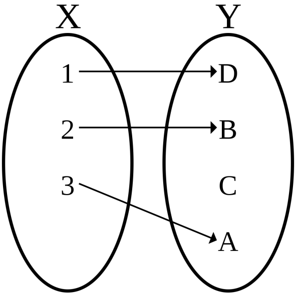
EECS 1028 E: Discrete Mathematics for Engineers

Suprakash Datta
Office: LAS 3043

Course page: http://www.eecs.yorku.ca/course/1028
Also on eClass

Relations - Intuition

Generalizations of Functions:



Why do we need such a concept?

- "Is a brother of", over a set of people
- "Is greater than", over a set of integers

Relations -Definition

Ch 9.1 in the text

- A relation R from A to B is a subset of $A \times B$
  - no restriction that all elements in the domain are covered
  - no restriction that one element map to exactly one element
- A relation R on a set X is a subset of $X \times X$
  - easier to visualize as a "relationship" between elements in the same set than as a mapping
- Notation: xRy OR $(x, y) \in R$ OR "x is related to y"

Relations: Examples

- $A =$ All EECS students in W 2018
  $B =$ All EECS courses in W 2018
  aRb: Student a is enrolled in course b
- $A =$ Universities in Ontario
  $B =$ All degrees offered in Ontario
  aRb: University a offers degree b
- $A =$ All current EECS faculty members
  $B =$ All current IEEE Journals
  aRb: Professor a has published at least one paper in journal b
- $A =$ All EECS students in W 2018
  aRb: Student a and student b are both enrolled in at least one EECS course

Properties of Relations over a Set - 1

A relation on a set X is

- Reflexive: aRa for all $a \in X$
  - Example of a reflexive relation: $\leq$ over $\mathbb{R}$
  - Example of a non-reflexive relation: $<$ over $\mathbb{R}$
- Symmetric: aRb implies bRa for all $a, b \in X$
  - Example of a symmetric relation: "is a sibling of" over a set of people
  - Example of a non-symmetric relation: "is a brother of" over a set of people

Properties of Relations over a Set - 2

A relation on a set X is

- Transitive: aRb and bRc imply aRc for all $a, b, c \in X$
  - Example of a transitive relation: "is a sibling of" over a set of people
  - Example of a non-transitive relation: "in the same class as" over a set of students

- Equivalence relation: The relation is reflexive, symmetric and transitive.

  Notation: $a \equiv b$ or $a \sim b$.
  - Example of an equivalence relation: "is in the same engineering major as" over a set of Lassonde BEng students
  - Example of a non-equivalence relation: "divides" over the set of natural numbers

  Q: How does it relate to the notion of equality?

Representations of Relations

- Matrices: If R is a relation from $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$, then M is a $m \times n$ matrix with $m_{ij} = 1$ if $a_i R b_j$ and $m_{ij} = 0$ otherwise.

- Directed graphs (will be covered at the end of the course)

Equivalence Relations

Ch 9.5 of the text

Recall: A function R is an equivalence relation if it is reflexive, symmetric and transitive

Examples of Equivalence Relations:

- $R \subseteq \mathbb{W} \times \mathbb{W}$, mRn if $m \mod 2 = n \mod 2$
- $R \subseteq \mathbb{W} \times \mathbb{W}$, mRn if $m \mod k = n \mod k$
  Also expressed as $m \equiv n \pmod{k}$
- $A =$ set of EECS students, aRb if a, b have the same major

Not Equivalence Relations:

- $\mathbb{Z}$, aRb if $|a - b| \leq 5$
- $\mathbb{N}$, aRb if $a < b$

Equivalence Classes

- Consider the relation $R \subseteq \mathbb{W} \times \mathbb{W}$, mRn if $m \equiv n \pmod{k}$
  Every element of $\mathbb{W}$ is in one of k disjoint sets, "induced" by R

- Consider the equivalence relation "is in the same engineering major as" over a set of Lassonde BEng students
  Every member of that et of Lassonde BEng students is in one of several disjoint sets (possible BEng majors plus "undecided"), "induced" by the relation

- Define for any equivalence relation R on A: $[a]_R =$ the set of all elements equivalent to a
  Q: Does A get partitioned into a disjoint set of subsets, induced by R?
  A: Yes, and these subsets are called **equivalence classes**

Results on Equivalence Classes

- Theorem 1 (pg 612) If R is an equivalence relation on A, the following are equivalent
  - aRb
  - $[a]_R = [b]_R$
  - $[a]_R \cap [b]_R \neq \emptyset$
- Theorem 2 (pg 613) If R is an equivalence relation on S, the equivalence classes of R form a partition of S. Conversely, given a partition of S, there is an equivalence relation that corresponds to the partition sets.

Operations on Relations

If R, S are equivalence relations on A,

- $R \cup S$: the set union of R, S
- $R \cap S$: the set intersection of R, S
- Composition ($T = S \circ R$): aTc if there exists $b \in A$ such that aRb and bSc

Connections to Databases

- Consider some practical examples of databases:
  - Password database: Sets of **tuples** of the form $\langle \text{login-name}, \text{encrypted-passwd}, \text{timestamp of last change} \rangle$
  - YorkU Student database: Sets of **tuples** of the form $\langle \text{name}, \text{SID}, \text{Address}, \ldots \rangle$
- These are examples of relations
- The initial databases were designed as **Relational Databases**
- Computations on relational databases were made more efficient by thinking of properties of relations