Chapter 1

Some Elementary Informal Set Theory

Set theory is due to Georg Cantor. "Elementary" in the title above does not apply to the body of his work, since he went into considerable technical depth in this, his new theory. It applies however to *our* coverage as we are going to restrict ourselves to elementary topics only.

Cantor made many technical mistakes in the process of developing set theory, some of considerable consequence. The next section is about the easiest and most fundamental of his mistakes.

How come he made mistakes? The reason is that his theory was not based on axioms and rigid rules of reasoning —a state of affairs for a theory that we loosely characterise as "informal".

At the opposite end of informal we have the *formal* theories that are based on axioms *and* logic and are thus "safer" to develop (they do not lead to *obvious* contradictions).

One *cannot* fault Cantor for not using logic in arguing his theorems —that process was not invented when he built his theory— but then, *a fortiori*, mathematical logic was not invented in Euclid's time either, *and yet* he did use axioms that stated how his building blocks, *points*, *lines* and *planes* interacted and behaved!

Guess what: Euclidean Geometry leads to no contradictions.

The problem with Cantor's set theory is that anything goes as to what sets are and how they come about. He neglected to ask the most fundamental question: "How are sets formed?"[†] He just sidestepped this and simply said that a *set* is any collection. In fact he took the term "set" as just a synonym for "collection", "class", "aggregate", etc.

[†]It's amazing how much trouble could be avoided if he had done so!

Failure to ask and answer this question leads to “trouble”, which is the subject matter of the next section.

One can still do “safe” set theory —devoid of “trouble”, that is— within an *informal* (non axiomatic) setting, but we have to ask and answer how sets are built *first* and derive from our answer some *principles* that will guide (and protect!) the theory’s development! We will do so.

1.1. Russell’s “Paradox”

Cantor’s *naïve* (this adjective is not derogatory but is synonymous in the literature with *informal* and *non axiomatic*) set theory was plagued by *paradoxes*, the most famous of which (and the *least* “technical”) being pointed out by Bertrand Russell and thus nicknamed “Russell’s paradox”.[†]

His theory is the theory of collections (i.e., sets) of objects, as we mentioned above, terms that were neither defined nor how they were built.[‡]

This theory studies operations on sets, properties of sets, and aims to use set theory as the foundation *of all mathematics*. Naturally, mathematicians “do” set theory of *mathematical object collections* —not collections of birds and other beasts. We have learnt some elementary aspects of set theory at high school. We will learn more in this course.

1. **Variables**. Like any theory, informal or not, informal set theory —a safe variety of which we will develop here— uses *variables* just as algebra does. There is only *one type* of variable that varies over set and over atomic objects too, the latter being objects that have no set structure. For example integers. We use the names $A, B, C, \ldots$ and $a, b, c, \ldots$ for such variables, sometimes with primes (e.g., A'') or subscripts (e.g., x_{23}), or both (e.g., x'''_{22}, Y'_{42}).

2. **Notation**. *Sets given by listing.* For example, $\{1, 2\}$ is a set that contains precisely the objects 1 and 2, while $\{1, \{5, 6\}\}$ is a set that contains precisely the objects 1 and $\{5, 6\}$. The braces $\{$ and $\}$ are used to show the collection/set by outright listing.

3. **Notation**. *Sets given by “defining property”*. But what if we cannot (or will not) explicitly list all the members of a set? Then we may define

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[†]From the Greek word “paradoxo” (παράδοξο) meaning against one’s belief or knowledge; a contradiction.

[‡]This is not a problem *in itself*. Euclid too did not say *what* points and lines *were*; but his axioms did characterise their nature and interrelationships: For example, he started from these (among a few others) *a priori truths* (axioms): *a unique line passes through two distinct points*; also, *on any plane, a unique line l can be drawn parallel to another line k on the plane if we want l to pass through a given point A that is not on k.*

The point is:

You cannot leave out *both* what the nature of your objects is and *how* they behave/interrelate and get away with it! Euclid omitted the former but provided the latter, so all worked out.

what objects x get in the set/collection by having them to *pass an entrance requirement*, $P(x)$:

> **An object x gets in the set *iff* (*if and only if*) $P(x)$ is true of said object.**

Let us parse "iff":

(a) The *IF*: So, IF $P(x)$ is true, then x gets in the set (it passed the "admission requirement").

