大学共同利用機関法人 情報・システム研究機構
国立情報学研究所
National Institute of Informatics

Continuous Domain Analysis of Graph Laplacian Regularization for Image Denoisng

Presenter: Gene Cheung

National Institute of Informatics

[1] Jiahao Pang, Gene Cheung, Wei Hu, Oscar C. Au, "**Redefining Self-Similarity in Natural Images for Denoising Using Graph Signal Gradient**," in *APSIPA ASC*, Siem Reap, Cambodia, December, 2014.

Outline

- Introduction
- Convergence of the Graph Laplacian Regularizer
- Justification of the Graph Laplacian Regularizer
- Formulation and Algorithm
- Experimental Results
- Towards the Optimal Graph Laplacian Regularizer
- Conclusion

Lena, $\sigma = 30$

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Motivation (I)

- Image denoising—a basic restoration problem:

observation → $$\mathbf{y} = \mathbf{x} + \mathbf{e}$$ ← noise, ← desired signal

- It is under-determined, needs image priors for regularization

fidelity term → $$\min_{\mathbf{x}} \|\mathbf{y} - \mathbf{x}\|_2^2 + \lambda \ \mathrm{prior}(\mathbf{x})$$ ← prior term

- Graph Laplacian regularizer: should be small for target patch $\mathbf{x}$

$$S_{\mathrm{G}}(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}\mathbf{L}\mathbf{x} \qquad \mathbf{L} = \mathbf{D} - \mathbf{A}$$ ← graph Laplacian matrix

- Many works use Gaussian kernel to compute graph weights [2]:

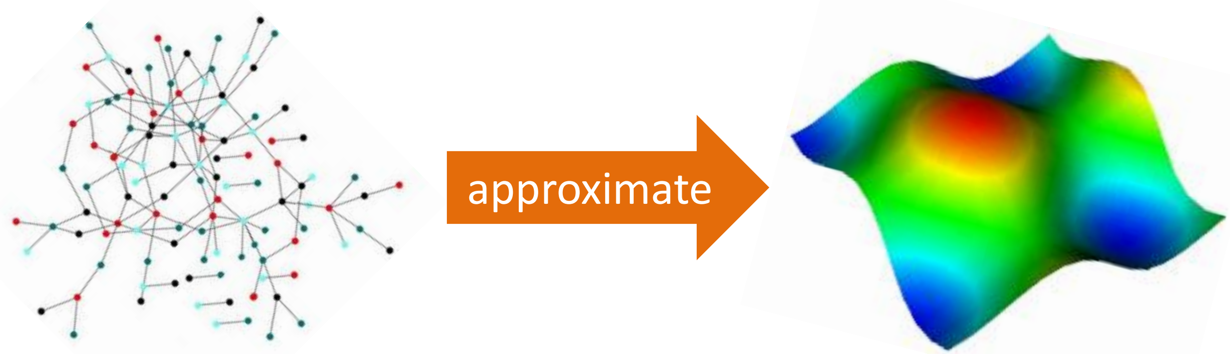
$$w_{ij} = \exp\left(\frac{dist(i,j)^2}{\sigma^2}\right)$$

$\mathrm{dist}(i, j)$ is some distance metric between pixels i and j

[2] D. Shuman, S. Narang, P. Frossard, A. Ortega, and P. Vandergheynst, "**The emerging field of signal processing on graphs: extending high-dimensional data analysis to networks and other irregular domains**," *IEEE Signal Processing Magazine*, vol. 30, no. 3, pp. 83–98, 2013.

Motivation (II)

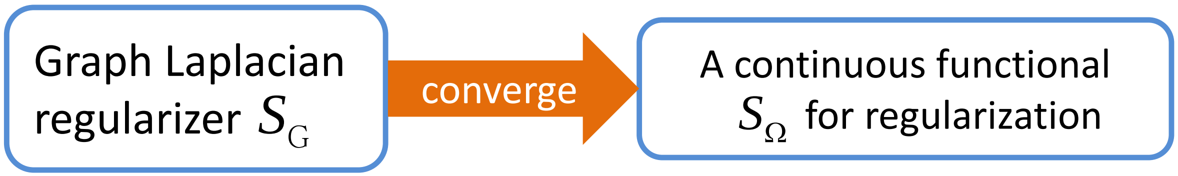
- However…
  a. Why is $S_G(\mathbf{x}) = \mathbf{x}^T\mathbf{L}\mathbf{x}$ a good prior?
  b. Why using Gaussian kernel for edge weights?
  c. How to design a discriminant $\mathbf{x}^T\mathbf{L}\mathbf{x}$ for restoration?
- We answer these by viewing
  - discrete graph as **samples** of high-dimensional manifold.



discrete graph continuous manifold

Our Contributions

1. Using Gaussian kernel to compute graph weights, $S_G(\mathbf{x}) = \mathbf{x}^T \mathbf{L} \mathbf{x}$ converges to a continuous functional S_Ω, which can be interpreted as regularizer in continuous domain.



2. Analysis of functional S_Ω provides understanding of what signals are being discriminated and to what extent, on a point-by-point basis in the continuous domain.

