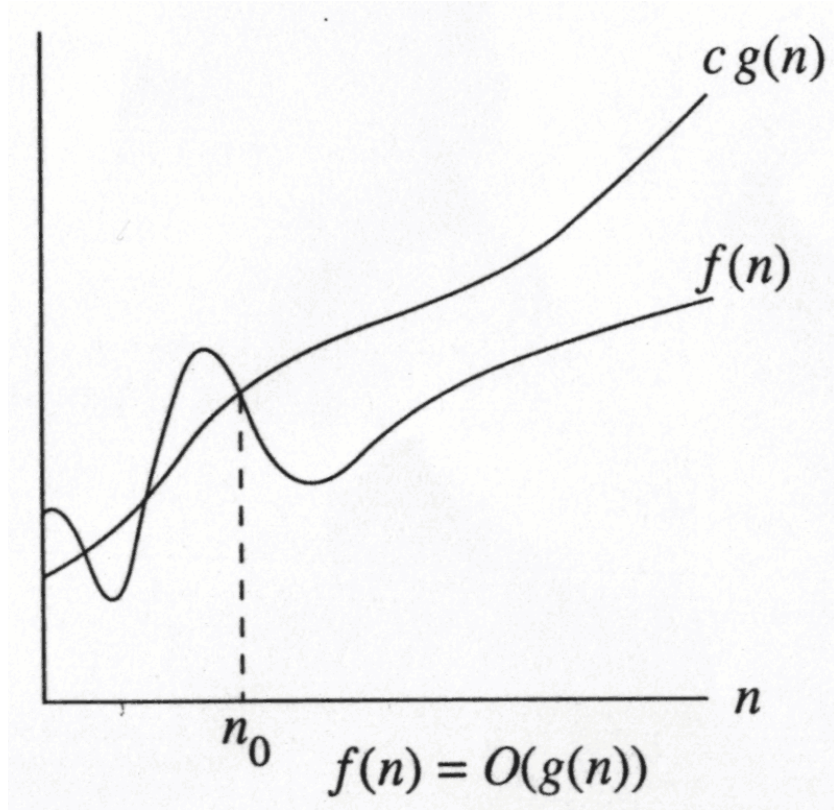
ALGORITHM ANALYSIS (PART 2)

EECS 2011

Growth Rate



- The idea is to establish a relative order among functions for large n.
- $\exists$ c , $n_0 > 0$ such that $f(N) \leq c\, g(N)$ when $N \geq n_0$
- f(N) grows no faster than g(N) for “large” N

Asymptotic Notation: Big-Oh

- f(N) is O(g(N)) if
- There are positive constants c and n_0 such that

  $$f(N) \leq c.g(N) \text{ when } N \geq n_0$$

  where c is a real number.

- The growth rate of f(N) is *less than or equal to* the growth rate of g(N).

- g(N) is an upper bound on f(N).

Big-Oh: Examples

- Let $f(N) = 2N^2$. Then
  - $f(N)$ is $O(N^4)$
  - $f(N)$ is $O(N^3)$
  - $f(N)$ is $O(N^2)$ (best answer, asymptotically tight)

- $O(N^2)$: reads “order N-squared” or “Big-Oh N-squared”

Example

- Show that $7N^2 + 10N + 5N\log N + 3$ is $O(N^2)$.

- Find c and n_0 such that when $N \geq n_0$
  $7N^2 + 10N + 5N\log N + 3 \leq cN^2$

- $7N^2 + 10N + 5N\log N + 3 \leq 7N^2 + 10N^2 + 5N^2 + 3N^2$
  $\leq 25N^2$ when $N \geq 1$
  So $c = 25$ and $n_0 = 1$.

- Use the same "technique" for the following problems.

