Solutions to the Exercises in
An Introduction to Metric Semantics: Operational and Denotational Models for Programming and Specification Languages

Franck van Breugel

Technical Report CS-2000-02

August 1, 2000

Department of Computer Science

4700 Keele Street, Toronto, Ontario M3J 1P3, Canada

Solutions to the Exercises in

An Introduction to Metric Semantics:

Operational and Denotational Models for Programming and Specification Languages

Franck van Breugel

York University, Department of Computer Science
4700 Keele Street, Toronto, Canada M3J 1P3

franck@cs.yorku.ca

Abstract

This report contains solutions to the exercises in [Bre].

Solutions

Solution to Exercise 5 We define

$$\mathcal{X} = \{\, X \subseteq (Stat_{\mathrm{E}} \times \Sigma) \times \Sigma \times (Stat_{\mathrm{E}} \times \Sigma) \mid X \text{ satisfies the axioms and rules of Definition 4} \,\}.$$

Next, we show that $\bigcap \mathcal{X}$ also satisfies the axioms and rules of Definition 4. Obviously, this is the smallest set satisfying them.

Since every $X \in \mathcal{X}$ satisfies the axioms (1) and (2), that is, for all $v \in Var$, $e \in Exp$, and $\varsigma \in \Sigma$, $\langle [v := e,\, \varsigma], \varsigma\{n/v\}, [\mathrm{E},\, \varsigma\{n/v\}] \rangle \in X$, where $n = \mathcal{E}\,(e)(\varsigma)$, and $\langle [\mathsf{skip},\, \varsigma], \varsigma, [\mathrm{E},\, \varsigma] \rangle \in X$, also $\bigcap \mathcal{X}$ satisfies them.

$\bigcap \mathcal{X}$ also satisfies the rules (3) and (4). For example, assume that $\langle [s_1,\, \varsigma], \varsigma', [\mathrm{E},\, \varsigma''] \rangle \in \bigcap \mathcal{X}$ for some $s_1 \in Stat$ and $\varsigma, \varsigma', \varsigma'' \in \Sigma$. Then, $\langle [s_1,\, \varsigma], \varsigma', [\mathrm{E},\, \varsigma''] \rangle \in X$ for all $X \in \mathcal{X}$. Since X satisfies rule (3), we can conclude that $\langle [s_1 \,;\, s_2,\, \varsigma], \varsigma', [s_2,\, \varsigma''] \rangle \in X$. Hence, $\langle [s_1 \,;\, s_2,\, \varsigma], \varsigma', [s_2,\, \varsigma''] \rangle \in \bigcap \mathcal{X}$. ⌟

Solution to Exercise 7 We prove (1) by induction on the proof of

$$[\bar{s},\, \varsigma] \xrightarrow{\varsigma'} [\bar{s}',\, \varsigma''].$$

We consider only one typical case. Assume the proof is of the form

$$\frac{\begin{array}{c}\vdots\\ [s_1,\, \varsigma] \xrightarrow{\varsigma'} [s_1',\, \varsigma'']\end{array}}{[s_1 \,;\, s_2,\, \varsigma] \xrightarrow{\varsigma'} [s_1' \,;\, s_2,\, \varsigma'']}$$

By induction, $\varsigma' = \varsigma''$.

The implication from right to left of (2) follows immediately from the fact that there is no axiom or rule for the empty statement E. The other implication is proved by showing that if $\bar{s} \neq \mathrm{E}$ then $[\bar{s},\, \varsigma] \rightarrow$. This is shown by structural induction on $\bar{s}$. For example, assume that $\bar{s} = \mathsf{if}\ b\ \mathsf{then}\ s_1\ \mathsf{else}\ s_2\ \mathsf{fi}$ and suppose that $\mathcal{B}\,(b)(\varsigma) = true$. By induction, $[s_1,\, \varsigma] \rightarrow$. Hence, we can conclude that $[\mathsf{if}\ b\ \mathsf{then}\ s_1\ \mathsf{else}\ s_2\ \mathsf{fi},\, \varsigma] \rightarrow$. ⌟

Solution to Exercise 10 We prove (1) by induction on the proof. First observe that for all $s \in Stat$, $comp\,(s) > 0$. We only consider two typical cases.

* Let the proof be of the form

$$\dfrac{\begin{array}{c}\vdots\\ [s_1,\,\varsigma] \xrightarrow{\varsigma'} [\mathrm{E},\,\varsigma']\end{array}}{[s_1\,;\,s_2,\,\varsigma] \xrightarrow{\varsigma'} [s_2,\,\varsigma']}$$

  Then

$$\begin{array}{ll} \multicolumn{2}{l}{\mathit{comp}\,(s_1\,;\,s_2)} \\ = & \mathit{comp}\,(s_1) + \mathit{comp}\,(s_2) \\ > & \mathit{comp}\,(s_2) \quad [\mathit{comp}\,(s_1) > 0] \end{array}$$

* Assume the proof is of the form

$$\dfrac{\begin{array}{c}\vdots\\ [s_1,\,\varsigma] \xrightarrow{\varsigma'} [s_1',\,\varsigma']\end{array}}{[s_1\,;\,s_2,\,\varsigma] \xrightarrow{\varsigma'} [s_1'\,;\,s_2,\,\varsigma']}$$

  In this case,

$$\begin{array}{ll} \multicolumn{2}{l}{\mathit{comp}\,(s_1\,;\,s_2)} \\ = & \mathit{comp}\,(s_1) + \mathit{comp}\,(s_2) \\ > & \mathit{comp}\,(s_1') + \mathit{comp}\,(s_2) \quad [\text{induction}] \\ = & \mathit{comp}\,(s_1'\,;\,s_2). \end{array}$$

┘

SOLUTION TO EXERCISE 14 We define the function $\mathcal{A} : \mathit{Stat} \to \Sigma \to \Sigma$ by

$$\mathcal{A}\,(s)(\varsigma) = \varsigma' \text{ if } [s,\,\varsigma] = [\bar{s}_0,\,\varsigma_0] \xrightarrow{\varsigma_1} [\bar{s}_1,\,\varsigma_1] \xrightarrow{\varsigma_2} \cdots \xrightarrow{\varsigma_n} [\mathrm{E},\,\varsigma_n] = [\mathrm{E},\,\varsigma'].$$

From Exercise 7(2) we can conclude that for all $s \in \mathit{Stat}$ and $\varsigma \in \Sigma$, $\mathcal{O}\,(s)(\varsigma) \in \Sigma^+$. To link $\mathcal{O}$ and $\mathcal{A}$ we introduce the function $\mathit{last} : \Sigma^+ \to \Sigma$ defined by

$$\mathit{last}\,(\sigma) = \begin{cases} \varsigma & \text{if } \sigma = \varsigma \\ \mathit{last}\,(\sigma') & \text{if } \sigma = \varsigma\sigma'. \end{cases}$$

