EECS 1028 E: Discrete Mathematics for Engineers

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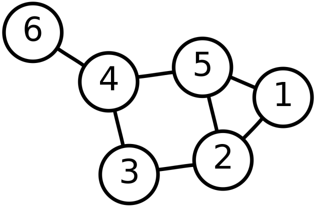
Course page: http://www.eecs.yorku.ca/course/1028
Also on eClass

Graphs: Motivations and Basic Idea

Sec 10.1, 10.2

- Tool for modeling many real applications
- Abstract model that throws away many non-essential aspects of a problem
- Nodes, connected by edges



- No geographical locations attached to node positions, no significance of edge lengths

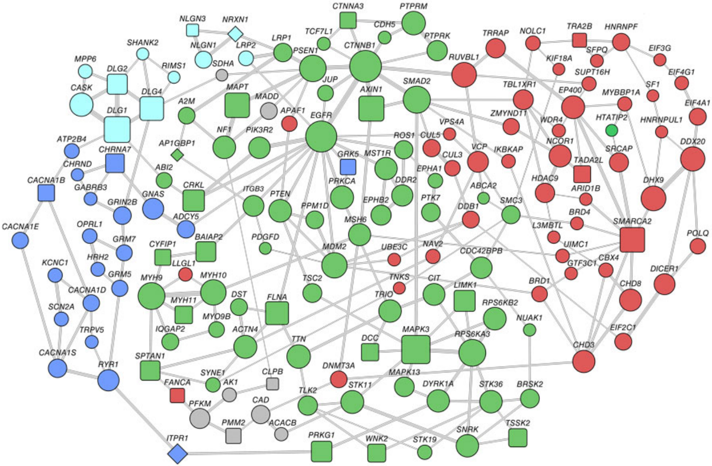
Graphs: Applications

Many Applications, including:

- Road networks
- Subway/Train networks
- Airline networks
- Social Networks
- Power Grid
- Electronic Communication Networks
- Electrical Circuits
- Biological Networks
- Ecological Networks

Graphs: Applications - 2

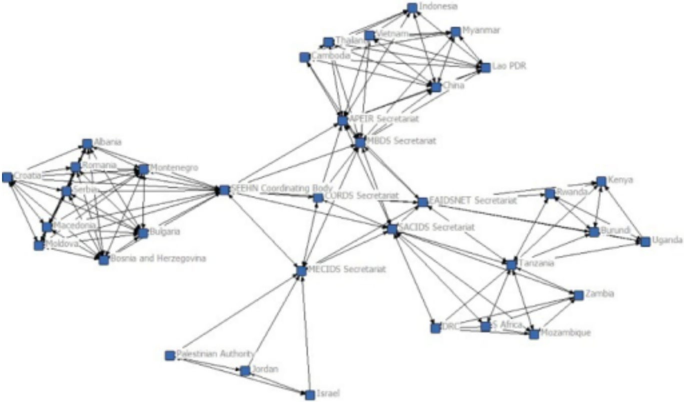
- The web graph
- Software module dependencies
- Computation structure
- Scheduling constraints
- Collaboration graphs
- State graphs of machines and protocols
- Many, many others



Article: Gene networks offer entry point to unraveling autism

From https://spectrumnews.org/news/gene-networks-offer-entry-point-to-unraveling-autism/

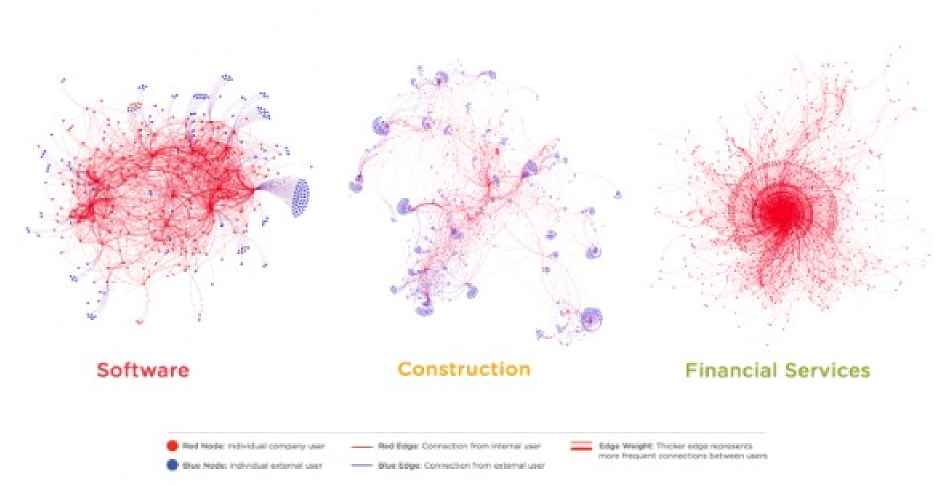
Graphs: Applications - 4



A social network graph illustrating the connections among countries and regional networks in CORDS (CORDS=Connecting Organizations for Regional Disease Surveillance;