3. We design a discriminant S_Ω for regularization in continuous domain, then obtain the graph Laplacian regularizer S_G

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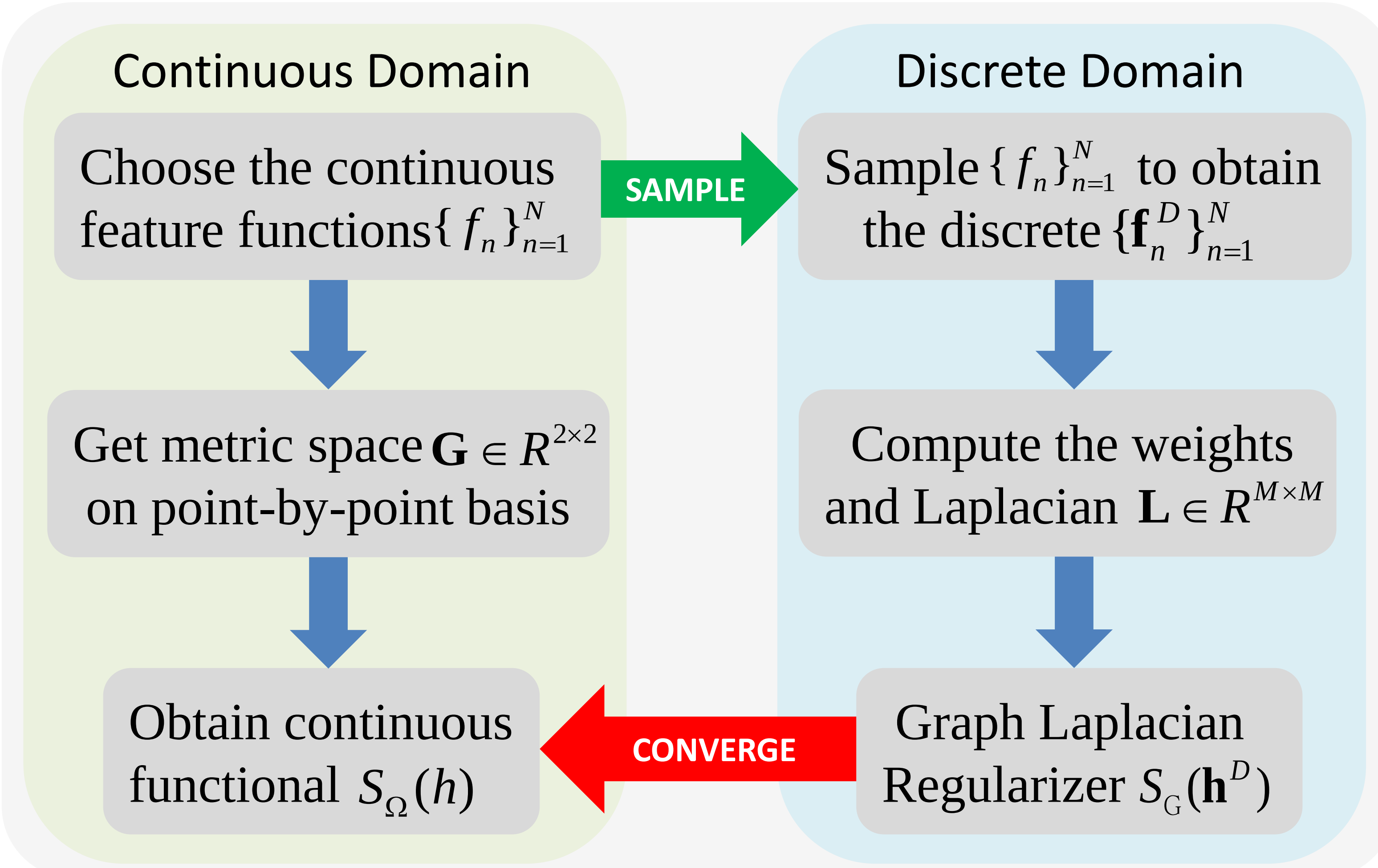
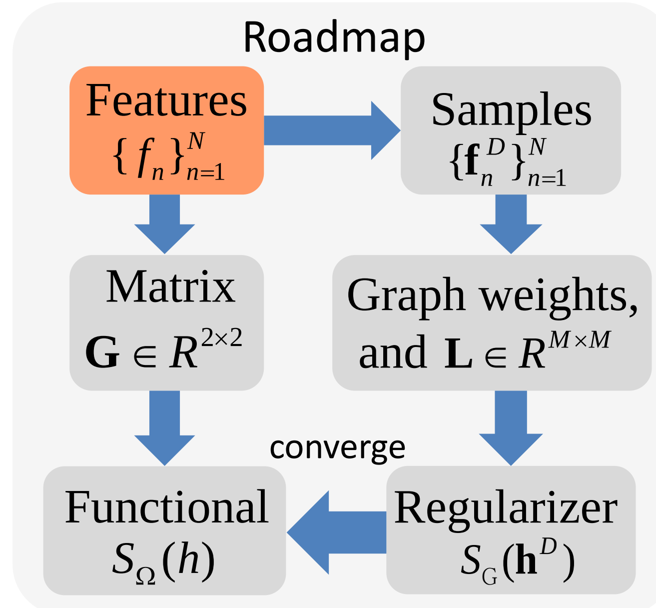
Road Map



- Different $\{f_n\}_{n=1}^N$ leads to different regularization behavior!