From the definitions of $\mathcal{O}$ and $\mathcal{A}$ immediately follows that for all $s \in \mathit{Stat}$ and $\varsigma \in \Sigma$,

$$\mathit{last}\,(\mathcal{O}\,(s)(\varsigma)) = \mathcal{A}\,(s)(\varsigma).$$

┘

SOLUTION TO EXERCISE 20 Let $\bar{s} = s_1\,;\,s_2$ and assume that $[s_1,\,\varsigma] \xrightarrow{\varsigma'} [s_1',\,\varsigma']$. In this case,

$$\begin{array}{ll} \multicolumn{2}{l}{\mathcal{O}\,([s_1\,;\,s_2,\,\varsigma])} \\ = & \varsigma'\,\mathcal{O}\,([s_1'\,;\,s_2,\,\varsigma']) \quad [\text{Proposition 12}] \\ = & \varsigma'\,\mathcal{D}\,([s_1'\,;\,s_2,\,\varsigma']) \quad [\text{induction}] \\ = & \varsigma'\,(\mathcal{D}\,([s_1',\,\varsigma'])\,;_{\varsigma'}\,\mathcal{D}\,(s_2)) \\ = & (\varsigma'\,\mathcal{D}\,([s_1',\,\varsigma']))\,;_{\varsigma}\,\mathcal{D}\,(s_2) \\ = & (\varsigma'\,\mathcal{O}\,([s_1',\,\varsigma']))\,;_{\varsigma}\,\mathcal{D}\,(s_2) \quad [\text{induction}] \\ = & \mathcal{O}\,([s_1,\,\varsigma])\,;_{\varsigma}\,\mathcal{D}\,(s_2) \quad [\text{Proposition 12}] \\ = & \mathcal{D}\,([s_1,\,\varsigma])\,;_{\varsigma}\,\mathcal{D}\,(s_2) \quad [\text{induction}] \\ = & \mathcal{D}\,([s_1\,;\,s_2,\,\varsigma]). \end{array}$$

┘

Solution to Exercise 24 We show that a transition can be proved by (1)–(5) if and only if it can be proved by (1)–(4), (6).

Assume that a transition can be proved by (1)–(5). Then there also exists a proof of the transition using (1)–(4), (6) as we show next by induction on the proof. We distinguish the following three cases.

* Let $\mathcal{B}(b) = \mathit{true}$ and consider the proof

$$\frac{\begin{array}{c}\vdots\\ [s,\ \varsigma] \xrightarrow{\varsigma'} [\mathrm{E},\ \varsigma'']\end{array}}{[\mathsf{while}\ b\ \mathsf{do}\ s\ \mathsf{od},\ \varsigma] \xrightarrow{\varsigma'} [\mathsf{while}\ b\ \mathsf{do}\ s\ \mathsf{od},\ \varsigma'']}$$

  By induction,

$$\frac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ [s,\ \varsigma] \xrightarrow{\varsigma'} [\mathrm{E},\ \varsigma'']\end{array}}{[s\ ;\ \mathsf{while}\ b\ \mathsf{do}\ s\ \mathsf{od},\ \varsigma] \xrightarrow{\varsigma'} [\mathsf{while}\ b\ \mathsf{do}\ s\ \mathsf{od},\ \varsigma'']}}{[\mathsf{if}\ b\ \mathsf{then}\ s\ ;\ \mathsf{while}\ b\ \mathsf{do}\ s\ \mathsf{od}\ \mathsf{else}\ \mathsf{skip}\ \mathsf{fi},\ \varsigma] \xrightarrow{\varsigma'} [\mathsf{while}\ b\ \mathsf{do}\ s\ \mathsf{od},\ \varsigma'']}}{[\mathsf{while}\ b\ \mathsf{do}\ s\ \mathsf{od},\ \varsigma] \xrightarrow{\varsigma'} [\mathsf{while}\ b\ \mathsf{do}\ s\ \mathsf{od},\ \varsigma'']}$$

* Let $\mathcal{B}(b) = \mathit{true}$ and consider the proof

$$\frac{\begin{array}{c}\vdots\\ [s,\ \varsigma] \xrightarrow{\varsigma'} [s',\ \varsigma'']\end{array}}{[\mathsf{while}\ b\ \mathsf{do}\ s\ \mathsf{od},\ \varsigma] \xrightarrow{\varsigma'} [s'\ ;\ \mathsf{while}\ b\ \mathsf{do}\ s\ \mathsf{od},\ \varsigma'']}$$

  By induction,

$$\frac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ [s,\ \varsigma] \xrightarrow{\varsigma'} [s',\ \varsigma'']\end{array}}{[s\ ;\ \mathsf{while}\ b\ \mathsf{do}\ s\ \mathsf{od},\ \varsigma] \xrightarrow{\varsigma'} [s'\ ;\ \mathsf{while}\ b\ \mathsf{do}\ s\ \mathsf{od},\ \varsigma'']}}{[\mathsf{if}\ b\ \mathsf{then}\ s\ ;\ \mathsf{while}\ b\ \mathsf{do}\ s\ \mathsf{od}\ \mathsf{else}\ \mathsf{skip}\ \mathsf{fi},\ \varsigma] \xrightarrow{\varsigma'} [s'\ ;\ \mathsf{while}\ b\ \mathsf{do}\ s\ \mathsf{od},\ \varsigma'']}}{[\mathsf{while}\ b\ \mathsf{do}\ s\ \mathsf{od},\ \varsigma] \xrightarrow{\varsigma'} [s'\ ;\ \mathsf{while}\ b\ \mathsf{do}\ s\ \mathsf{od},\ \varsigma'']}$$

* Let $\mathcal{B}(b) = \mathit{false}$ and consider the proof

$$[\mathsf{while}\ b\ \mathsf{do}\ s\ \mathsf{od},\ \varsigma] \xrightarrow{\varsigma} [\mathrm{E},\ \varsigma]$$

  In this case we have the corresponding proof

$$\frac{\dfrac{[\mathsf{skip},\ \varsigma] \xrightarrow{\varsigma} [\mathrm{E},\ \varsigma]}{[\mathsf{if}\ b\ \mathsf{then}\ s\ ;\ \mathsf{while}\ b\ \mathsf{do}\ s\ \mathsf{od}\ \mathsf{else}\ \mathsf{skip}\ \mathsf{fi},\ \varsigma] \xrightarrow{\varsigma} [\mathrm{E},\ \varsigma]}}{[\mathsf{while}\ b\ \mathsf{do}\ s\ \mathsf{od},\ \varsigma] \xrightarrow{\varsigma} [\mathrm{E},\ \varsigma]}$$

The other implication can be proved similarly. For example, assume that $\mathcal{B}(b) = \mathit{false}$ and consider the proof

$$\frac{\vdots}{\dfrac{[\mathsf{if}\ b\ \mathsf{then}\ s\ ;\ \mathsf{while}\ b\ \mathsf{do}\ s\ \mathsf{od}\ \mathsf{else}\ \mathsf{skip}\ \mathsf{fi},\ \varsigma] \xrightarrow{\varsigma'} [\bar{s},\ \varsigma'']}{[\mathsf{while}\ b\ \mathsf{do}\ s\ \mathsf{od},\ \varsigma] \xrightarrow{\varsigma'} [\bar{s},\ \varsigma'']}}$$

From (1)–(4), (6) we can deduce that this proof is of the form

$$\dfrac{\dfrac{[\mathsf{skip},\ \varsigma] \xrightarrow{\varsigma} [\mathrm{E},\ \varsigma]}{[\mathsf{if}\ b\ \mathsf{then}\ s\ ;\ \mathsf{while}\ b\ \mathsf{do}\ s\ \mathsf{od}\ \mathsf{else}\ \mathsf{skip}\ \mathsf{fi},\ \varsigma] \xrightarrow{\varsigma} [\mathrm{E},\ \varsigma]}}{[\mathsf{while}\ b\ \mathsf{do}\ s\ \mathsf{od},\ \varsigma] \xrightarrow{\varsigma} [\mathrm{E},\ \varsigma]}$$

Obviously, we can also prove this transition by means of (1)-(5). ⌟

SOLUTION TO EXERCISE 33 Let $(x_n)_n$ and $(y_n)_n$ be converging sequences in a metric space X. Let $\epsilon \geq 0$. Assume that for all $n \in \mathbb{N}$, $d_X(x_n, y_n) \leq \epsilon$. To conclude that $d_X(\lim_n x_n, \lim_n y_n) \leq \epsilon$, it suffices to show that for all $\delta > 0$, $d_X(\lim_n x_n, \lim_n y_n) \leq \epsilon + \delta$. Let $\delta > 0$. We have that

$$\begin{array}{l} \exists M \in \mathbb{N} : \forall m \geq M : d_X(x_m, \lim_n x_n) \leq \frac{\delta}{2} \\ \exists N \in \mathbb{N} : \forall n \geq N : d_X(y_n, \lim_n y_n) \leq \frac{\delta}{2}. \end{array}$$

