York University EECS 4115A/5115A Eric Ruppert October 3, 2025

Homework Assignment #3
Due: October 10, 2025 at 5:00 p.m.

1. Consider the COLOURABILITY decision problem:

INPUT: A graph G and an integer k.
OUTPUT: YES if it is possible to colour the nodes of G with k colours so that no two nodes connected by an edge get the same colour; NO otherwise.

This question looks at the relationship between the decision problem and other problems about colouring. In both of the following parts, you should explain why the algorithm you design works and why it runs in polynomial time.

[3] **(a)** Show that if COLOURABILITY is solvable in polynomial time, then there is a polynomial-time algorithm that finds the minimum number of colours needed to colour a given graph.

[6] **(b)** Show that if COLOURABILITY is solvable in polynomial time, then there is a polynomial-time algorithm to *find* a colouring of a given graph using the minimum possible number of colours.

Hint: Suppose G is colourable with k colours. Let G' be G with one extra edge (u, v) added. If G' is still colourable with k colours, what does that tell you? If G' is not colourable with k colours, what does that tell you?

2. Consider the SURFACES problem.

INPUT: A collection of equalities involving variables $x_1, \ldots, x_m$, each of the form $x_i^2 + x_j^2 = c$ (where c is a natural number) or $x_i \cdot x_j \cdot x_k = c$ (where c is an integer).
OUTPUT: YES if there are integer values for the variables $x_1, \ldots, x_m$ that satisfy all of the equalities; or NO otherwise.

[1] **(a)** Show that if there is a solution that satisfies all the equalities, then there is a solution that uses only integers between $-C$ and C, where C is the largest constant that appears on the right hand side of any of the equalities.

[2] **(b)** Show that SURFACES is in NP.
Hint: Use part (a).

[5] **(c)** Show that SURFACES is NP-complete.
Hint: focus on equalities that have just a few simple constants on the right hand sides.

> This comment is not useful for answering this question, but it explains the name of this problem. If an instance of SURFACES just has three variables, each equality defines a surface in 3-dimensional space, and the problem is to find points with integer coordinates that are on the intersection of all the surfaces. The first type of equality defines a cylinder. There is a picture of the other type of surface at `https://mathcurve.com/surfaces.gb/titeica/titeica.shtml` . In general, the equalities define hypersurfaces in m-dimensional space.