EECS 1028 E: Discrete Mathematics for Engineers

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Course page: `http://www.eecs.yorku.ca/course/1028`
Also on eClass

Elementary Counting

Ch 6, Sec 1, 3, 4

Many Applications:

- How many factors does an integer have?
- How many case-sensitive alphanumeric passwords are there of length k?
- How many binary functions with n binary inputs are there?
- Computing probabilities

Ch 6.2, Pigeonhole Principle was covered earlier

The Product Rule

If 2 independent subtasks can be done in m, n ways (resp.) then the task can be done in mn ways. E.g.,

- If I have 2 keyboard players and 3 percussionists, I can choose a keyboard-percussion duo in 6 ways.
- Q: How many 2 digit numbers are there?
  9 choices for the first digit, 10 choices for the second
- W: How many k character alphanumeric passwords are there?
  62 choices for each of k positions

Counting functions

Boolean output:

- One boolean input: 2^2 functions
- One unsigned integer $(0 \ldots MAXINT - 1)$ input:
  2^{MAXINT} functions
  Caveat: NOT $MAXINT^2$
- n boolean inputs: 2^{2^n} functions

Integer output:

- One integer input: $MAXINT^{MAXINT}$ functions

Counting Binary Strings

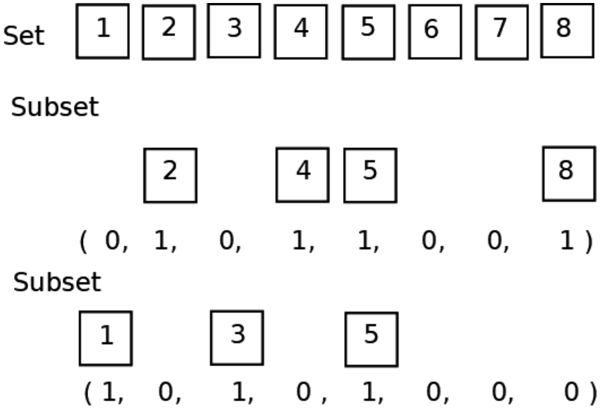
What is the number of binary strings of length n?

- each position can be 0 or 1 (2 choices)
- each position represents an independent choice
- Using the product rule, the number of strings is 2^n

Counting using Bijections

If we can find a bijection $f : A \to B$, then $|A| = |B|$

Claim: there is a bijection between the set of binary strings of length n and the power set of a set with n elements



Cardinality of Power Sets

What is the number of subsets of a set of n elements?

Proof: From the previous diagram,

- Each subset corresponds to a unique binary indicator string of length n
- Each binary string of length n corresponds to a unique subset
- Thus the map is a bijection
- Therefore, each set has the same cardinality, 2^n

Counting Number of Factors

Special Case: How many factors of 2^n are there?

- Wrong argument: each 2 may or may not be chosen

- Correct argument: we can take $0, 1, .., n$ of the 2's. Therefore there are $n + 1$ factors (including 1 and 2^n itself).

Counting Number of Factors

General Case: How many factors of $m = p_1^{a_1} p_2^{a_2} \dots p_K^{a_K}$ are there?

- Claim: the number of factors (including 1 and m itself) is $(a_1 + 1)(a_2 + 1) \dots (a_k + 1)$
- Proof: we can take $0, 1, \dots, a_1$ of the p_1's, $0, 1, \dots, a_2$ of the p_2's and so on.

The Factorial Function

Used in many counting techniques

- Definition: $n! = n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1$

- $0! = 1$ by definition.

Counting Powers of 2

Q: How many factors of 2 are there in $n!$?

- Claim: $(n \div 2) + (n \div 4) + (n \div 8) + \ldots + (n \div 2^k)$ where $2^k \leq n < 2^{k+1}$. Here $n \div m$ is the integer quotient when n is divided by m.
- Proof:
  - Each multiple of 2 gives a factor of 2
  - Each multiple of $2^2 = 4$ gives an **extra** factor of 2
  - Each multiple of $2^3 = 8$ gives **another** extra factor of 2
  - and so on

Counting Number of Trailing Zeroes

Q:How many trailing zeroes in 150!?