Consequently,

$$\begin{array}{ll} & d_X(\lim\limits_n x_n, \lim\limits_n y_n) \\ = & d_X(\lim\limits_n x_n, x_{\max\{M,N\}}) + d_X(x_{\max\{M,N\}}, y_{\max\{M,N\}}) + d_X(y_{\max\{M,N\}}, \lim\limits_n y_n) \\ = & \frac{\delta}{2} + \epsilon + \frac{\delta}{2}. \end{array}$$

Next, we prove Proposition 32. A function $\phi : \Sigma \times \Sigma^\infty \times (\Sigma \to \Sigma^\infty) \to \Sigma^\infty$ satisfies the property $P(\phi)$ if for all $\varsigma \in \mathit{State}$, $\sigma \in \Sigma^\infty$, and $f_1,\ f_2 \in \Sigma \to \Sigma^\infty$,

$$d(\phi(\varsigma, \sigma, f_1), \phi(\varsigma, \sigma, f_2)) \leq \begin{cases} d(f_1, f_2) & \text{if } \sigma = \varepsilon \\ \frac{1}{2} \cdot d(f_1, f_2) & \text{otherwise.} \end{cases}$$

In order to prove $P(;)$ we exploit Banach's theorem and the above fact. Let

$$\phi_n = \begin{cases} \lambda\langle\varsigma, \sigma, f\rangle.\varepsilon & \text{if } n = 0 \\ \Phi(\phi_{n-1}) & \text{otherwise.} \end{cases}$$

According to Banach's theorem and the above fact, it suffices to show that for all $n \in \mathbb{N}$, $P(\phi_n)$ holds. This is shown by induction on n. Obviously, $P(\phi_0)$ is valid. Assume $P(\phi_n)$ holds. In order to verify that $P(\phi_{n+1})$ is satisfied, we distinguish the following two cases.

* Let $\sigma = \varepsilon$. Then

$$\begin{array}{ll} & d(\phi_{n+1}(\varsigma, \varepsilon, f_1), \phi_{n+1}(\varsigma, \varepsilon, , f_2)) \\ = & d(\Phi(\phi_n)(\varsigma, \varepsilon, f_1), \Phi(\phi_n)(\varsigma, \varepsilon, f_2)) \\ = & d(f_1(\varsigma), f_2(\varsigma)) \\ \leq & d(f_1, f_2). \end{array}$$

* If $\sigma = \varsigma'\sigma'$ then

$$\begin{array}{ll} & d(\phi_{n+1}(\varsigma, \varsigma'\sigma', f_1), \phi_{n+1}(\varsigma, \varsigma'\sigma', f_2)) \\ = & d(\Phi(\phi_n)(\varsigma, \varsigma'\sigma', f_1), \Phi(\phi_n)(\varsigma, \varsigma'\sigma', f_2)) \\ = & d(\varsigma'\, \phi_n(\varsigma', \sigma', f_1), \varsigma'\, \phi_n(\varsigma', \sigma', f_2)) \\ \leq & \frac{1}{2} \cdot d(\phi_n(\varsigma', \sigma', f_1), \phi_n(\varsigma', \sigma', f_2)) \quad \text{[Example 120(3)]} \\ \leq & \frac{1}{2} \cdot d(f_1, f_2) \quad \text{[induction]} \end{array}$$

⌟

SOLUTION TO EXERCISE 37 We prove this fact by structural induction on s. We only consider the two most important cases.

* Let $s = s_1 \,;\, s_2$. We have that

$$\mathcal{D}\,(s_1 \,;\, s_2)(\varsigma) = \mathcal{D}\,(s_1)(\varsigma) \,;_\varsigma\, \mathcal{D}\,(s_2).$$

  By induction, $\mathcal{D}\,(s_1)(\varsigma) \neq \varepsilon$. From the definition of the semantic sequential composition we can conclude that $\mathcal{D}\,(s_1 \,;\, s_2)(\varsigma) \neq \varepsilon$.

* For $\mathsf{while}\ b\ \mathsf{do}\ s\ \mathsf{od}$ and $\mathcal{B}\,(b)(\varsigma) = \mathit{true}$ we have that

$$\begin{array}{cl} & \mathcal{D}\,(\mathsf{while}\ b\ \mathsf{do}\ s\ \mathsf{od})(\varsigma) \\ = & \mathit{fix}\,(\Psi\,(\mathcal{B}\,(b), \mathcal{D}\,(s)))(\varsigma) \\ = & \mathcal{D}\,(s)(\varsigma) \,;_\varsigma\, \mathit{fix}\,(\Psi\,(\mathcal{B}\,(b), \mathcal{D}\,(s))). \end{array}$$

  The rest of the proof is similar to the previous case. If $\mathcal{B}\,(b)(\varsigma) = \mathit{true}$ then

$$\begin{array}{cl} & \mathcal{D}\,(\mathsf{while}\ b\ \mathsf{do}\ s\ \mathsf{od})(\varsigma) \\ = & \mathit{fix}\,(\Psi\,(\mathcal{B}\,(b), \mathcal{D}\,(s)))(\varsigma) \\ = & \varsigma, \end{array}$$

  which is a nonempty sequence.

⌟

SOLUTION TO EXERCISE 47 We distinguish the following cases.

* Let $\bar{s} = \mathrm{E}$. Obviously, the set $\mathcal{S}\,([\mathrm{E},\, \varsigma]) = \emptyset$ is finite.

* Let $\bar{s} = v := e$ and assume $n = \mathcal{E}\,(e)(\varsigma)$. Clearly, also the set $\mathcal{S}\,([v := e,\, \varsigma]) = \{\langle \varsigma\{{}^n\!/\!_v\}, [\mathrm{E},\, \varsigma\{{}^n\!/\!_v\}]\rangle\}$ is finite.

* If $\bar{s} = \mathsf{skip}$ then $\mathcal{S}\,([\mathsf{skip},\, \varsigma]) = \{\langle \varsigma, [\mathrm{E},\, \varsigma]\rangle\}$. This is a finite set.

* Let $\bar{s} = s_1 \,;\, s_2$. Then

$$\begin{array}{rll} \mathcal{S}\,([s_1 \,;\, s_2,\, \varsigma]) = & \{\, \langle \varsigma', [s_2,\, \varsigma']\rangle & \mid \langle \varsigma', [\mathrm{E},\, \varsigma']\rangle \in \mathcal{S}\,([s_1,\, \varsigma])\,\} \cup \\ & \{\, \langle \varsigma', [s_1' \,;\, s_2,\, \varsigma']\rangle & \mid \langle \varsigma', [s_1',\, \varsigma']\rangle \in \mathcal{S}\,([s_1,\, \varsigma])\,\}. \end{array}$$

  By induction, the set $\mathcal{S}\,([s_1,\, \varsigma])$ is finite. Consequently, also the above set is finite.

* Let $\bar{s} = \mathsf{if}\ b\ \mathsf{then}\ s_1\ \mathsf{else}\ s_2\ \mathsf{fi}$ and suppose $\mathcal{B}\,(b)(\varsigma) = \mathit{true}$. Then

$$\mathcal{S}\,([\mathsf{if}\ b\ \mathsf{then}\ s_1\ \mathsf{else}\ s_2\ \mathsf{fi},\, \varsigma]) = \mathcal{S}\,([s_1,\, \varsigma]).$$

  By induction, $\mathcal{S}\,([s_1,\, \varsigma])$ is a finite set.