(b) The *ONLY IF*: So, IF x gets in the set, then the ***only*** *way for this to happen* is for it to pass the "admission requirement"; that is, $P(x)$ is true.

In other words, "iff" (as we probably learnt in high school or some previous university course such as calculus) is the same thing as "is equivalent":

"x is in the set" is equivalent to "$P(x)$ is true".

We denote the collection/set[†] defined by the entrance condition $P(x)$ by

$$\{x : P(x)\} \tag{1}$$

but also as

$$\{x \,|\, P(x)\} \tag{1'}$$

reading it "the set of all x *such that* (this "such that" is the ":" or "|") $P(x)$ is true [or holds]"

4. "$x \in A$" is the assertion that "object x is in the set A". Of course, this assertion may be true or false or "it depends", just like the assertions of algebra $2 = 2$, $3 = 2$ and $x = y$ are so (respectively).

5. $x \notin A$ is the negation of the assertion $x \in A$.

6. **Properties**

   - Sets are *named* by letters of the Latin alphabet (cf. **Variables**, above). Naming is pervasive in mathematics as in, e.g., "let $x = 5$" in algebra.

     So we can write "let $A = \{1, 2\}$" and let "$c = \{1, \{5, 6\}\}$" to give the names A and c to the two example sets above, ostensibly because we are going to discuss these sets, and refer to them often, and it is cumbersome to keep writing things like $\{1, \{5, 6\}\}$. Names are *not permanent*;[‡] they are *local* to a discussion (argument).

[†]We have not yet reached Russell's result, so keeping an open mind and humouring Cantor we still allow ourselves to call said collection a "set".

[‡]OK, there *are* exceptions: $\emptyset$ is the permanent name for the *empty set* —the set with no elements at all— and for that set only; $\mathbb{N}$ is the permanent name of the set of all *natural numbers*.

- **Equality of sets** (repetition and permutation do not matter!)

  Two sets A and B are equal iff they have the same members. Thus order and multiplicity do not matter! E.g., $\{1\} = \{1,1,1\}$, $\{1,2,1\} = \{2,1,1,1,1,2\}$.

- The fundamental equivalence pertaining to definition of sets by "defining property": So, if we name the set in (1) above, S, that is, if we say "let $S = \{x : P(x)\}$", then "$x \in S$ iff $P(x)$ is true"

  By the way, we almost *never say* "is true" unless we want to shout out this fact. We would say instead: "$x \in S$ iff $P(x)$".

  Equipped with the knowledge of the previous bullet, we see that the symbol $\{x : P(x)\}$ defines a *unique* set/collection: Well, say A and B are so defined, that is, $A = \{x : P(x)\}$ and $B = \{x : P(x)\}$. Thus

$$x \in A \overset{A=\{x:P(x)\}}{\text{iff}} P(x) \overset{B=\{x:P(x)\}}{\text{iff}} x \in B$$

  thus

$$x \in A \text{ iff } x \in B$$

  and thus $A = B$.

Let us pursue, as Russell did, the point made in the last bullet above. Take $P(x)$ to be specifically the assertion $x \notin x$. He then gave a name to

$$\{x : x \notin x\}$$

say, R. But then, by the last bullet above,

$$x \in R \text{ iff } x \notin x \tag{2}$$

If we now *believe*,† as *Cantor*, the father of set theory did not question and went ahead with it, that every $P(x)$ defines a *set*, *then R is a set.*

What is wrong with that?

Well, if R is a set then this object has the proper *type* to be plugged into the *variable of type "math object"*, namely, x, throughout the equivalence (2) above. But this yields the contradiction

$$R \in R \text{ iff } R \notin R \tag{3}$$

This contradiction is called the Russell's Paradox.