- Equal to the number of factors of 10.
- There are many more 2's than 5's so it is enough to count the number of 5's in the factorization.
- So the answer is $(150 \div 5) + (150 \div 25) + (150 \div 125)$
- Easily generalizes to n!

The Sum Rule

If a job can be done in one of m ways or (exclusive or) in one of n ways, the total number of ways is $m + n$. E.g.

- If you must take 3 credits of Math or 3 credits of Physics (but not both) an there are m Math courses and p Physics courses, there is a total of $m + p$ courses to choose from.

- Often used together with the product rule

Counting Strings

- Number of binary strings of length 4 with exactly one 1?
  - There are 4 choices (cases) for placing the 1
  - For each case, the number of ways of placing the 0's is 1
  - By the sum rule the answer is $1 + 1 + 1 + 1 = 4$
- DNA sequences: strings using the characters A, C, T, G

  Number of DNA sequences of length 4 containing exactly 1 A?
  - There are 4 choices (cases) for placing the A
  - For each case, the number of ways of placing the others is 3^3 (using the product rule)
  - By the sum rule, the answer is $3^3 + 3^3 + 3^3 + 3^3 = 4 \cdot 3^3 = 108$

More Complex Problems

- Q: How many 2 digit numbers are multiples of 11 or 13?
  A: 9 (multiples of 11) + 7 (multiples of 13)
- Harder question: How many 3 digit numbers are multiples of 11 or 13?
- The problem is 143 (and its multiples) are multiples of both!
- How to avoid duplication?

Inclusion-Exclusion (or the subtraction rule) Ch 8.5

$$|A \cup B| = |A| + |B| - |A \cap B|$$

- Q: How many 3 digit numbers are multiples of 11 or 13?
  A: Let $A =$ 3 digit multiples of 11, $B =$ No of 3 digit multiples of 13. So $A \cap B =$ No of 3 digit multiples of 143.

- Q: In how many ways can you toss two dice, so that the first toss is a 1 OR the last toss is a 6?
  A: Let $A =$ No. of possible outcomes with the first toss being 1, $B =$ No. of possible outcomes with the second toss being 6. So $A \cap B =$ No. of possible outcomes with the first toss being 1 and the second toss being 6. So,
  $|A \cup B| = |A| + |B| - |A \cap B| = 6 + 6 - 1 = 11$

Complementary Counting

Instead of computing the cardinality of a set, it may be easier to compute the cardinality of the complement

- Q1: How many DNA sequences of length 5 do not contain a C?
  A: Each position has 3 choices, so by the product rule, the answer is $3^5 = 243$

- Q2: How many DNA sequences of length 5 contain at least 1 C?
  The number of all possible DNA sequences of length 5 is $4^5 = 1024$ (4 choices for each position)
  The number of DNA sequences of length 5 with no 1's is 243.
  So the answer is $1024 - 243 = 781$

- Q3: What is the number of length 5 alphanumeric strings with at least one digit?

Combinatorics

Ch 6.3

- Counts arrangements of objects
- Used extensively in discrete probability computations
- Primary tools: Permutations, Combinations

Permutations

Q: In how many ways can n objects be arranged in a line (order matters)?

- The answer is $n! = n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1$
- Reason: n choices for the first place, $n-1$ choices for the second place etc.
- Generalization: $P(n, r)$ number of ways in which r students (out of a class of n) can be lined up for a picture.

$$
\begin{aligned}
P(n,n) &= n! \\
P(n,r) &= n \cdot (n-1) \cdot \ldots \cdot (n-r+1) = n!/(n-r)!
\end{aligned}
$$

Combinations

Different from permutations: order does not matter

Q: In how many ways can a team of r players be chosen from a set of n players (order does not matter)?
A: Define $\binom{n}{r}$ or $C(n,r)$: Number of ways r objects can be chosen from a set of n objects

- Claim: $P(n,r) = C(n,r)P(r,r)$
  Proof: To generate r-permutations from n objects, we first choose a set of r objects (ignoring order) and then permute the r objects in all possible ways