* Let $\bar{s} = \mathsf{if}\ b\ \mathsf{then}\ s_1\ \mathsf{else}\ s_2\ \mathsf{fi}$ and let $\mathcal{B}\,(b)(\varsigma) = \mathit{false}$. Similar to the previous case.

* Let $\bar{s} = \mathsf{while}\ b\ \mathsf{do}\ s\ \mathsf{od}$ and assume $\mathcal{B}\,(b)(\varsigma) = \mathit{true}$. In this case, we have that

$$\begin{array}{rll} \mathcal{S}\,([\mathsf{while}\ b\ \mathsf{do}\ s\ \mathsf{od},\, \varsigma]) = & \{\, \langle \varsigma', [\mathsf{while}\ b\ \mathsf{do}\ s\ \mathsf{od},\, \varsigma']\rangle & \mid \langle \varsigma', [\mathrm{E},\, \varsigma']\rangle \in \mathcal{S}\,([s,\, \varsigma])\,\} \cup \\ & \{\, \langle \varsigma', [s' \,;\, \mathsf{while}\ b\ \mathsf{do}\ s\ \mathsf{od},\, \varsigma']\rangle & \mid \langle \varsigma', [s',\, \varsigma']\rangle \in \mathcal{S}\,([s,\, \varsigma])\,\}. \end{array}$$

  Since the set $\mathcal{S}\,([s,\, \varsigma])$ is finite by induction, the above set is also finite.

* If $\bar{s} = \mathsf{while}\ b\ \mathsf{do}\ s\ \mathsf{od}$ and $\mathcal{B}(b)(\varsigma) = \mathit{false}$ then

$$\mathcal{S}([\mathsf{while}\ b\ \mathsf{do}\ s\ \mathsf{od},\ \varsigma]) = \{\langle \varsigma, [\mathrm{E}, \varsigma] \rangle\}.$$

Obviously, this set is finite.

* Let $\bar{s} = s_1 \parallel s_2$. Then

$$\begin{aligned}\mathcal{S}([s_1 \parallel s_2, \varsigma]) = & \{ \langle \varsigma', [s_2, \varsigma'] \rangle & \mid \langle \varsigma', [\mathrm{E}, \varsigma'] \rangle \in \mathcal{S}([s_1, \varsigma]) \} \cup \\ & \{ \langle \varsigma', [s_1, \varsigma'] \rangle & \mid \langle \varsigma', [\mathrm{E}, \varsigma'] \rangle \in \mathcal{S}([s_2, \varsigma]) \} \cup \\ & \{ \langle \varsigma', [s_1' \parallel s_2, \varsigma'] \rangle & \mid \langle \varsigma', [s_1', \varsigma'] \rangle \in \mathcal{S}([s_1, \varsigma]) \} \cup \\ & \{ \langle \varsigma', [s_1 \parallel s_2', \varsigma'] \rangle & \mid \langle \varsigma', [s_2', \varsigma'] \rangle \in \mathcal{S}([s_2, \varsigma]) \}.\end{aligned}$$

Because the sets $\mathcal{S}([s_1, \varsigma])$ and $\mathcal{S}([s_2, \varsigma])$ are finite by induction, the above set is also finite.

┘

SOLUTION TO EXERCISE 49 Assume that the operational semantics $\mathcal{O}$ is compositional. Then there exists a semantic parallel composition

$$\parallel : (\Sigma \to \mathcal{P}_n(\Sigma^\infty)) \times (\Sigma \to \mathcal{P}_n(\Sigma^\infty)) \to (\Sigma \to \mathcal{P}_n(\Sigma^\infty))$$

such that for all $s_1, s_2 \in \mathit{Stat}$,

$$\mathcal{O}(s_1 \parallel s_2) = \mathcal{O}(s_1) \parallel \mathcal{O}(s_2).$$

Clearly

$$\mathcal{O}(v := 1 ; v := 2) = \mathcal{O}(v := 1 ; v := v + 1)$$

and

$$\mathcal{O}((v := 1 ; v := 2) \parallel v := 3) \neq \mathcal{O}((v := 1 ; v := v + 1) \parallel v := 3).$$

The existence of a semantic parallel composition leads to

$$\begin{aligned}&\mathcal{O}((v := 1 ; v := 2) \parallel v := 3)\\ &= \mathcal{O}(v := 1 ; v := 2) \parallel \mathcal{O}(v := 3)\\ &= \mathcal{O}(v := 1 ; v := v + 1) \parallel \mathcal{O}(v := 3)\\ &= \mathcal{O}((v := 1 ; v := v + 1) \parallel v := 3),\end{aligned}$$

a contradiction. ┘

SOLUTION TO EXERCISE 54 Let $\epsilon > 0$. Since $(\langle \varsigma'_{s(n)}, p'_{s(n)} \rangle)_n$ converges to $\langle \varsigma', p' \rangle$,

$$\exists N \in \mathbb{N} : \forall n \geq N : d(\langle \varsigma'_{s(n)}, p'_{s(n)} \rangle, \langle \varsigma', p' \rangle) \leq \epsilon.$$

Let $n \geq N$. Then

$$\begin{aligned}&d(\langle \varsigma'_{s(n)}, \phi(p'_{s(n)}, q) \rangle, \langle \varsigma', \phi(p', q) \rangle)\\ &= \max\{d(\varsigma'_{s(n)}, \varsigma'), \tfrac{1}{2} \cdot d(\phi(p'_{s(n)}, q), \phi(p', q))\}\\ &\leq \max\{d(\varsigma'_{s(n)}, \varsigma'), \tfrac{1}{2} \cdot d(\langle p'_{s(n)}, q \rangle, \langle p', q \rangle)\} \quad [\phi \text{ is nonexpansive}]\\ &= \max\{d(\varsigma'_{s(n)}, \varsigma'), \tfrac{1}{2} \cdot d(p'_{s(n)}, p')\}\\ &= d(\langle \varsigma'_{s(n)}, p'_{s(n)} \rangle, \langle \varsigma', p' \rangle)\\ &\leq \epsilon.\end{aligned}$$

Hence, $(\langle \varsigma'_{s(n)}, \phi(p'_{s(n)}, q) \rangle)_n$ converges to $\langle \varsigma', \phi(p', q) \rangle$. ┘

SOLUTION TO EXERCISE 58 We show that for all $\phi_1, \phi_2 \in \mathbb{P} \times \mathbb{P} \xrightarrow[1]{} \mathbb{P}$, and $p, q \in \mathbb{P}$,

$$d(\Phi(\phi_1)(p, q), \Phi(\phi_2)(p, q)) \leq \tfrac{1}{2} \cdot d(\phi_1, \phi_2).$$

We distinguish the following two cases.

* If $p = \surd$ then

$$\begin{array}{ll} & d\,(\Phi\,(\phi_1)(\surd, q), \Phi\,(\phi_2)(\surd, q)) \\ = & d\,(q, q) \\ = & 0 \\ \leq & \frac{1}{2} \cdot d\,(\phi_1, \phi_2). \end{array}$$

* If $p \neq \surd$, then for all $\varsigma \in \Sigma$,

$$\begin{array}{ll} & d\,(\Phi\,(\phi_1)(p, q)(\varsigma), \Phi\,(\phi_2)(p, q)(\varsigma)) \\ = & d\,(\{\, \langle \varsigma', \phi_1\,(p', q) \rangle \mid \langle \varsigma', p' \rangle \in p\,(\varsigma)\,\}, \{\, \langle \varsigma', \phi_2\,(p', q) \rangle \mid \langle \varsigma', p' \rangle \in p\,(\varsigma)\,\}). \end{array}$$

The observation that for all $\langle \varsigma', p' \rangle \in p\,(\varsigma)$,

$$\begin{array}{ll} & d\,(\langle \varsigma', \phi_1\,(p', q) \rangle, \langle \varsigma', \phi_2\,(p', q) \rangle) \\ = & \max\,\{d\,(\varsigma', \varsigma'), \frac{1}{2} \cdot d\,(\phi_1\,(p', q), \phi_2\,(p', q))\} \\ = & \frac{1}{2} \cdot d\,(\phi_1\,(p', q), \phi_2\,(p', q)) \\ \leq & \frac{1}{2} \cdot d\,(\phi_1, \phi_2) \end{array}$$

completes the proof.