Graph Construction (I)

- First, define:
  - 2D domain $\Omega \subset R^2$
    — the shape of an image
  - $\Gamma = \left\{ \mathbf{s}_i = [x_i \ y_i]^{\mathrm{T}} \mid \mathbf{s}_i \in \Omega, 1 \le i \le M \right\}$
    — a set of M random samples uniformly distributed on Ω, construed as pixel locations

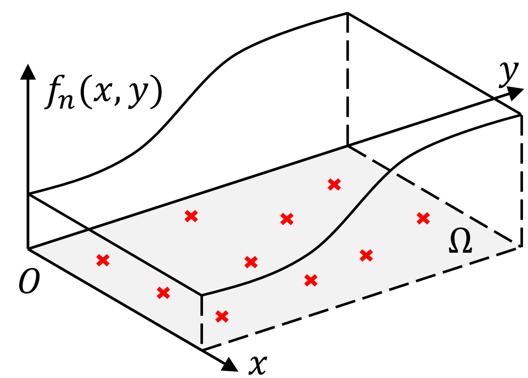


- (Freely) choose N continuous functions

$$f_n(x, y) : \Omega \to R, \ 1 \le n \le N$$

called feature functions, can be

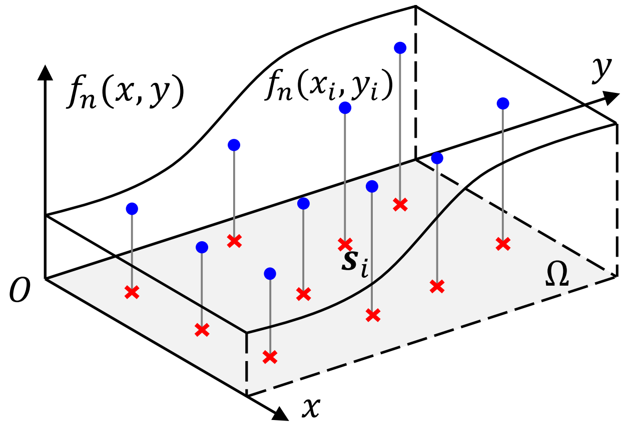
  - intensity for gray-scale image $(N = 1)$
  - R, G, B channels for color image $(N = 3)$

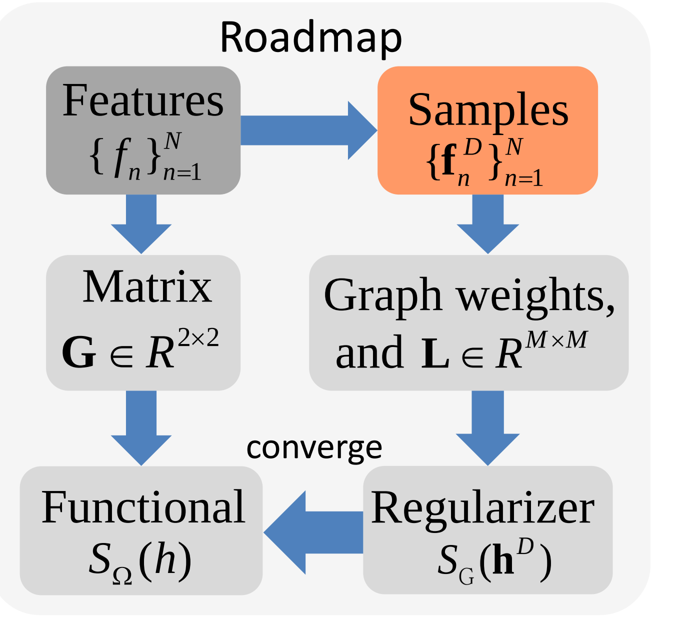


Graph Construction (II)

- Sampling f_n at positions in Γ gives N discretized feature functions

$$\mathbf{f}_n^D = [\, f_n(x_1, y_1)\ f_n(x_2, y_2) \ldots f_n(x_M, y_M)]^{\mathrm{T}}$$





- For each pixel location $\mathbf{s}_i \in \Gamma$, define a length $N+2$ vector

$$\mathbf{v}_i = [x_i\ y_i\ \beta \mathbf{f}_1^D(i)\ \beta \mathbf{f}_2^D(i) \ldots \beta \mathbf{f}_N^D(i)]^{\mathrm{T}}$$

β is a tunable constant

- Build a graph G with M vertices, each pixel location $\mathbf{s}_i \in \Gamma$ have a vertex V_i

Graph Construction (III)