- So, $C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{n!}{r!(n-r)!}$

Combinations - 2

- $C(n, r) = C(n, n - r)$: Choosing r objects from n is the same as choosing the $n - r$ items to leave out (the rest have to be included)
- Alternative way to think about combinations: Suppose we are choosing 3 objects out of n, and order does not matter.
  - There are $P(n, 3)$ ways of choosing them if order does not matter.
  - Consider when objects 1,2,3 are chosen. These objects will show up as $123, 132, 213, 231, 312, 321$, i.e., in all 3! ways.
  - This is true for every other set of 3 objects
  - So we must divide $P(n, 3)$ by 3! to get the number of combinations

Problems

- Q22, pg 435: How many permutations of the letters ABCDEFG contain the string BCD?
  Hint: Treat BCD as one "letter"
- How many binary strings of length n contain exactly k 1's?
  Hint: think of choosing positions for the k 1's
- Q 32, pg 436: How many strings of 6 lowercase letters contain the letter a?
  Hint: use complementary counting
- Suppose a group of 5 men and 7 women want to pick a 5-person team. How many teams can they make with 3 men and 2 women?

Sec 6.4. Pascal's Identity

$$C(n, r) + C(n, r-1) = C(n+1, r)$$

Direct proof:

$$\begin{aligned} C(n, r) + C(n, r-1) &= \frac{n!}{(n-r)!r!} + \frac{n!}{(n-r+1)!(r-1)!} \\ &= \frac{n!}{(n-r)!(r-1)!}\left(\frac{1}{r} + \frac{1}{n-r+1}\right) \\ &= \frac{n!}{(n-r)!(r-1)!}\left(\frac{n+1}{r(n-r+1)}\right) \\ &= \frac{(n+1)!}{(n-r+1)!r!} \end{aligned}$$

Note: This identity uses only additions for computing $C(n, r)$, and avoids overflow issues

Pascal's Identity - A Combinatorial Proof

$$C(n+1, r) = C(n, r) + C(n, r-1)$$

- LHS = Number of ways of choosing r objects from $n+1$ objects
- Alternative way: think about a particular (say, the first) object
  - Case 1: the first item is NOT chosen. So r objects must be chosen for the n remaining objects. There are $C(n, r)$ ways of doing this
  - Case 2: the first item IS chosen. So $r-1$ more objects must be chosen for the n remaining objects. There are $C(n, r-1)$ ways of doing this
- These are disjoint cases, and the Sum Rule is applicable. Using the Sum Rule, we get the RHS

The Binomial Theorem

Page 441

$$(x+y)^n = \sum_{r=0}^{n} C(n,r) x^{n-r} y^r,\ n = 0, 1, 2, \ldots$$

Intuition: think of $(x+y) \cdot (x+y) \cdot \ldots \cdot (x+y)$

- The set of terms of the form $x^{n-r}y^r$ have a 1-1 correspondence with the set of binary strings of length n, having exactly r 1's.
- There are $C(n,r)$ or $\binom{n}{r}$ such strings
- Proof by induction on n

The Binomial Theorem - Implications

$$(x+y)^n = \sum_{r=0}^{n} C(n,r)x^{n-r}y^r$$

It follows that

- $\sum_{r=0}^{n} C(n,r) = 2^n$ (substituting $x = y = 1$)
- $\sum_{r=0}^{n}(-1)^r C(n,r) = 0$ (substituting $x = 1, y = -1$)
- $\sum_{r=0}^{n} C(n,r)2^r = 3^n$ (substituting $x = 1, y = 2$)

Page 419, Pascal's Triangle

$$\binom{0}{0}$$

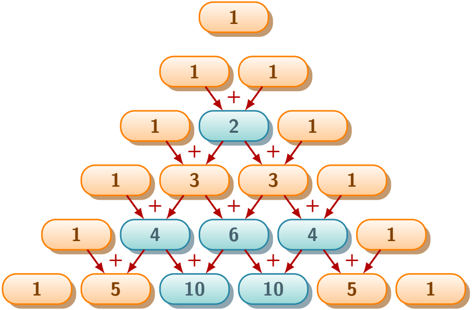
$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$



Problems

- How many binary strings of length 10 with at least 8 1's can we form?