$\lrcorner$

SOLUTION TO EXERCISE 70 Let $\bar{s} = s_1 \parallel s_2$. Then

$$\begin{array}{ll} & \Omega\,(\mathcal{D})(s_1 \parallel s_2)(\varsigma) \\ = & \{\, \langle \varsigma', \mathcal{D}\,(s_2) \rangle \mid [s_1,\, \varsigma] \xrightarrow{\varsigma'} [\mathrm{E},\, \varsigma']\,\} \cup \{\, \langle \varsigma', \mathcal{D}\,(s_1' \parallel s_2) \rangle \mid [s_1,\, \varsigma] \xrightarrow{\varsigma'} [s_1',\, \varsigma']\,\} \cup \\ & \{\, \langle \varsigma', \mathcal{D}\,(s_1) \rangle \mid [s_2,\, \varsigma] \xrightarrow{\varsigma'} [\mathrm{E},\, \varsigma']\,\} \cup \{\, \langle \varsigma', \mathcal{D}\,(s_1 \parallel s_2') \rangle \mid [s_2,\, \varsigma] \xrightarrow{\varsigma'} [s_2',\, \varsigma']\,\} \\ = & \{\, \langle \varsigma', \mathcal{D}\,(\mathrm{E}) \parallel \mathcal{D}\,(s_2) \rangle \mid [s_1,\, \varsigma] \xrightarrow{\varsigma'} [\mathrm{E},\, \varsigma']\,\} \cup \{\, \langle \varsigma', \mathcal{D}\,(s_1') \parallel \mathcal{D}\,(s_2) \rangle \mid [s_1,\, \varsigma] \xrightarrow{\varsigma'} [s_1',\, \varsigma']\,\} \cup \\ & \{\, \langle \varsigma', \mathcal{D}\,(s_1) \parallel \mathcal{D}\,(\mathrm{E}) \rangle \mid [s_2,\, \varsigma] \xrightarrow{\varsigma'} [\mathrm{E},\, \varsigma']\,\} \cup \{\, \langle \varsigma', \mathcal{D}\,(s_1) \parallel \mathcal{D}\,(s_2') \rangle \mid [s_2,\, \varsigma] \xrightarrow{\varsigma'} [s_2',\, \varsigma']\,\} \\ = & \{\, \langle \varsigma', \mathcal{D}\,(\bar{s}_1) \parallel \mathcal{D}\,(s_2) \rangle \mid [s_1,\, \varsigma] \xrightarrow{\varsigma'} [\bar{s}_1,\, \varsigma']\,\} \cup \{\, \langle \varsigma', \mathcal{D}\,(s_1) \parallel \mathcal{D}\,(\bar{s}_2) \rangle \mid [s_2,\, \varsigma] \xrightarrow{\varsigma'} [\bar{s}_2,\, \varsigma']\,\} \\ = & \{\, \langle \varsigma', \mathcal{D}\,(\bar{s}_1) \parallel \Omega\,(\mathcal{D})(s_2) \rangle \mid [s_1,\, \varsigma] \xrightarrow{\varsigma'} [\bar{s}_1,\, \varsigma']\,\} \cup \{\, \langle \varsigma', \Omega\,(\mathcal{D})(s_1) \parallel \mathcal{D}\,(\bar{s}_2) \rangle \mid [s_2,\, \varsigma] \xrightarrow{\varsigma'} [\bar{s}_2,\, \varsigma']\,\} \\ & \text{[induction]} \\ = & (\Omega\,(\mathcal{D})(s_1) \parallel \Omega\,(\mathcal{D})(s_2))\,(\varsigma) \\ = & (\mathcal{D}\,(s_1) \parallel \mathcal{D}\,(s_2))\,(\varsigma) \quad \text{[induction]} \\ = & \mathcal{D}\,(s_1 \parallel s_2)(\varsigma). \end{array}$$

$\lrcorner$

SOLUTION TO EXERCISE 83 Clearly, the set

$$\mathcal{S}\,([t \text{ in } [0,\, 1],\, \varsigma,\, \tau]) = \{\, \langle \langle \varsigma, \tau\{r/t\} \rangle, [\mathrm{E},\, \varsigma,\, \tau\{r/t\}] \rangle \mid r \in [0,\, 1]\,\}$$

is infinite. $\lrcorner$

SOLUTION TO EXERCISE 88 We show that for all $\langle \varsigma, \tau \rangle \in \Sigma$, $\sigma \in A^\infty$ and $f_1,\, f_2 \in \Sigma \xrightarrow[1]{} \mathcal{P}_{nc}\,(A^\infty)$,

$$d\,(\Phi\,(\phi)(\varsigma, \tau, \sigma)(f_1), \Phi\,(\phi)(\varsigma, \tau, \sigma)(f_2)) \leq d\,(f_1, f_2).$$

We distinguish the following cases.

* Let $\sigma = \varepsilon$. Then

$$\begin{aligned}
& d\,(\Phi\,(\phi)(\varsigma, \tau, \varepsilon)(f_1), \Phi\,(\phi)(\varsigma, \tau, \varepsilon)(f_2)) \\
& \quad = \quad d\,(f_1\,(\varsigma, \tau), f_2\,(\varsigma, \tau)) \\
& \quad \leq \quad d\,(f_1, f_2).
\end{aligned}$$

* If $\sigma = \langle \varsigma', \tau' \rangle \sigma'$ then

$$\begin{aligned}
& d\,(\Phi\,(\phi)(\varsigma, \tau, \langle \varsigma', \tau' \rangle \sigma')(f_1), \Phi\,(\phi)(\varsigma, \tau, \langle \varsigma', \tau' \rangle \sigma')(f_2)) \\
& \quad = \quad d\,(\langle \varsigma', \tau' \rangle\, \phi\,(\varsigma', \tau', \sigma')(f_1), \langle \varsigma', \tau' \rangle\, \phi\,(\varsigma', \tau', \sigma')(f_2)) \\
& \quad \leq \quad \tfrac{1}{2} \cdot d\,(\phi\,(\varsigma', \tau', \sigma')(f_1), \phi\,(\varsigma', \tau', \sigma')(f_2)) \\
& \quad \leq \quad \tfrac{1}{2} \cdot d\,(f_1, f_2) \quad [\phi\,(\varsigma', \tau', \sigma') \text{ is nonexpansive}]
\end{aligned}$$

* The case that $\sigma = r\sigma'$ is similar to the previous one.

┘

SOLUTION TO EXERCISE 98 We show that for all $s \in \mathit{Stat}$,

* for all $\langle \varsigma, \tau \rangle \in \Sigma$, the set $\mathcal{D}\,(s)(\varsigma, \tau)$ is compact, and
* for all $\langle \varsigma, \tau_1 \rangle$, $\langle \varsigma, \tau_2 \rangle \in \Sigma$, $d\,(\mathcal{D}\,(s)(\varsigma, \tau_1), \mathcal{D}\,(s)(\varsigma, \tau_2)) \leq d\,(\tau_1, \tau_2)$

by structural induction on s. We distinguish the following cases.