- Weight between vertices V_i and V_j

degree before normalization

$$\rho_i = \sum_{j=1}^{m} \psi(d_{ij})$$

$$w_{ij} = (\rho_i \rho_j)^{-\gamma} \psi(d_{ij})$$

normalization factor γ

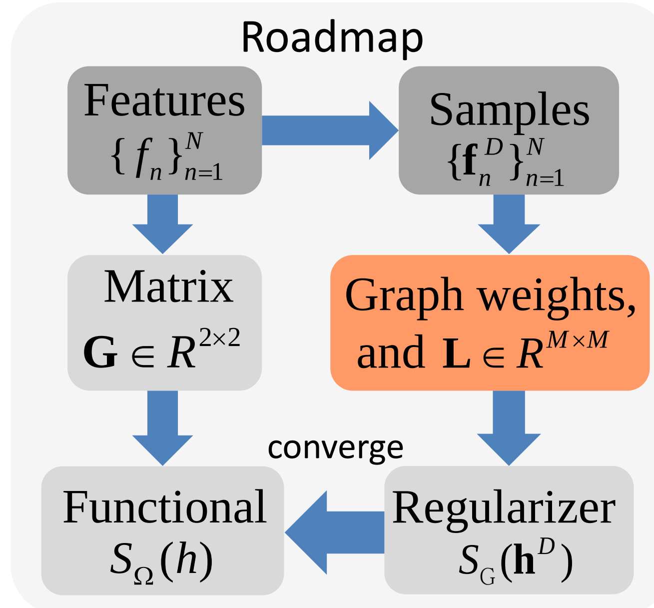
clipped Gaussian kernel

$$\psi(d) = \begin{cases} \exp\left(-\dfrac{d^2}{2\varepsilon^2}\right) & |d| \le r, \\ 0 & \text{otherwise} \end{cases}$$

where $r = \varepsilon C_r$ and C_r is a constant

distance $d_{ij}^2 = \left\| \mathbf{v}_i - \mathbf{v}_j \right\|_2^2$

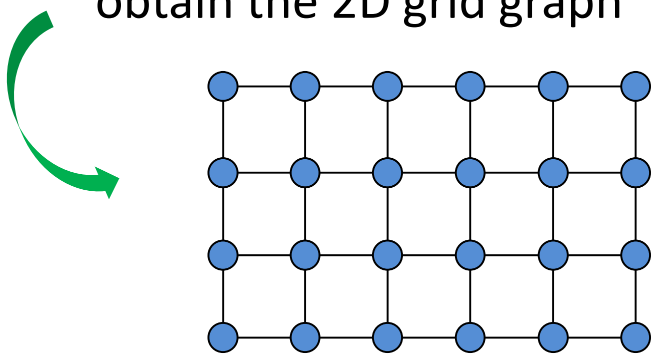
$$= \left\| \mathbf{s}_i - \mathbf{s}_j \right\|_2^2 + \beta^2 \sum_{n=1}^{N} (\mathbf{f}_n^D(i) - \mathbf{f}_n^D(j))^2$$

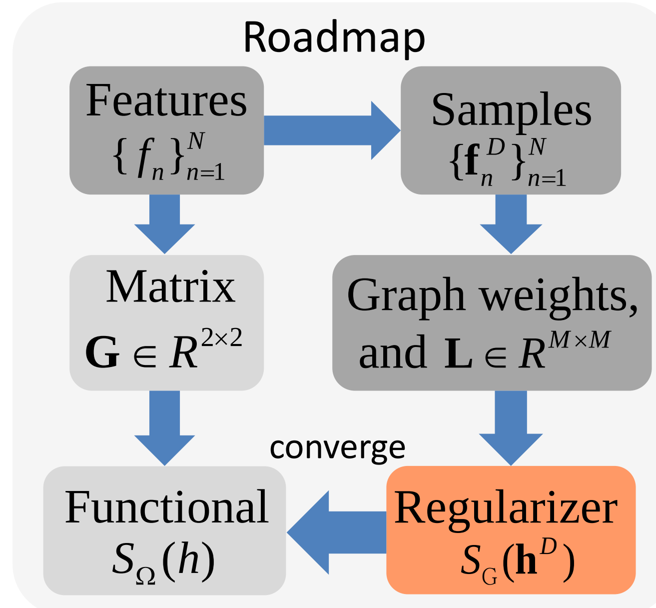


- G is an *r*-neighborhood graph, *i.e.*, no edge connecting two vertices with distance greater than r

Graph Construction (IV)

- Our graph G is very general
  - *e.g.*, choose a small β with proper r, obtain the 2D grid graph

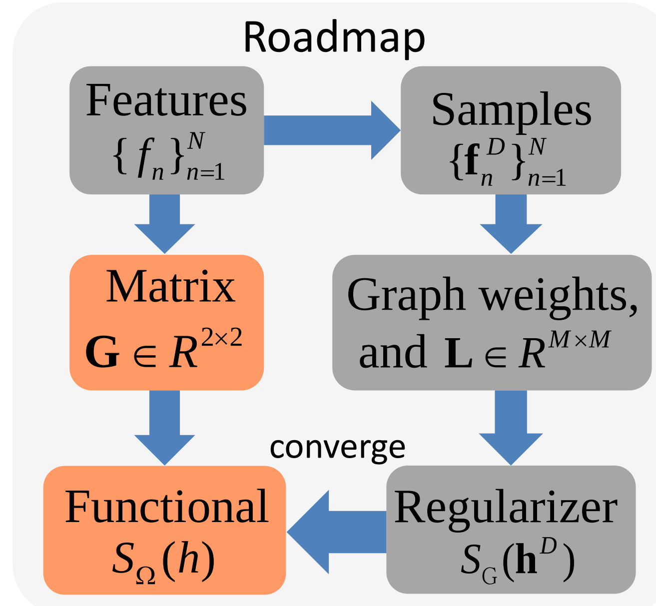




- $\mathbf{A}$ — its (i, j) entry is w_{ij}; $\mathbf{D}$ — its (i, j) entry is $\sum_{j=1}^{m} w_{ij}$ } unnormalized Graph Laplacian $\mathbf{L} = \mathbf{D} - \mathbf{A}$
- $h(x, y): \Omega \to R$ is a continuous **candidate function**

