Quiz

First Name: _____

Last Name: _____

Student Number: _____

*This test lasts **60 minutes**. No aids allowed.*
Make sure your test has 4 pages, including this cover page.
*Answer in the space provided. (If you need more space, use the reverse side of the page and indicate **clearly** which part of your work should be marked.)*
Write legibly.

Question 1	/4
Question 2	/4
Question 3	/2
Question 4	/4
Question 5	/5
Total	/19

[4] **1.** Let $A = \{w : w \text{ is a string of } \texttt{0}\text{'s and } \texttt{1}\text{'s of length } 4\}$.
Let $B = \{\texttt{0101}, \texttt{1001}, \varepsilon\}$.

(a) Give an explicit listing of all elements of $A \cap B$.

(b) Give three subsets of B.

(c) How many elements are in $B \times B$? Briefly explain why your answer is correct.

[4] **2.** Are the following statements true or false? Prove your answers are correct.

(a) $\forall x$ in $\mathbb{N}, \exists y$ in $\mathbb{N}$ such that $x < y$.

(b) $\exists y$ in $\mathbb{N}$ such that $\forall x$ in $\mathbb{N}$, $x \leq y$.

[2] **3.** Egbert claims that every positive integer n is odd and proves it using strong induction as follows.

1 **Base case** ($n = 1$): 1 is odd.
2 **Induction step**: Let n be any positive integer.
3 The goal of our induction step is to prove that if $1, 2, \ldots, n$ are all odd, then $n + 1$ is also odd.
4 So, assume $1, 2, \ldots, n$ are all odd (as our induction hypothesis).
5 In particular, $n - 1$ is odd.
6 Adding 2 to an odd number produces another odd number.
7 Thus, $(n - 1) + 2 = n + 1$ is odd.
8 The claim follows by strong induction.

Egbert's claim is clearly false, so there must be at least one mistake somewhere in his proof. For each mistake in Egbert's proof, state precisely where the mistake is (including its line number) and explain why it is a mistake.
(Note: if you describe mistakes that are not actually mistakes, you will lose marks.)

[4] **4.** Recall that if X is a set, $\mathcal{P}(X)$ denotes the power set of X.
Prove that for all sets A and B, $\mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B)$.

[5] **5.** Define a function $f : \mathbb{N} \to \mathbb{N}$ recursively as follows.

$$\begin{aligned} f(0) &= 7 \\ f(n) &= 2 \cdot f(n-1) - 1, \text{ for } n \geq 1 \end{aligned}$$

Give a careful proof that for all $n \in \mathbb{N}$, $f(n) = 6 \cdot 2^n + 1$.