* Let $s = v := e$ and $n = \mathcal{E}\,(e)(\varsigma)$. Clearly, the set $\{\langle \varsigma\{n/v\}, \tau \rangle\}$ is compact. Furthermore,

$$\begin{aligned}
& d\,(\mathcal{D}\,(v := e)(\varsigma, \tau_1), \mathcal{D}\,(v := e)(\varsigma, \tau_2)) \\
& \quad = \quad d\,(\{\langle \varsigma\{n/v\}, \tau_1 \rangle\}, \{\langle \varsigma\{n/v\}, \tau_2 \rangle\}) \\
& \quad = \quad d\,(\tau_1, \tau_2).
\end{aligned}$$

* The case $s = \mathsf{skip}$ is similar to the previous one.
* The case $t\,\mathsf{in}\,[r_1,\, r_2]$ is dealt with as follows. One can easily verify that the function $\lambda r. \langle \varsigma, \tau\{r/t\} \rangle$ is nonexpansive. Because the set $[r_1,\, r_2]$ is compact (Proposition 127), we can deduce from Alexandroff's theorem that the set $\{\, \langle \varsigma, \tau\{r/t\} \rangle \mid r \in [r_1,\, r_2] \,\}$ is compact as well. For all $r \in [r_1,\, r_2]$,

$$\begin{aligned}
& d\,(\langle \varsigma, \tau_1\{r/t\} \rangle, \langle \varsigma, \tau_2\{r/t\} \rangle) \\
& \quad = \quad d\,(\tau_1\{r/t\}, \tau_2\{r/t\}) \\
& \quad \leq \quad d\,(\tau_1, \tau_2).
\end{aligned}$$

  Consequently,

$$\begin{aligned}
& d\,(\mathcal{D}\,(t\,\mathsf{in}\,[r_1,\, r_2])(\varsigma, \tau_1), \mathcal{D}\,(t\,\mathsf{in}\,[r_1,\, r_2])(\varsigma, \tau_2)) \\
& \quad = \quad d\,(\{\, \langle \varsigma, \tau_1\{r/t\} \rangle \mid r \in [r_1,\, r_2] \,\}, \{\, \langle \varsigma, \tau_2\{r/t\} \rangle \mid r \in [r_1,\, r_2] \,\}) \\
& \quad \leq \quad d\,(\tau_1, \tau_2).
\end{aligned}$$

* Let $s = \mathsf{wait}\, t$. Obviously, the set $\{\tau\,(t)\}$ is compact. Furthermore,

$$\begin{aligned}
& d\,(\mathcal{D}\,(\mathsf{wait}\, t)(\varsigma, \tau_1), \mathcal{D}\,(\mathsf{wait}\, t)(\varsigma, \tau_2)) \\
& \quad = \quad d\,(\{\tau_1\,(t)\}, \{\tau_2\,(t)\}) \\
& \quad \leq \quad d\,(\tau_1, \tau_2).
\end{aligned}$$

* Let $s = s_1 \,;\, s_2$. The compactness of the set $\mathcal{D}(s_1 \,;\, s_2)(\varsigma, \tau)$ can be proved along the lines of the second part of the proof of Proposition 94. Furthermore,

$$\begin{aligned}
& d(\mathcal{D}(s_1 \,;\, s_2)(\varsigma, \tau_1), \mathcal{D}(s_1 \,;\, s_2)(\varsigma, \tau_2)) \\
= \;& d(\bigcup \{\, \sigma_1 \,;_{\langle \varsigma, \tau_1 \rangle} \mathcal{D}(s_2) \mid \sigma_1 \in \mathcal{D}(s_1)(\varsigma, \tau_1) \,\}, \bigcup \{\, \sigma_2 \,;_{\langle \varsigma, \tau_2 \rangle} \mathcal{D}(s_2) \mid \sigma_2 \in \mathcal{D}(s_1)(\varsigma, \tau_2) \,\}) \\
\leq \;& d(\{\, \sigma_1 \,;_{\langle \varsigma, \tau_1 \rangle} \mathcal{D}(s_2) \mid \sigma_1 \in \mathcal{D}(s_1)(\varsigma, \tau_1) \,\}, \{\, \sigma_2 \,;_{\langle \varsigma, \tau_2 \rangle} \mathcal{D}(s_2) \mid \sigma_2 \in \mathcal{D}(s_1)(\varsigma, \tau_2) \,\}) \\
& \text{[Michael's theorem]}
\end{aligned}$$

Let $\sigma_1 \in \mathcal{D}(s_1)(\varsigma, \tau_1)$. Then there exists a $\sigma_2 \in \mathcal{D}(s_1)(\varsigma, \tau_2)$ such that

$$\begin{aligned}
& d(\sigma_1, \sigma_2) \\
\leq \;& d(\mathcal{D}(s_1)(\varsigma, \tau_1), \mathcal{D}(s_1)(\varsigma, \tau_2)) \\
\leq \;& d(\tau_1, \tau_2) \quad \text{[induction]}
\end{aligned}$$

Hence,

$$\begin{aligned}
& d(\sigma_1 \,;_{\langle \varsigma, \tau_1 \rangle} \mathcal{D}(s_2), \sigma_2 \,;_{\langle \varsigma, \tau_2 \rangle} \mathcal{D}(s_2)) \\
\leq \;& \max\{d(\sigma_1, \sigma_2), d(\tau_1, \tau_2)\} \quad \text{[; is nonexpansive]} \\
\leq \;& d(\tau_1, \tau_2) \quad \text{[see above]}
\end{aligned}$$

* Assume $s = \mathsf{if}\ b\ \mathsf{then}\ s_1\ \mathsf{else}\ s_2\ \mathsf{fi}$. The compactness of $\mathcal{D}(\mathsf{if}\ b\ \mathsf{then}\ s_1\ \mathsf{else}\ s_2\ \mathsf{fi})(\varsigma, \tau)$ follows immediately by induction. If $\mathcal{B}(b)(\varsigma) = \mathit{true}$ then

$$\begin{aligned}
& d(\mathcal{D}(\mathsf{if}\ b\ \mathsf{then}\ s_1\ \mathsf{else}\ s_2\ \mathsf{fi})(\varsigma, \tau_1), \mathcal{D}(\mathsf{if}\ b\ \mathsf{then}\ s_1\ \mathsf{else}\ s_2\ \mathsf{fi})(\varsigma, \tau_2)) \\
= \;& d(\mathcal{D}(s_1)(\varsigma, \tau_1), \mathcal{D}(s_1)(\varsigma, \tau_2)) \\
\leq \;& d(\tau_1, \tau_2) \quad \text{[induction]}
\end{aligned}$$

The case that $\mathcal{B}(b)(\varsigma_1) = \mathit{false}$ can be dealt with similarly.

* For the case $\mathsf{while}\ b\ \mathsf{do}\ s\ \mathsf{od}$ we first have to check that for all $\langle \varsigma, \tau \rangle \in \Sigma$, $\varepsilon \notin \mathcal{D}(s)(\varsigma, \tau)$. This can be proved by structural induction on s (see Exercise 37). The compactness and nonexpansiveness follow from the definition of Ψ.