  $\mathbf{h}^D = [h(x_1, y_1)\; h(x_2, y_2) \ldots h(x_M, y_M)]^\mathrm{T}$ — samples of $h(x, y)$
- $S_\mathrm{G}(\mathbf{h}^D) = (\mathbf{h}^D)^\mathrm{T} \mathbf{L} \mathbf{h}^D$ — graph Laplacian regularizer, a functional on R^M

Convergence of the Graph Laplacian Regularizer (I)



- The continuous counterpart of S_G is a functional S_Ω on domain Ω

$$S_\Omega(h) = \iint_\Omega (\nabla h)^{\mathrm{T}} \mathbf{G}^{-1} (\nabla h) \left(\sqrt{\det \mathbf{G}}\right)^{2\gamma-1} dxdy$$

$\nabla h = [\partial_x h \;\; \partial_y h]^{\mathrm{T}}$ is the gradient of h

- $\mathbf{G}$ is a 2-by-2 matrix:

$$\mathbf{G} = \mathbf{I} + \beta^2 \begin{bmatrix} \sum_{n=1}^N (\partial_x f_n)^2 & \sum_{n=1}^N \partial_x f_n \cdot \partial_y f_n \\ \sum_{n=1}^N \partial_x f_n \cdot \partial_y f_n & \sum_{n=1}^N (\partial_y f_n)^2 \end{bmatrix} = \mathbf{I} + \beta^2 \sum_{n=1}^N \nabla f_n \cdot (\nabla f_n)^{\mathrm{T}}$$

2x2 identity matrix

Structure tensor [3] of the gradients $\{\nabla f_n(x,y)\}_{n=1}^N$

- $\mathbf{G}$ is computed from $\{\nabla f_n\}_{n=1}^N$ on a point-by-point basis

[3] H. Knutsson, C.-F. Westin, and M. Andersson, "**Representing local structure using tensors ii**," in *Image Analysis*. Springer, 2011, vol. 6688, pp. 545–556.

Convergence of the Graph Laplacian Regularizer (II)

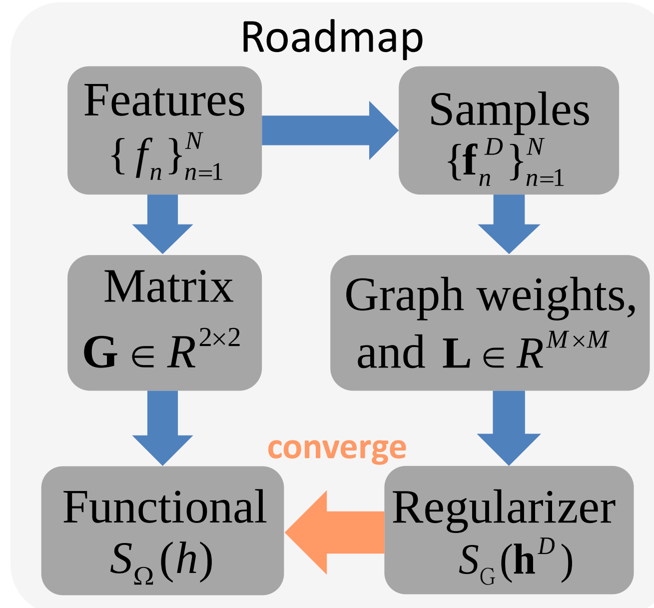
- ***Theorem :** convergence of* S_G *to* S_Ω

$$\lim_{\substack{M\to\infty \\ \varepsilon\to 0}} \frac{M^{2\gamma-1}}{\varepsilon^{4(1-\gamma)}(M-1)} S_G\left(\mathrm{h}^D\right) \sim S_\Omega(h)$$

number of samples M increases

neighborhood $r = \varepsilon C_r$ shrinks

"~" means there exist a constant such that equality holds.



- With results of [4], we proved it by viewing a graph as proxy of an $N+2$-dimensional Riemannian manifold

Vertex	Coordinate on **Ω**	Coordinate on (N+2)-D manifold
V_i	$\mathbf{s}_i = (x_i, y_i)$	$\mathbf{v}_i = [x_i \; y_i \; \beta\mathbf{f}_1^D(i) \; \beta\mathbf{f}_2^D(i) \ldots \beta\mathbf{f}_N^D(i)]^T$

[4] M. Hein, "**Uniform convergence of adaptive graph-based regularization**," in *Learning Theory*. Springer, 2006, pp. 50–64.

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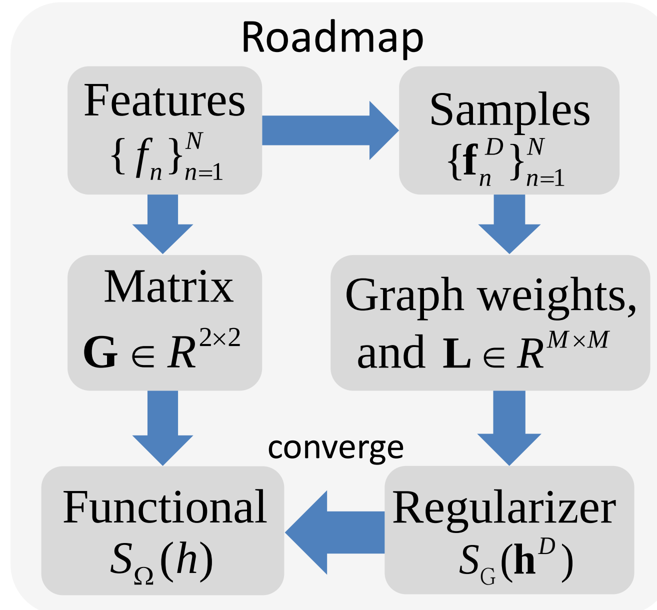
Justification of Graph Laplacian Regularizer (I)

