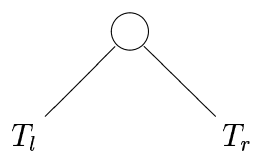
PROPOSITION 7.5. *Let $T$ be a binary tree with $n$ nodes and height $h$.*

1. *If the levels 0, 1, ..., $h$ have the maximum number of nodes then $n = 2^{h+1} - 1$.*
2. *If the levels 0, 1, ..., $h - 1$ have the maximum number of nodes and level $h$ has one nodes then $n = 2^h$.*
3. *If $T$ is complete then $2^h \leq n \leq 2^{h+1} - 1$.*
4. *If $T$ is complete then $h = \lfloor \log(n) \rfloor$.*

PROOF

1. We prove this proposition by structural induction on $T$. The base case, where $T$ consists of a single node, is trivial: $h = 0$ and $n = 1$. Next we consider the induction step. The binary tree $T$ is of the form




where the left subtree $T_l$ and the right subtree $T_r$ are both smaller than $T$. Note that $T_l$ and $T_r$ have height $h - 1$ and that both have the maximum number of nodes for levels 0, ..., $h - 1$. By the induction hypothesis, the number of nodes of $T_l$ and $T_r$ (denoted by $n_l$ and $n_r$) are both $2^{(h-1)+1} - 1 = 2^h - 1$. Therefore

$$\begin{aligned} n &= n_l + n_r + 1 \\ &= (2^h - 1) + (2^h - 1) + 1 \\ &= 2^{h+1} - 1. \end{aligned}$$

2. Immediate consequence of 1.

3. Immediate consequence of 1. and 2.

4. According to 3.,

$$\begin{aligned} & 2^h \leq n \\ & \Rightarrow \; 2^h \leq n \\ & \Rightarrow \; h \leq \log(n). \end{aligned} \tag{1}$$

According to 3.,

$$\begin{aligned} & 2^{h+1} - 1 \geq n \\ & \Rightarrow \; 2^{h+1} \geq n + 1 \\ & \Rightarrow \; h + 1 \geq \log(n + 1) \\ & \Rightarrow \; h \geq \log(n + 1) - 1 \\ & \Rightarrow \; h > \log(n) - 1 \quad [\log(n + 1) > \log(n)]. \end{aligned} \tag{2}$$

Combining (1) and (2) we get $h = \lfloor \log(n) \rfloor$.

□