- Use a combinatorial proof to show that

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

- Prove the following inequality (a) by induction on n, and (b) by using the Binomial Theorem

$$\binom{2n}{n} < 4^n$$

Advanced Counting

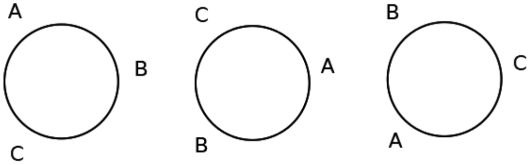
Ch 6, Section 5

- Circular Permutations
- Counting with repetitions
- Counting with identical objects

Circular Permutations

Postions have no absolute value, but clockwise and anti-clockwise order are taken as different.

- Number of Circular Permutations of n different objects: $(n-1)!$
  - Reasoning 1: Fix the first object
  - Reasoning 2: Divide by the *equivalent* configurations (figure for $n = 3$)



- Number of Circular Permutations of n different objects taken r at a time: $P(n, r)/r$

Permutations with Repetitions

- Theorem 1, Page 446: The number of r-permutations of a set with n objects with repetitions allowed is n^r

- Proof: n choices for the first position, n choices for the second position, ..., n choices for the r^{th} position.

Combinations with Repetitions

Example: How many ways can I select 5 pieces of fruit from apples, oranges, strawberries and pears (I have at least 5 of each)?

- Theorem 2, Page 448: There are $C(n+r-1, n-1)$ r-combinations from a set with n elements when repetition is allowed.
- Proof 1: Balls and separators argument
- Proof 2: There is a bijection between the set we are trying to count and the set of permutations of $n+r-1$ of which r are identical and the other $n-1$ objects are identical.

Comparison table

If n is the total number of items and r is the number of items selected, then:

	without repetition	with repetition
permutations	$C(n, r)$	$C(n + r - 1, r)$
combinations	$P(n, r)$	n^r

Permutations with Identical Objects

- Theorem 3 (page 450): The number of different permutations of n objects where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2, ..., n_k indistinguishable objects of type k, and $n = n_1 + n_2 + \ldots + n_k$ is

$$\frac{n!}{n_1!n_2!...n_k!}$$

- Reasoning 1: Choose places for the indistinguishable objects and then they can be arranged in only one way, which gives $C(n, n_1)C(n-n_1, n_2)C(n-n_1-n_1, n_3)...C(n-n_1-\ldots-n_{k-1}, n_k)$
- Reasoning 2: Assume distinguishable and then divide by number of times the same arrangement is counted

Distributing Distinguishable Objects into Distinguishable Boxes

- Theorem 4 (page 452): The number of ways to distribute n distinguishable objects into k distinguishable boxes so that n_i objects are placed in box i, $i = 1, 2, ..., k$ is

$$\frac{n!}{n_1!n_2!...n_k!}$$

- Reasoning: Choose elements for the first box, then the second box, and so on and use the product rule.

Distributing Indistinguishable Objects into Distinguishable Boxes

Page 452

- The number of ways to distribute n indistinguishable objects into k distinguishable boxes is $C(n+k-1, k-1)$

- Reasoning: Use $k-1$ separators. Each permutation of $n+k-1$ objects of which $k-1$ are identical and the other n are identical, corresponds to a different arrangement of n indistinguishable objects into k distinguishable boxes

Problems

- How many strings can we form with the letters of MISSISSIPPI?
- How many solutions does the following equation have over the non-negative integers?

$$x_1 + x_2 + x_3 = 7$$

  Answer: 7 identical objects in 3 distinguishable boxes; $C(9, 2)$

- In how many ways can 5 boys and 5 girls be seated at a round table so that no two girls may be together ?
  Answer: Keep one seat vacant between two boys, 5 boys may be seated in 4! ways. The 5 girls can sit in the 5 seats 5! ways. So the answer is $4!5! = 2880$ ways.