$$S_\Omega(h) = \iint_\Omega (\nabla h)^{\mathrm{T}} \mathbf{G}^{-1} (\nabla h) \left(\sqrt{\det \mathbf{G}}\right)^{2\gamma-1} dxdy$$

$$\mathbf{G} = \mathbf{I} + \beta^2 \sum_{n=1}^{N} \nabla f_n \cdot (\nabla f_n)^{\mathrm{T}}$$

$$S_{\mathrm{G}}(\mathbf{h}^D) = (\mathbf{h}^D)^{\mathrm{T}} \mathbf{L} \mathbf{h}^D$$

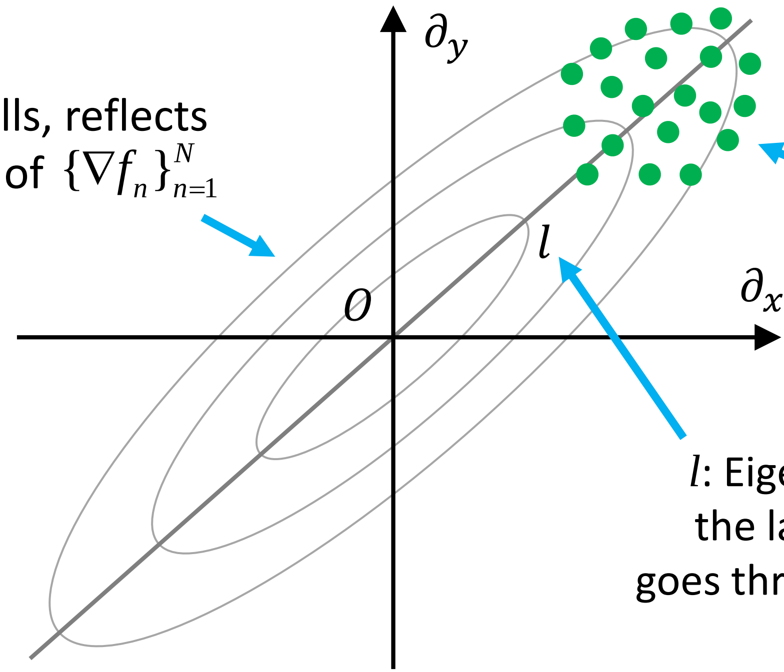


- S_{G} converges to S_Ω, With S_Ω, any new *insights* we can gain on S_{G} ??
    - The eigen-space of $\mathbf{G}$ reflects statistics of $\{\nabla f_n\}_{n=1}^N$
    - $(\nabla h)^{\mathrm{T}} \mathbf{G}^{-1} (\nabla h)$ measures length of ∇h in a metric space established by $\mathbf{G}$!
    - S_Ω integrates the gradient norm

Justification of Graph Laplacian Regularizer (II)

- Metric space defiend by $\mathbf{G}$

Ellipses are norm-balls, reflects how concentration of $\{\nabla f_n\}_{n=1}^{N}$



Green dots are $\{\nabla f_n(x, y)\}_{n=1}^{N}$

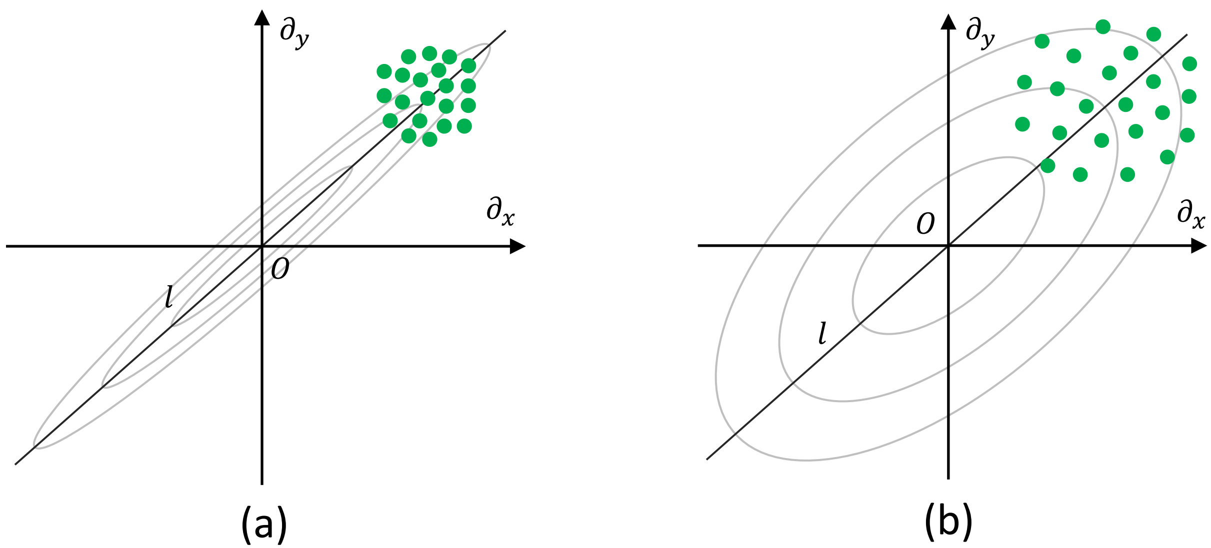
l: Eigenvector corresponds to the largest eigenvalue of $\mathbf{G}$, goes through the cluster of $\{\nabla f_n\}_{n=1}^{N}$

