Computing for Math and Stats

Lecture 19

Ellipses

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} a^2 & \\ & b^2 \end{bmatrix}^{-1} \begin{bmatrix} x \\ y \end{bmatrix} = 1$$

Ellipses

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}^T M^{-1} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 1$$

$$\vec{x}^T M^{-1} \vec{x} = 1$$

$$(\vec{x} - \mu)^T M^{-1} (\vec{x} - \mu) = 1$$

Ellipses

- The equation of an ellipse can be written in a matrix form
- The matrix involved has to be symmetric
  - If it is not we take the symmetric part of it.
- The matrix involved has to be positive definite
  - If it is not it is a parabola or hyperbola
- Can be extended to higher dimensions
  - We can use the term ellipsoid for 3-D
  - We can stick the prefix hyper- for higher dimensions

Ellipses

- Drawing an ellipse
  - Create a set of points that satisfy the equation of an ellipse
- It is easy to take care of the center of the ellipse
  - For now the center is at the origin
- We know how to draw a unit circle
  - We start from there

Ellipses

$$M = L^T L$$

$$x_c^{\,T} x_c = 1$$

$$x = L^T x_c$$

$$x^T M^{-1} x = (L^T x_c)^T L^{-1} L^{-T} L^T x_c = ¿$$

$$x_c^T L L^{-1} L^{-T} L^T x_c = x_c^T x_c = 1$$

Ellipses

- Here is how we draw the ellipse:
  - Create the points to draw a circle
  - Multiply these points by the transpose of matrix L
    - Which we get by decomposing the matrix M
  - The resulting points form an ellipse
- The same exact procedure can be used for 3-D ellipsoids (or higher but then we cannot plot them)

Ellipses

- This procedure
  - Makes drawing easy
  - Given the matrix we can draw the ellipse
- But
  - Cannot draw hyperbolas/parabolas
    - The Cholesky decomposition does not work for matrices representing hyperbolas

Ellipsoids and Hyperboloids

- We talked about ellipsoids
- These techniques for ellipsoids do not apply to hyperboloids
- There are techniques to plot hyperbolas
  - We will not discuss them here
- We discuss now a very different technique to “plot” ellipsoids, paraboloids, hyperboloids using scatter-plots (scatter3)

Ray-Tracing

- It is a very versatile technique used extensively in graphics to render photorealistic CGI (instead of polygons)
- Here we introduce a **very** simplified version.

Ray Tracing

- Our ray-tracing will return a large number of 3-D points the lie on the surface of the ellipsoid or hyperboloid.
- We find these points by generating random “rays” (i.e. lines in 3-D) and find their intersection with the hyperboloid.