† Informal mathematics often relies on "I know so" or "I believe" or "it is 'obviously' true". Some people call "proofs" like this —i.e., "proofs" without justification(s)— "proofs by intimidation". Nowadays, with the ubiquitousness of the qualifier "fake", one could also call them "fake proofs".

This and similar paradoxes motivated mathematicians to develop formal symbolic logic and look to axiomatic set theory[†] as a means to avoid paradoxes like the above.

Other mathematicians who did not care to use mathematical logic and axiomatic theories found a way to do set theory *informally*, yet *safely*.

You see, they asked *and* answered "how are sets formed?"[‡]

Read on!

[†]There are many flavours or axiomatisations of set theory, the most frequently used being the "ZF" set theory, due to Zermelo and Fraenkel.

[‡]Actually, axiomatic set theory —in particular, its axioms are— is built upon the answers this group came up with. This story is told at an advanced level in [Tou03].

Chapter 2

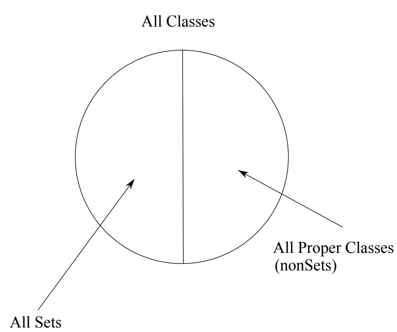
Safe Set Theory

So, *some* collections are *not* —technically— sets, as the Russell Paradox taught us! How do we tell them apart?

From now one we will deal with collections that *may or may not* be sets, with a promise of learning how to create sets if we want to!

The modern literature uses the terminology "**class**" for any such collection (and uses the term "collection" non technically and sparsely).

The above is captured by the following picture:



2.0.1 Definition. (Classes and sets)
From now on we call *all* collections **classes**.

Definitions by defining property "Let $\mathbb{A} = \{x : P(x)\}$" always define a **class**, but as we saw, sometimes —e.g., if "$P(x)$" is specifically "$x \notin x$"— that class is *not* a set (Section 1.1). *Classes that are not sets* are called **proper classes**. We will normally use what is known as "blackboard bold" notation and capital latin letters to denote classes by names such as $\mathbb{A}, \mathbb{B}, \mathbb{X}$. If we determine that some class $\mathbb{A}$ *is* a set, we would rather write it as A, but we make an exception for the following **sets**: Mathematicians use notation and results from set theory in their everyday practice. We call the sets that mathematicians use the "real sets" of our mathematical *intuition*, like the set of natural numbers, $\mathbb{N}$ (also denoted by ω), integers $\mathbb{Z}$, rationals $\mathbb{Q}$ and reals $\mathbb{R}$. $\square$

In forming the class $\{x : P(x)\}$ for any property $P(x)$ we say that we apply *comprehension*. It was the Frege/Cantor who believed (explicitly or implicitly) that comprehension was *safe* —i.e., always produced a set. Russell proved that it was not.

It is known that set theory, using as primitives the notions of *set*, *atom* (an object that is not sub-divisible; not a collection of objects), and the relation *belongs to* ($\in$), is sufficiently strong to serve as the foundation of all mathematics.

Mathematicians use notation and results from set theory in their everyday practice. We call the sets that mathematicians use the "real sets" of our mathematical *intuition*, like the set of natural numbers, $\mathbb{N}$ (also denoted by *omega*), integers $\mathbb{Z}$, rationals $\mathbb{Q}$ and reals $\mathbb{R}$.

2.1. The "real sets"

So, how can we tell, or indeed *guarantee*, that a certain class is a *set*?

Russell proposed this "recovery" from his Paradox:

Make sure that sets are built by stages, where at stage 0 all atoms are available. Atoms are also called *urelements* in the literature from the German *Urelemente*, which in analogy with the word "*urtext*" —meaning *the earliest text*— would mean that they are the "earliest" mathematical objects. Witness that they are available at stage 0!