$$S_\Omega(h) = \iint_\Omega (\nabla h)^{\mathrm{T}} \mathbf{G}^{-1} (\nabla h) \left(\sqrt{\det \mathbf{G}}\right)^{2\gamma-1} dxdy$$

$$\mathbf{G} = \mathbf{I} + \beta^2 \sum_{n=1}^{N} \nabla f_n \cdot \left(\nabla f_n\right)^{\mathrm{T}}$$

Justification of Graph Laplacian Regularizer (III)

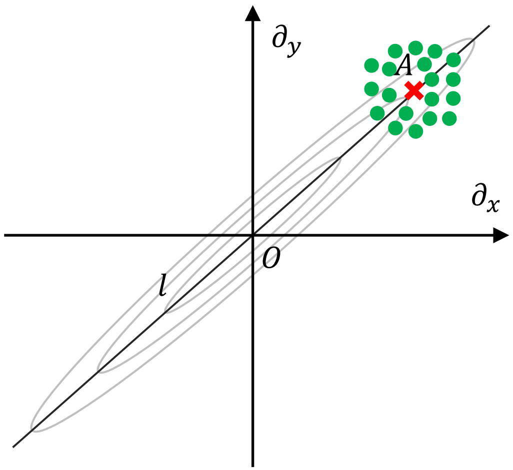
- *The 2D metric space provides a clear picture of what signals are being discriminated and to what extent, on a point-by-point basis in the continuous domain!*



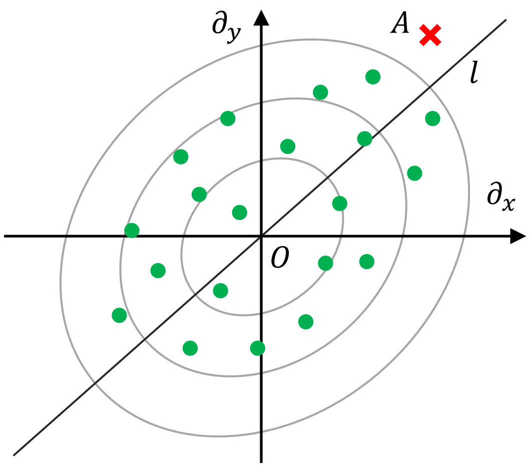
- (a) is more skewed, or discriminant, than (b)
- In (a), a small distance away from the direction orthogonal to l brings large metric distance

Justification of Graph Laplacian Regularizer (IV)

- **Lesson**: Select feature functions properly!
- Suppose A is the truth gradient, choose $\{f_n\}_{n=1}^{N}$ such that
  - (i) l goes through A; (ii) Ellipses stretched flat along l.



(a) A good scheme, $\{\nabla f_n\}_{n=1}^{N}$ are similar to the ground-truth A



(b) A bad scheme...

- For the case of discrete images, one can seek for similar patches in terms of gradient!

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Problem Formulation and Algorithm Development

- Adopt a patch-based recovery framework to denoise the image
- For a noisy patch $\mathbf{p}_0$ on the image
  1. Assume a "**self-similar-in-gradient**" image model, search for $K-1$ patches similar to $\mathbf{p}_0$ in terms of gradient in *pre-filtered* image.
  2. Compute graph Laplacian from the similar patches.
  3. Solve the unconstrained quadratic optimization iteratively:

$$\mathbf{q}^{\cdot} = \arg\min_{\mathbf{q}} \ \|\mathbf{p}_0 - \mathbf{q}\|_2^2 + \lambda \mathbf{q}^{\mathrm{T}} \mathbf{L} \mathbf{q}$$

  to obtain the denoised patch $\mathbf{q}^{\cdot}$

- Aggregate denoised patches to form an updated image.
- Denoise the given image iteratively to gradually enhance its quality.
- Our denoising method is named

  Graph-based Denoising using Gradient-based Self-similarity (GDGS)

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Experimental Results (I)

- Test images: *Lena*, *Barbara*, *Boats* and *Peppers*
- i.i.d. Additive White Gaussian Noise (AWGN)
- Non-Local GBT (NLGBT) – an existing graph-based denoising method [5]
- Compared to BF, NLM and NLGBT

1.4 dB better than NLM!