From https://openi.nlm.nih.gov/detailedresult.php?img=PMC3557911_EHTJ-6-19913-g001&req=4

Graphs: Applications - 5



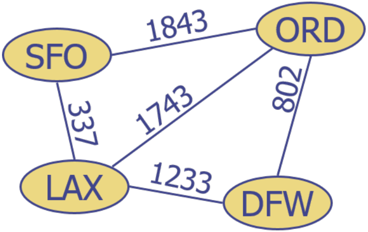
Collaboration graph among people in the same company

From `https://linkurio.us/blog/visualizing-business-organizations-the-collaboration-graph/`

Definitions

$G = (V, E)$, V = set of nodes/vertices, E = set of edges

- Edges incident on a vertex
- Adjacent vertices
- degree of a node
- neighborhood of a node
- Self-loops
- Edge weights



Definitions - 2

- Edge Types:
  - Directed edge: ordered pair of vertices (u, v)
    - u : origin, v : destination
  - Undirected edge: unordered pair of vertices (u, v)
- Graph Types:
  - Directed graph: all the edges are directed
  - Undirected graph: all the edges are undirected
- Paths:
  - Simple Paths
  - Cycles
  - Simple cycles: no vertex repeated

Graph Representations

- Adjacency list
- Adjacency matrix
- Incidence matrix

Elementary Properties

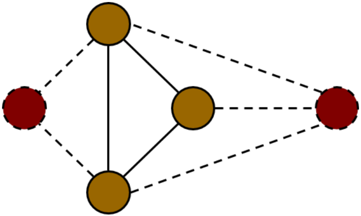
- **Handshaking Theorem**, Thm 1 (pg 686): Sum of degrees equals twice the number of edges in an undirected graph

- Thm 3 (pg 687) Sum of indegrees equals sum of outdegrees in a digraph, which in turn equals $|E|$

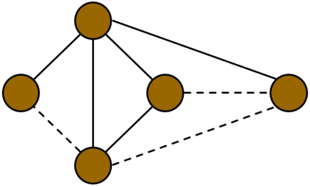
- In an undirected graph $m \leq \frac{n(n-1)}{2}$
  What is the bound for directed graphs?

Subgraphs

- A subgraph S of a graph G is a graph such that
  - The vertices of S are a subset of the vertices of G
  - The edges of S are a subset of the edges of G
- A spanning subgraph of G is a subgraph that contains all the vertices of G



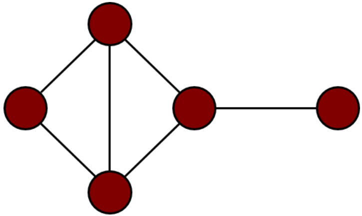
Subgraph



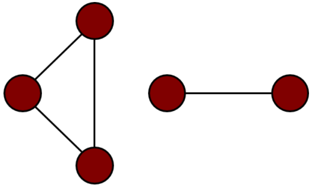
Spanning subgraph

Connected graphs

- A graph is connected if there is a path between every pair of vertices
- A connected component of a graph G is a maximal connected subgraph of G



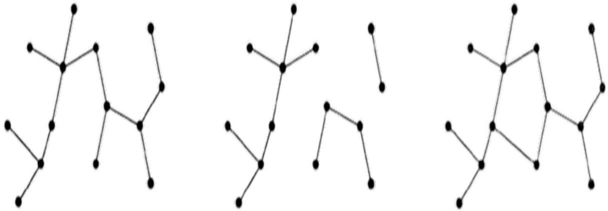
Connected graph



Disconnected graph with two connected components

Trees (Ch 11.1)

- Defn 1 (pg 782) A tree is a connected, acyclic, undirected graph
- A forest is a set of trees
- Theorem [pg 782] An undirected graph is a tree iff there is a unique, simple path between any 2 of its vertices



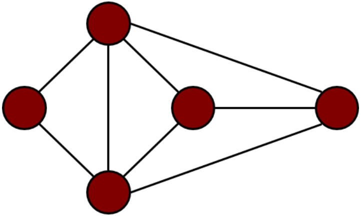
Tree, forest, a cyclic graph

Rooted Trees

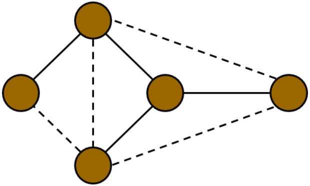
- Internal vertices, leaves
- Defn 2 (pg 783) Rooted tree: tree with a designated vertex called root; all edges directed away from the root.
- Defn 2 (pg 784) A rooted tree is called a m-ary tree if every internal vertex has no more than m children.
  $m = 2$: **binary** tree
- If every internal vertex has exactly m children, the tree is called **full**.
- If a full tree has every leaf at the same depth, it is called **complete**

