EECS 1028 E: Discrete Mathematics for Engineers

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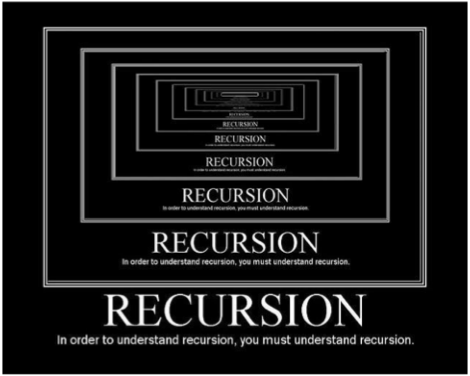
Course page: `http://www.eecs.yorku.ca/course/1028`
Also on eClass

Recursive definitions (Ch 5.3)

- Recursion: definition in terms of itself
- Design Pattern in Software Engineering
- Algorithm design strategy



From https://www.cse.buffalo.edu/ hartloff/CSE250/images/recursion.png

Recursive Definitions

Recursively defined sequence

Fibonacci series.

$f_0 = f_1 = 1, \forall n > 1, f_n = f_{n-1} + f_{n-2}$

Recursively defined function

```
// recursive factorial function
public static int recursiveFactorial(int n) {
    // base case
    if (n == 0)||(n == 1) return 1;
    // recursive case
    else return n * recursiveFactorial(n - 1);
}
```

Recursively Definitions - 2

- Non-recursive function: $f : \mathbb{R} \to \mathbb{R}, f(x) = 2x + x^2 - 3$

- Recursive function: $f : \mathbb{N} \to \mathbb{N}$,
  $f(1) = 1$
  $\forall n > 1, f(n) = 2f(n-1) + 3.$

Relationship with Sums and Products

- Non-recursive:

$$S_n = \sum_{k=1}^{n} a_k$$

Recursive:

$$S_1 = a_1, \forall n > 1, S_n = S_{n-1} + a_n$$

- Non-recursive:

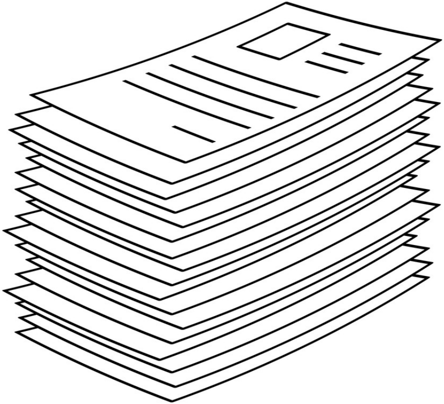
$$f(n) = n! = \prod_{k=1}^{n} k$$

Recursive:

$$f(1) = 1, \forall n > 1, f(n) = f(n-1) * n$$

Recursively Defined Problem

Sort the tests from this class by names (MergeSort)



Recursively Defined Sets

Set of prime numbers: the unique set of positive integers satisfying

- 1 is not a prime number
- any other positive integer is a prime number if and only if it is not divisible by any prime number smaller than itself

The set of even numbers:

- 0 is an even number
- For any even number n, $n + 2$ is even
- No number is even unless it is obtained from the previous rules

Recursively Defined Sets - 2

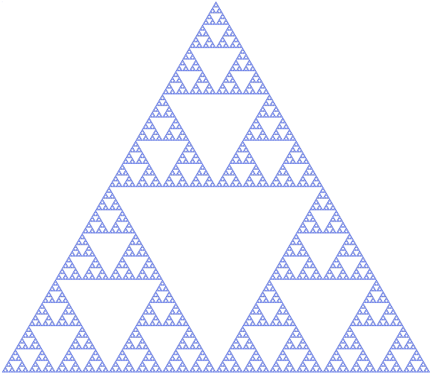
Set of natural numbers $\mathbb{N}$: the smallest set satisfying

- $1 \in \mathbb{N}$
- $\forall m \in \mathbb{N}, \forall n \in \mathbb{N}, m + n \in \mathbb{N}$

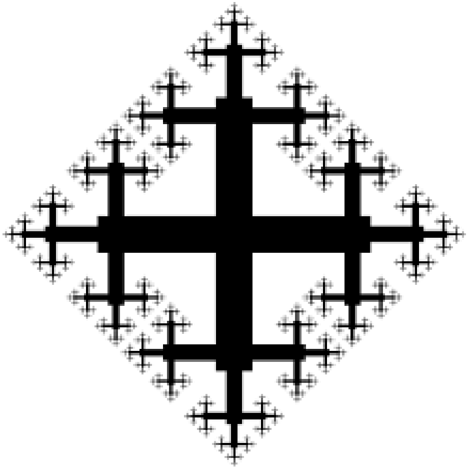
Set of all binary strings B:
Assume that λ is the empty string. Then

- $\lambda \in B$
- $\forall x \in B, (0x \in B) \wedge (1x \in B)$

Another Recursively Defined Set

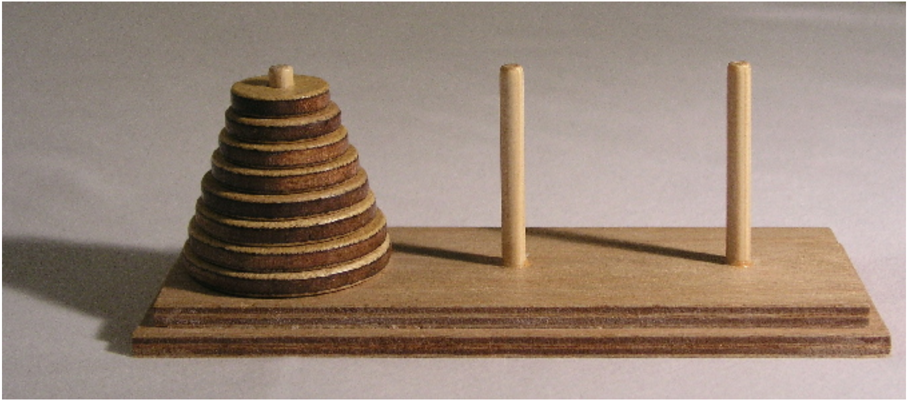


Yet Another Recursively Defined Set



From https://imagej.nih.gov/ij/plugins/fraclac/FLHelp/cross.htm

A Game that is Played Recursively



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Making Recursive Definitions Non-recursive

- Used a lot in the analysis of recursive programs
- Difficult for many recurrences, e.g. Fibonacci series definition
- Unrolling, or repeated expansion. E.g. $f : \mathbb{N} \to \mathbb{N}$, $f(1) = 1, \forall n > 1, f(n) = 2f(n-1)$.

$$\begin{aligned}
f(n) &= 2f(n-1) \\
&= 2(2f(n-2)) \\
&= 2^2 f(n-2) \\
&= 2^2 2f(n-3) \\
&= 2^3 f(n-3) \\
&= \ldots \\
&= 2^{n-1} f(1) \\
&= 2^{n-1}
\end{aligned}$$

Exercises

- Pg 379, Q9
- Pg 379, Q 25
- Write a recursive definition for the set of all binary strings of even length

Applications

- Recursion is a key concept in computation
- Fascinating connections to fractals
- Fractals are useful in computer graphics, image compression

Induction on Recursively Defined Objects

- Prove the following equation for fibonacci numbers $[f_1 = f_2 = 1, \forall n > 2, f_n = f_{n-1} + f_{n-2}]$:

$$\sum_{i=1}^{n} f_i^2 = f_n f_{n+1}$$

- Later: Structural Induction