We may then collect atoms to form all sorts of "first level" *sets*. We may proceed to collect any mix of atoms and first-level sets to build new collections —second-level sets— *and so on.* Much of what set theory does is attempting to remove the ambiguity from this "and so on". See below, **Principles** 0–2.

Thus, at the beginning we have all the level-0, or type-0, objects available to us. For example, atoms such as $1, 2, 13, \sqrt{2}$ are available. At the next level we can include any number of such atoms (from none at all, to all) to build a set, that is, a new mathematical object. Allowing the usual notation, i.e., listing of what is included within braces, we may cite a few examples of level-1 sets:

L1-1. $\{1\}$.

L1-2. $\{1, 1\}$.

L1-3. $\{1, \sqrt{2}\}$.

L1-4. $\{\sqrt{2}, 1\}$.

We already can identify a few level-2 objects, using what (we already know) *is* available:

L2-1. $\{\{\sqrt{2}, 1\}\}$.

Note how the level of nesting of { }-brackets matches the level or stage of the formation of these objects!

2.1.1 Definition. (Class and set *equality*) This definition applies to any classes, hence, in particular, to any *sets* as well.

Two classes $\mathbb{A}$ and $\mathbb{B}$ are *equal* —written $\mathbb{A} = \mathbb{B}$— means

$$x \in \mathbb{A} \text{ iff } x \in \mathbb{B}$$

That is, an object is in $\mathbb{A}$ iff it is also in $\mathbb{B}$.

$\mathbb{A}$ is a *subclass* of $\mathbb{B}$ —written $\mathbb{A} \subseteq \mathbb{B}$— means that every element of the first class occurs also in the second, or

$$\text{If } x \in \mathbb{A}, \text{ then } x \in \mathbb{B}$$

If $\mathbb{A}$ is a set, then we say it is a *subset* of $\mathbb{B}$.

If we have $\mathbb{A} \subseteq \mathbb{B}$ but $\mathbb{A} \neq \mathbb{B}$, then we write $\mathbb{A} \underset{\neq}{\subseteq} \mathbb{B}$ (some of the literature uses $\mathbb{A} \subsetneq \mathbb{B}$ or even $\mathbb{A} \subset \mathbb{B}$ nstead) and say that $\mathbb{A}$ is a *proper subclass* of $\mathbb{B}$.

Caution. In the terminology "*proper subclass*" the "proper" refers to the fact that $\mathbb{A}$ is not all of $\mathbb{B}$. It does *NOT* say that $\mathbb{A}$ is not a set! It *may* be a set and then we say that it is "*proper subset*" of $\mathbb{B}$. □

If n is an integer-valued variable, then what do you understand by "$2n$ is even"? The normal understanding is that "no matter what the value of n is, $2n$ is even", or "for all values of n, $2n$ is even".

When we get into our logic topic in the course we will see that we *can* write "for all values of n, $2n$ is even" with less English as "$(\forall n)(2n$ is even)". So "$(\forall n)$" says "for all (values of) n".

Mathematicians often prefer to have statements like "$2n$ is even" with the "for all" *always implied*.[†] You can write a whole math book without writing $\forall$ even once, and without overdoing the English.

2.1.2 Remark. Since "iff" between two statements S_1 and S_2 means that we have *both* directions

$$\text{If } S_1, \text{ then } S_2$$

and

$$\text{If } S_2, \text{ then } S_1$$

we have that "$\mathbb{A} = \mathbb{B}$" is the same as (equivalent to) "$\mathbb{A} \subseteq \mathbb{B}$ and $\mathbb{B} \subseteq \mathbb{A}$". □

2.1.3 Example. In the context of the "$\mathbb{A} = \{x : P(x)\}$" notation we should remark that notation-by-listing can be simulated by notation-by-defining-property: For example, $\{a\} = \{x : x = a\}$ —here "$P(x)$" is $x = a$.

[†]An exception occurs in Induction that we will study later, where you *fix* an n (but keep it as a variable, not as 5 or 42) and assume the "induction hypothesis" $P(n)$. But do not worry about this now!

