecimal:

```
  3B6F
+ 5743
------
```

In Decimal:

```
  4357
+  585
------
```

```
       123
        45
      3682
        12
+      654
----------
```

Conversions between Decimal and Binary

Binary to Decimal

Technique

- use the definition of a number in a positional number system with base 2
- evaluate the definition formula using decimal arithmetic

Example

```
543210 ← position = corresponding power of 2
||||||
```

$101011 = 1\times 2^5 + 0\times 2^4 + 1\times 2^3 + 0\times 2^2 + 1\times 2^1 + 1\times 2^0 = 43$ (decimal)

$d_5d_4d_3d_2d_1d_0$

generalize: Octal to Decimal, Hexadecimal to Decimal, any base to Decimal

Decimal to Binary

Technique

- repeatedly divide by 2
- remainder is the next digit
- binary number is developed right to left

Example

173 ÷ 2	86	1	1
86 ÷ 2	43	0	01
43 ÷ 2	21	1	101
21 ÷ 2	10	1	1101
10 ÷ 2	5	0	01101
5 ÷ 2	2	1	101101
2 ÷ 2	1	0	0101101
1 ÷ 2	0	1	10101101

generalize: Decimal to Octal, Decimal to Hexadecimal, Decimal to any base