┘

Solution to Exercise 99 One has that

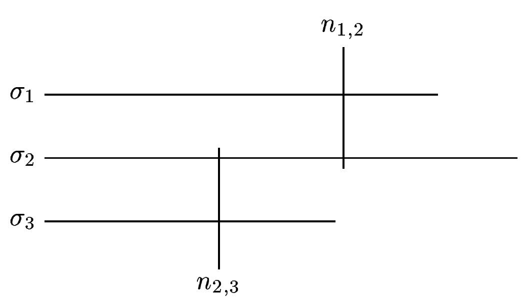
$$\mathcal{D}(t\ \mathsf{in}\ [0,\,1])(\varsigma, \tau) = \{\, \langle \varsigma, \tau\{r/t\} \rangle \mid r \in [0, 1] \,\}.$$

Clearly, this set is not a compact subset of $(\Sigma \cup \mathbb{R}_+)^\infty$ endowed with the Baire metric, since, for example, the sequence $(\langle \varsigma, \tau\{\frac{1}{n}/t\} \rangle)_n$ does not have a converging subsequence (all elements of the sequence are distance 1 apart). ┘

Solution to Exercise 108 Clearly, d_{A^∞} satisfies (1) and (2). We have left to prove that for all σ_1, σ_2, $\sigma_3 \in A^\infty$,

$$d(\sigma_1, \sigma_3) \leq \max\{d(\sigma_1, \sigma_2), d(\sigma_2, \sigma_3)\}. \tag{3}$$

Assume that σ_1, σ_2, and σ_3 are all different (otherwise (3) is vacuously true). Let $n_{1,2}$ and $n_{2,3}$ be the length of the longest common prefix of σ_1 and σ_2, and of σ_2 and σ_3, respectively.



The longest common prefix of σ_1 and σ_3 is at least $\min\{n_{1,2}, n_{2,3}\}$. Since

$$2^{-\min\{n_{1,2}, n_{2,3}\}} = \max\{2^{-n_{1,2}}, 2^{-n_{2,3}}\},$$

we can conclude (3). ┘

SOLUTION TO EXERCISE 121 Let the function $f : X \to Y$ be α-Lipschitz. If $\alpha = 0$ then f is a constant function which is clearly continuous. Let $\alpha > 0$. Assume that the sequence $(x_n)_n$ converges to x. Let $\epsilon > 0$. Since $(x_n)_n$ converges to x,

$$\exists N \in \mathbb{N} : \forall n \geq N : d_X(x_n, x) \leq \tfrac{\epsilon}{\alpha}.$$

Because f is α-Lipschitz,

$$\exists N \in \mathbb{N} : \forall n \geq N : d_Y(f(x_n), f(x)) \leq \epsilon.$$

Hence, the sequence $(f(x_n))_n$ converges to $f(x)$. ┘

SOLUTION TO EXERCISE 124 For all $n \in \mathbb{N}$,

$$\inf\{\, d_{A^\infty}(a^n, a^m) \mid m \in \mathbb{N} \cup \{\omega\}\,\} = \inf\{1, \tfrac{1}{2}, \tfrac{1}{4}, \ldots, 2^{-(n-1)}, 0, 2^{-n}\} = 0$$

and for all $m \in \mathbb{N}$,

$$\inf\{\, d_{A^\infty}(a^m, a^n) \mid n \in \mathbb{N}\,\} = \inf\{1, \tfrac{1}{2}, \tfrac{1}{4}, \ldots, 2^{-(m-1)}, 0, 2^{-m}\} = 0.$$

Furthermore,

$$\inf\{\, d_{A^\infty}(a^\omega, a^n) \mid n \in \mathbb{N}\,\} = \inf\{1, \tfrac{1}{2}, \tfrac{1}{4}, \ldots\} = 0.$$

Consequently,

$$d_{\mathcal{P}_n(A^\infty)}(\{\, a^n \mid n \in \mathbb{N}\,\}, \{\, a^n \mid n \in \mathbb{N}\,\} \cup \{a^\omega\}) = 0.$$

┘

SOLUTION TO EXERCISE 131 Because the sequence $(a_{m,n})_n$ converges to $\bar{a}_m$, there exists a strictly increasing sequence $(N_{m,n})_n$ such that for all n,

$$d_X(a_{m,N_{m,n}}, \bar{a}_m) \leq 2^{-n+1}. \tag{1}$$

According to (A.3), we can find an $a'_{m,n} \in A_n$ such that

$$d_X(a'_{m,n}, a_{m,N_{m,n}}) \leq 2^{-n+1}. \tag{2}$$

The sequence $(a'_{m,n})_n$ is Cauchy, since

$$\begin{array}{ll} d_X(a'_{m,n}, a'_{m,n+1}) & \\ \quad \leq & d_X(a'_{m,n}, a_{m,N_{m,n}}) + d_X(a_{m,N_{m,n}}, \bar{a}_m) + d_X(\bar{a}_m, a_{m,N_{m,n+1}}) + d_X(a_{m,N_{m,n+1}}, a'_{m,n+1}) \\ \quad \leq & 2^{-n+1} + 2^{-n+1} + 2^{-n} + 2^{-n} \quad [(1) \text{ and } (2)] \end{array}$$

Clearly, this sequence also converges to $\bar{a}_m$, because

$$\begin{array}{ll} d_X(a'_{m,n}, \bar{a}_m) & \\ \quad \leq & d_X(a'_{m,n}, a_{m,N_{m,n}}) + d_X(a_{m,N_{m,n}}, \bar{a}_m) \\ \quad \leq & 2^{-n+1} + 2^{-n+1} \quad [(1) \text{ and } (2)] \\ \quad = & 2^{-n+2}. \end{array}$$

┘

Solution to Exercise 133 Let $\mathcal{A}$, $\mathcal{B} \in \mathcal{P}_{nc}(\mathcal{P}_{nc}(X))$. We have to show that

$$d_{\mathcal{P}_{nc}(X)}(\bigcup \mathcal{A}, \bigcup \mathcal{B}) \leq d_{\mathcal{P}_{nc}(\mathcal{P}_{nc}(X))}(\mathcal{A}, \mathcal{B}).$$

Let $x \in \bigcup \mathcal{A}$. Then there exists an $A \in \mathcal{A}$ such that $x \in A$. Furthermore, there exist a $B \in \mathcal{B}$ satisfying $d_{\mathcal{P}_{nc}(X)}(A, B) \leq d_{\mathcal{P}_{nc}(\mathcal{P}_{nc}(X))}(\mathcal{A}, \mathcal{B})$. Hence, there exists a $y \in B$ such that

$$\begin{aligned} & d_X(x, y) \\ \leq\ & d_{\mathcal{P}_{nc}(X)}(A, B) \\ \leq\ & d_{\mathcal{P}_{nc}(\mathcal{P}_{nc}(X))}(\mathcal{A}, \mathcal{B}). \end{aligned}$$

┘

Solution to Exercise 155 The labelled transition system introduced in Example 146 induces the operational semantics $\mathcal{O}$ defined by

$$\mathcal{O}(c) = \{ a_n \mid n \in \mathbb{N} \}$$

and, for all $n \in \mathbb{N}$,

$$\mathcal{O}(c_n) = \{\varepsilon\}.$$

Because the set $\{ a_n \mid n \in \mathbb{N} \}$ is not compact, we can conclude that the above defined operational semantics is not compact. ┘

Solution to Exercise 165 The labelled transition system introduced in Example 146 induces a semantics transformation $\mathcal{T}$ which is not compactness preserving. Let $\mathcal{S}$ be a semantics satisfying

$$\mathcal{S}(c_n) = \{\varepsilon\}.$$

Then

$$\mathcal{T}(\mathcal{S})(c) = \{ a_n \mid n \in \mathbb{N} \}.$$

Since the set $\{ a_n \mid n \in \mathbb{N} \}$ is not compact, the semantics transformation $\mathcal{T}$ is not compactness preserving. ┘

Solution to Exercise 168 Consider the semantics transformation $\mathcal{T}$ induced by the labelled transition system of Example 137 and the semantics $\mathcal{S}_1, \mathcal{S}_2 : \{c_1, c_2\} \to \mathcal{P}_c(\{a_1, a_2\}^\infty)$ defined by

$$\begin{array}{ll} \mathcal{S}_1(c_1) = \emptyset & \mathcal{S}_2(c_1) = \{\varepsilon\} \\ \mathcal{S}_1(c_2) = \emptyset & \mathcal{S}_2(c_2) = \{\varepsilon\} \end{array}$$