Also $\{A, B\} = \{x : x = A \text{ or } x = B\}$. Let us verify the latter: Say $x \in$ lhs.[†] Then $x = A$ or $x = B$. Thus x must be A or B. But then the entrance requirement of the rhs[‡] is met, so $x \in$ rhs.

Conversely, say $x \in$ rhs. Then the entrance requirement is met so we have (at least) one of $x = A$ or $x = B$. Trivially, in the first case $x \in$ lhs and ditto for the second case. □

We now postulate the principles of formation of sets!

Principle 0. Sets and atoms are *the mathematical objects* of our (safe) set theory.

Sets are formed by stages. At stage 0 we acknowledge the *presence* of atoms. *They are given outright, they are not built.*

At *any* stage Σ we *may* build a *set*, collecting together other *mathematical objects* (sets or atoms) *provided* these (mathematical) objects we put into our set were available at stages before Σ.

Principle 1. *Every set is built at some stage.*

Principle 2. If Σ is a stage of set construction, then *there is* a stage Φ *after* it.

Principle 2 makes clear that we have infinitely many stages of set formation in our toolbox.

2.1.4 Remark. If some set is definable ("buildable") at some stage Σ, then it is also definable at any later stage as well, as **Principle** 0 makes clear.

The informal set-formation-by-stages will guide us to build, safely, all the sets we may need in order to do mathematics. □

2.2. What caused Russell's paradox

How would the set-building-by-stages doctrine avoid Russell's paradox?

Recall that *à la Cantor* we get a paradox (contradiction) because we insisted to believe that all classes are sets, that is, following Cantor we "believed" Russell's "R" was a *set*.

Principles 0–2 allow us to know *a priori* that R is a proper class. No contradiction!

How so?

[†] Left Hand Side.
[‡] Right Hand Side.

OK, is $x \in x$ true or false? Is there *any* mathematical object x —say, A— for which it *is* true?

$$A \in A? \tag{1}$$

Well, for atom A, (1) is false since atoms have no set structure, that is, are not collections of objects. An atom A *cannot contain anything*, in particular it cannot contain A.

What if A is a set and $A \in A$? Then in order to build A, the *set*, we have to wait until *after* its member, A is built (Principle 0). So, we need (the left) A to be built before (the right) A in (1).

Absurd!

So (1) is **false**. A being arbitrary, we demonstrated that

$$x \in x \text{ is false}$$

thus $x \notin x$ is true (forall x), therefore R of Section 1.1 is $\mathbb{U}$, the universe of *all sets and atoms.*

$$R = \mathbb{U}$$

So? Well this $\mathbb{U}$ is "far too big" to be built as a *set* and we should never have used $\{x : x \notin x\}$ so recklessly!

"Too Big" is bad in set theory; it intuitively means we ran out of stages after we built all the members of the class! No stages left to build the class as a set!

The "intuition", as always, is vague.

So here is why $\mathbb{U}$ is *not* a set. Well, <u>if it is</u>

- $\mathbb{U} \in \mathbb{U}$ since the rhs contains all sets and we believe the lhs to be a set.
- but we just saw that the above is false if $\mathbb{U}$ is a set!

So $\mathbb{U}$, aka R, is a *proper* class. Thus, the fact that R is not a set is neither a surprise, nor paradoxical. It is just a proper class as we just have recognised.

2.3. Some <u>useful</u> sets

2.3.1 Example. (Pair) By Principle 0, if A and B are sets or atoms, then let A be available at stage Σ and B at stage Σ'. Without loss of generality say Σ' is not later than Σ. Let then pick a stage Σ'' *after* Σ (Principle 2). This will be be after both (cf. Principle 2) Σ, Σ'.

At stage Σ'' we can build

$$\{A, B\} \tag{1}$$

as a *set* (cf. Principle 0).

We call (1) the (unordered) *pair set.*

Pause. Why "unordered"? See 2.1.1.◀ □