Spanning Trees

- A spanning tree of a connected graph is a spanning subgraph that is a tree
- A spanning tree is not unique unless the graph is a tree
- Spanning trees have applications to the design of communication networks
- A spanning forest of a graph is a spanning subgraph that is a forest

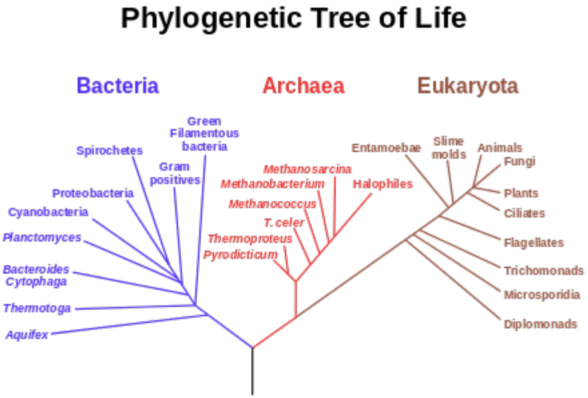


graph

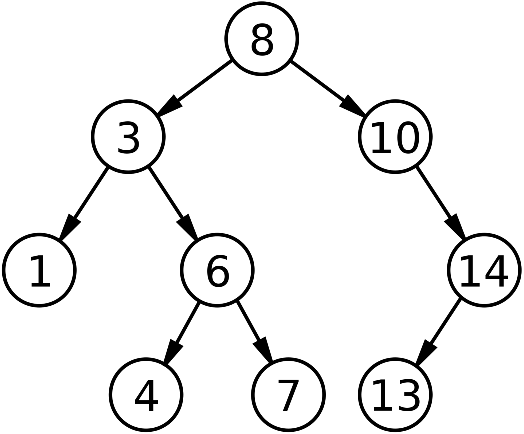


Spanning tree

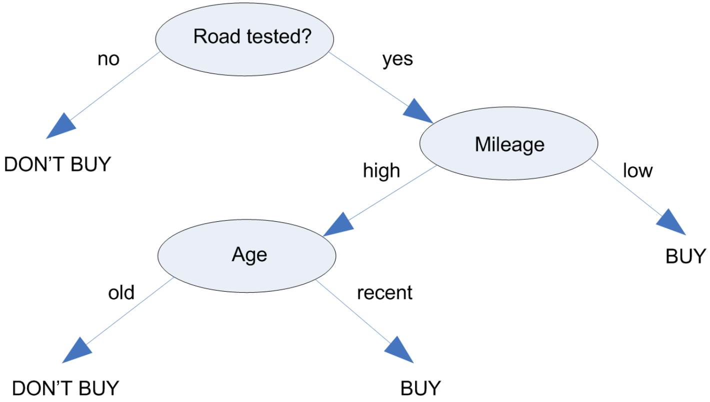
Sec 11.1


Phylogenetic Tree of Life
Bacteria
Archaea
Eukaryota
Green Filamentous bacteria
Spirochetes
Gram positives
Proteobacteria
Cyanobacteria
Planctomyces
Bacteroides Cytophaga
Thermotoga
Aquifex
Methanosarcina
Methanobacterium
Halophiles
Methanococcus
T. celer
Thermoproteus
Pyrodicticum
Entamoebae
Slime molds
Animals
Fungi
Plants
Ciliates
Flagellates
Trichomonads
Microsporidia
Diplomonads

More on Trees: Binary Search Trees



More on Trees: Decision Trees



From `https://www.ibm.com/support/knowledgecenter/en/SS3RA7_15.0.0/com.ibm.spss.modeler.help/nodes_treebuilding.htm`

Other Examples of Trees

- Org charts
- Directory structures
- Organic molecules

Properties of Trees

- **Theorem** (page 782): An undirected graph is a tree iff there is a unique simple path between any two vertices
- **Theorem** (page 788): A tree with n nodes has $n - 1$ edges
- **Theorem** (page 789): A full m-ary tree with i internal vertices has $mi + 1$ vertices
- **Theorem** (page 790): There are at most m^h leaves in a m-ary tree of height h