Then

$$\begin{aligned} & d(\mathcal{T}(\mathcal{S}_1), \mathcal{T}(\mathcal{S}_2)) \\ \geq\ & d(\mathcal{T}(\mathcal{S}_1)(c_1), \mathcal{T}(\mathcal{S}_2)(c_1)) \\ =\ & d(\emptyset, \{a_1, a_2\}) \\ =\ & 1 \\ \geq\ & d(\mathcal{S}_1, \mathcal{S}_2), \end{aligned}$$

that is, $\mathcal{T}$ is not contractive. ┘

Solution to Exercise 179 We define the function $d_{A^\infty} : A^\infty \times A^\infty \to [0, 1]$ by

$$d_{A^\infty}(\sigma_1, \sigma_2) = \max\{ 2^{-n-1} \cdot d_{A+\{\perp\}}(\sigma_1(n), \sigma_2(n)) \mid n \geq 1 \},$$

where $\sigma(n)$ is the n-th element of σ if $n \leq |\sigma|$ and $\perp$ (undefined) otherwise. One can easily verify that this function is a metric.

Next, we define the function $f : A^{\infty} \to (\{\varepsilon\} + (A \times \frac{1}{2} \cdot A^{\infty}))$ by

$$f(\sigma) = \begin{cases} \varepsilon & \text{if } \sigma = \varepsilon \\ \langle a, \sigma' \rangle & \text{if } \sigma = a\sigma'. \end{cases}$$

Clearly, this function is bijective. We have left to prove that for all $\sigma_1, \sigma_2 \in A^{\infty}$,

$$d(f(\sigma_1), f(\sigma_2)) = d(\sigma_1, \sigma_2).$$

If $\sigma_1 = \sigma_2$ this is of course the case. Assume that $\sigma_1 \neq \sigma_2$. We consider the following three cases.

* If $\sigma_1 = \varepsilon$ and $\sigma_2 = a_2\sigma_2'$ then

$$\begin{aligned} & d(f(\varepsilon), f(a_2\sigma_2')) \\ = \;& d(\varepsilon, \langle a_2, \sigma_2' \rangle) \\ = \;& 1 \\ = \;& d(\varepsilon, a_2\sigma_2'). \end{aligned}$$

* The case that $\sigma_1 = a_1\sigma_1'$ and $\sigma_2 = \varepsilon$ can be proved similar to the previous one.
* Let $\sigma_1 = a_1\sigma_1'$ and $\sigma_2 = a_2\sigma_2'$. Then

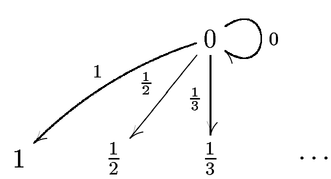
$$\begin{aligned} & d(f(a_1\sigma_1'), f(a_2\sigma_2')) \\ = \;& d(\langle a_1, \sigma_1' \rangle, \langle a_2, \sigma_2' \rangle) \\ = \;& \max\{d(a_1, a_2), \tfrac{1}{2} \cdot d(\sigma_1', \sigma_2')\} \\ = \;& \max\{d(a_1, a_2), \tfrac{1}{2} \cdot \max\{\, 2^{-n-1} \cdot d(\sigma_1'(n), \sigma_2'(n)) \mid n \geq 1 \,\}\} \\ = \;& d(a_1\sigma_1', a_2\sigma_2'). \end{aligned}$$

┘

SOLUTION TO EXERCISE 184 The compactly branching metric labelled transition system

$$\begin{cases} 0 \xrightarrow{0} 0 \\ 0 \xrightarrow{\frac{1}{n}} \frac{1}{n} & \text{for } n > 0 \end{cases}$$

depicted by



with the set of configurations $\{\, \frac{1}{n} \mid n > 0 \,\} \cup \{0\}$ and the set of actions $\{\, \frac{1}{n} \mid n > 0 \,\} \cup \{0\}$ both endowed with the Euclidean metric, does not induce a compact operational semantics. Note that the function $\mathcal{S}$ is not nonexpansive. ┘

SOLUTION TO EXERCISE 186 For a nonterminal configuration c, $\mathcal{S}(c) \neq \emptyset$ and for a terminal configuration c', $\mathcal{S}(c') = \emptyset$. Since the metric labelled transition system is nonexpansive,

$$1 = d(\mathcal{S}(c), \mathcal{S}(c')) \leq d(c, c').$$

┘

SOLUTION TO EXERCISE 191 Let $\sigma \in \mathcal{O}_n(c)$. We distinguish two cases.

* If $\sigma = a_1 a_2 \cdots a_k$ with $k \leq n$ and

$$c = c_0 \xrightarrow{a_1} c_1 \xrightarrow{a_2} \cdots \xrightarrow{a_k} a_k \not\rightarrow$$

then $\sigma \in \mathcal{O}(c)$.

* If $\sigma = a_1 a_2 \cdots a_n$ and

$$c = c_0 \xrightarrow{a_1} c_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} a_n \rightarrow$$

then $\sigma\sigma' \in \mathcal{O}(c)$ for some $\sigma' \in A^\infty$, and $d(\sigma, \sigma\sigma') = 2^{-n}$.

The fact that $\sigma \in \mathcal{O}(c)$ implies that there exists a $\sigma' \in \mathcal{O}_n(c)$ such that $d(\sigma, \sigma') \leq 2^{-n}$ can be proved similarly. ⌟

SOLUTION TO EXERCISE 206 We define the function $f : A^\infty \to (\{\varepsilon\} + (A \times \frac{1}{2} \cdot A^\infty))$ by

$$f(\sigma) = \begin{cases} \varepsilon & \text{if } \sigma = \varepsilon \\ \langle a, \sigma' \rangle & \text{if } \sigma = a\sigma'. \end{cases}$$

Clearly, this function is bijective. We have left to prove that for all σ_1, $\sigma_2 \in A^\infty$,

$$d(f(\sigma_1), f(\sigma_2)) = d(\sigma_1, \sigma_2).$$

If $\sigma_1 = \sigma_2$ this is of course the case. Assume that $\sigma_1 \neq \sigma_2$. We consider the following three cases.

* If $\sigma_1 = \varepsilon$ and $\sigma_2 = a_2\sigma_2'$ then

$$\begin{aligned} & d(f(\varepsilon), f(a_2\sigma_2')) \\ = \;& d(\varepsilon, \langle a_2, \sigma_2' \rangle) \\ = \;& 1 \\ = \;& d(\varepsilon, a_2\sigma_2'). \end{aligned}$$

* The case that $\sigma_1 = a_1\sigma_1'$ and $\sigma_2 = \varepsilon$ can be proved similar to the previous one.

* Let $\sigma_1 = a_1\sigma_1'$ and $\sigma_2 = a_2\sigma_2'$. Then

$$\begin{aligned} & d(f(a_1\sigma_1'), f(a_2\sigma_2')) \\ = \;& d(\langle a_1, \sigma_1' \rangle, \langle a_2, \sigma_2' \rangle) \\ = \;& \max\{d(a_1, a_2), \tfrac{1}{2} \cdot d(\sigma_1', \sigma_2')\} \\ = \;& \begin{cases} \frac{1}{2} \cdot d(\sigma_1', \sigma_2') & \text{if } a_1 = a_2 \\ 1 & \text{otherwise} \end{cases} \\ = \;& d(a_1\sigma_1', a_2\sigma_2'). \end{aligned}$$

⌟

References

[Bre] F. van Breugel. An Introduction to Metric Semantics: Operational and Denotational Models for Programming and Specification Languages. To appear in *Theoretical Computer Science.*