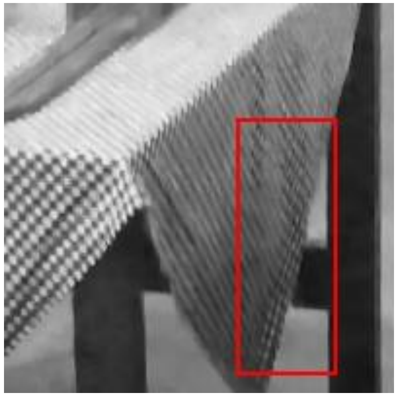
Performance comparisons in PSNR (dB)

Image	Method		Standard Deviation					
			15		20		30	
Lena	GDGS	NLM	**33.47**	32.03	**32.35**	31.51	**30.61**	29.45
	BF	NLGBT	27.00	33.22	24.80	31.90	21.52	30.19
Barbara	GDGS	NLM	**31.71**	30.76	**30.33**	30.15	**28.33**	27.91
	BF	NLGBT	25.78	31.22	23.86	29.62	21.03	27.67
Boats	GDGS	NLM	**31.59**	30.69	**30.30**	29.74	**28.55**	27.68
	BF	NLGBT	26.42	31.05	24.89	29.56	22.19	27.77
Peppers	GDGS	NLM	**33.30**	31.96	**32.38**	31.48	**30.83**	29.50
	BF	NLGBT	28.96	33.18	24.67	32.09	21.49	30.49

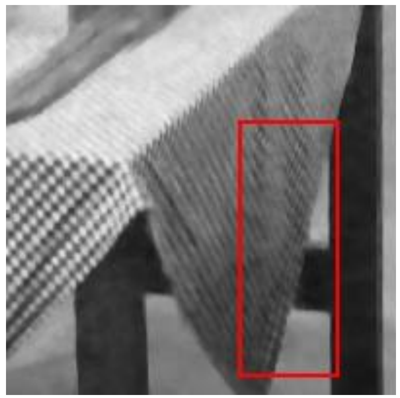
[5] W. Hu, X. Li, G. Cheung, and O. Au, "**Depth map denoising using graph-based transform and group sparsity**," in *IEEE Int'l Workshop on Multimedia Signal Processing*, 2013.

Experimental Results (II)

- GDGS vs NLGBT



GDGS



NLGBT

GDGS

NLGBT

- GDGS vs NLM

GDGS (31.39 dB)

NLM (30.38 dB)

GDGS (29.34 dB)

NLM (28.62 dB)

Noise standard deviation $\sigma = 25$

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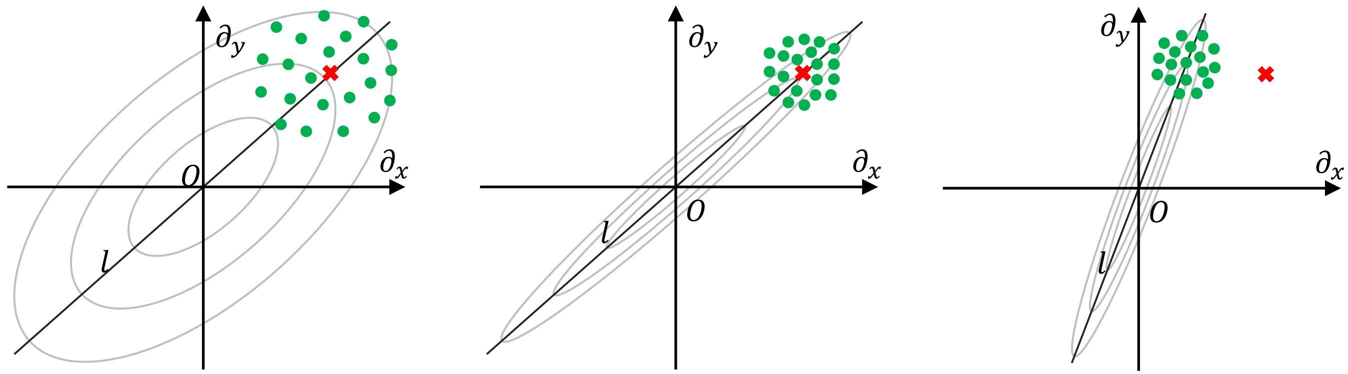
Towards Optimal Graph Laplacian Regularization

- Our latest work [6] derives the optimal metric space $\mathbf{G}^\cdot$, leading to optimal graph Laplacian regularization for denoising.



- *Metric space should be discriminant to the extent that estimates of ground-truth gradient are reliable.*

posterior prob. of ground truth

$$\mathbf{G}^\cdot = \arg\min_{\mathbf{G}} \iint_\Delta \left\|\mathbf{G} - \mathbf{G}_0(\mathbf{g})\right\|_F^2 Pr\left(\mathbf{g} \mid \{\mathbf{g}_k\}_{k=0}^{K-1}\right) d\mathbf{g}$$

Δ—whole gradient domain

ideal metric space given ground truth $\mathbf{g}$

[6] Jiahao Pang, Gene Cheung, Antonio Ortega, Oscar C. Au, "**Optimal Graph Laplacian Regularization for Natural Image Denoising**," submitted to *IEEE ICASSP*, Brisbane, Australia, April, 2015.

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Conclusion

- Image denoising is an ill-posed problem and requires good priors for regularization.
- graph Laplacian regularizer with Gaussian kernel weights converges to a continuous functional.
- Analysis of the continuous functional provides theoretical justification of why and under what conditions the graph Laplacian regularizer can be discriminant.
- Our denoising algorithm with graph Laplacian regularizer and gradient-based similarity out-performs NLM by up to 1.4 dB.
- Our latest work obtains the optimal graph Laplacian, which is discriminant when the estimates are accurate, and robust when the estimates are not.

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国立情報学研究所
National Institute of Informatics

Thank You!

Contact: Gene Cheung (cheung@nii.ac.jp)

Jiahao Pang (jpang@ust.hk)