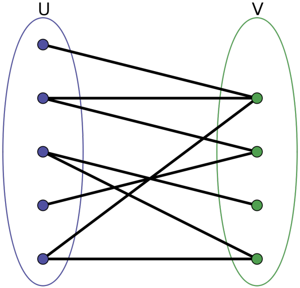
More on Graphs

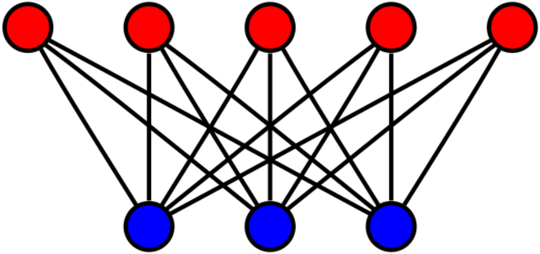
Sec 10.1, 10.2

- Some special families

- Computational problems on graphs

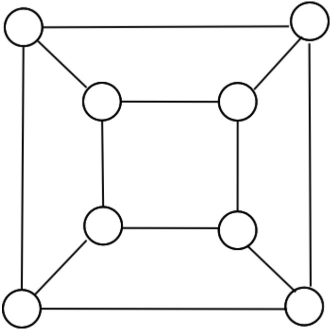
Bipartite Graphs



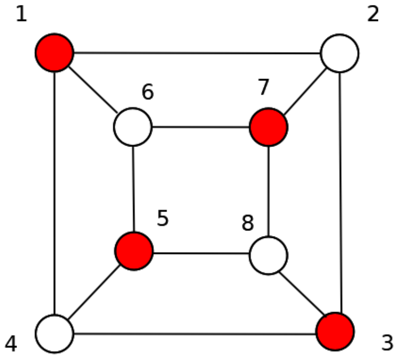


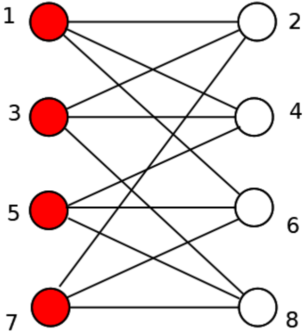
Bipartite Graphs - 2

- Vertex set V partitioned into V_1, V_2, all edges are between V_1, V_2
- How can you recognize a bipartite graph?



Bipartite Graphs - 3

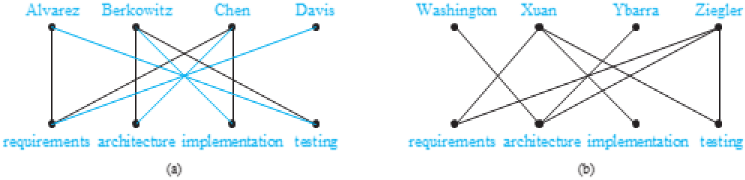




Bipartite Graphs - Coloring

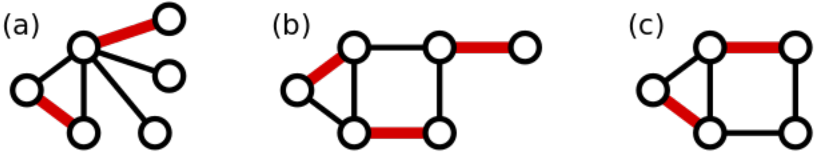
- **Theorem**(page 690): A graph is bipartite iff it is 2 colorable
- **Proof:** A given 2-coloring implies it is bipartite; it is easy to get a 2-coloring if it is known to be bipartite
- How to get a valid 2-coloring?
  Greedy algorithm: color a node white, its neighbours red and so on
- Proof of correctness: later courses
- Any graph containing an odd length cycle is **not** bipartite

Bipartite Graphs - Matching



- A matching is a subset of edges such that no 2 edges are incident on the same vertex
- A maximum matching is a matching with the maximum number of edges

General Graphs - Matching



- Same definitions hold

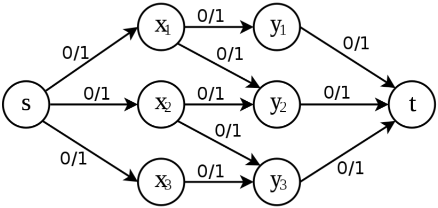
- Are these graphs bipartite?

Bipartite Graphs - Maximum Matching

- When does a maximum matching possible?
- Let A be a set of vertices; $N(A)$ is the set of its neighbour vertices
- Hall's Theorem (page 693): A bipartite graph has a complete matching iff $|N(A)| \geq |A|$ for all subsets $A \subseteq V_1$

Bipartite Graphs - Finding a Maximum Matching

- Hall's Theorem (page 693) only guarantees the existence of a matching
- It does not yield an efficient algorithm to find a maximum matching
- Most commonly used algorithm uses Network Flow to find a maximal matching



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Other Graph Problems

- Connectivity
- Graph Isomorphism
- Graph Coloring