Big Oh: More Examples

- $N^2 / 2 - 3N$ is $O(N^2)$
- $1 + 4N$ is $O(N)$
- $7N^2 + 10N + 3$ is $O(N^2)$, is also $O(N^3)$
- $\log_{10} N = \log_2 N / \log_2 10$ is $O(\log_2 N)$ or $O(\log N)$
- $\sin N$ is $O(1)$; 10 is $O(1)$, 10^{10} is $O(1)$
- $$\sum_{i=1}^{N} i \le N \cdot N = O(N^2)$$

$$\sum_{i=1}^{N} i^2 \le N \cdot N^2 = O(N^3)$$

- $\log N + N$ is $O(N)$
- $\log^k N$ is $O(N)$ for any constant k
- N is $O(2^N)$, but 2^N is not $O(N)$
- 2^{10N} is not $O(2^N)$

Math Review: Logarithmic Functions

$$x^a = b \quad iff \quad \log_x b = a$$

$$\log ab = \log a + \log b$$

$$\log_a b = \frac{\log_m b}{\log_m a}$$

$$\log a^b = b \log a$$

$$a^{\log n} = n^{\log a}$$

$$\log^b a = (\log a)^b \neq \log a^b$$

$$\frac{d \log_e x}{dx} = \frac{1}{x}$$

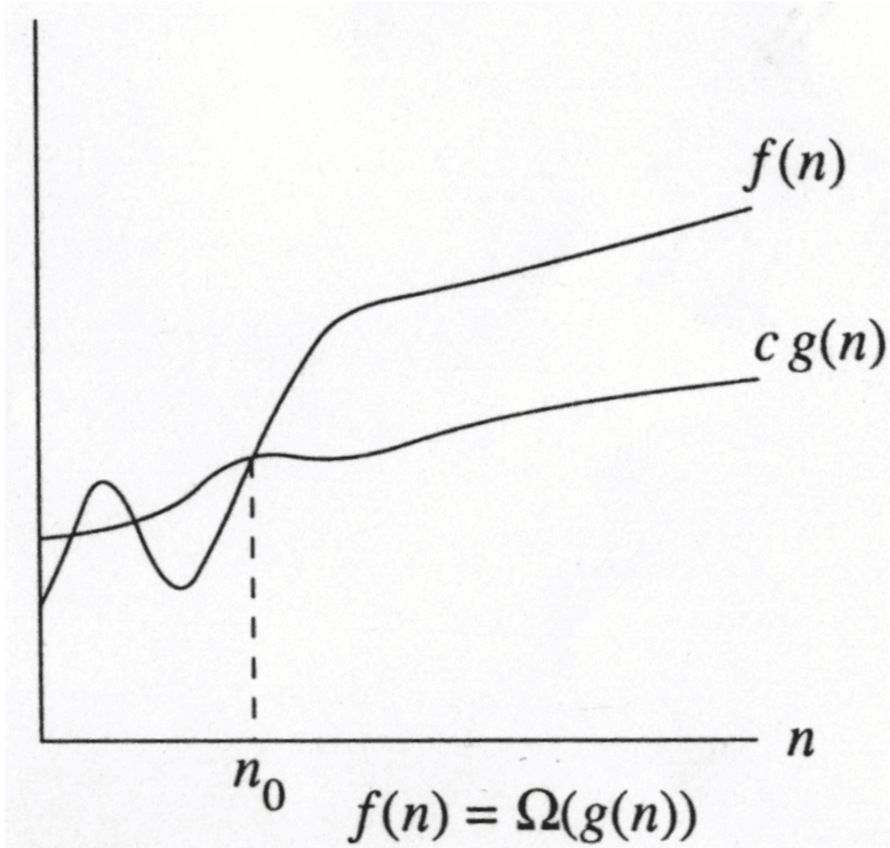
Some Rules

When considering the growth rate of a function using O()

- Ignore the lower order terms and the coefficients of the highest-order term
- No need to specify the base of logarithm
  - Changing the base from one constant to another changes the value of the logarithm by only a constant factor

- If $T_1(N)$ is O(f(N) and $T_2(N)$ is O(g(N)), then
  - $T_1(N) + T_2(N)$ is O(f(N) + g(N))
    (or less formally it is max (O(f(N)), O(g(N)))),
  - $T_1(N) * T_2(N)$ is O(f(N) * g(N))

Big-Omega



- $\exists$ c , n_0 > 0 such that f(N) $\geq$ c g(N) when N $\geq$ n_0
- f(N) grows no slower than g(N) for “large” N

Big-Omega

- f(N) is Ω(g(N)) if
- There are positive constants c and n_0 such that

  $$f(N) \geq c\, g(N) \text{ when } N \geq n_0$$

  where c is a real number.

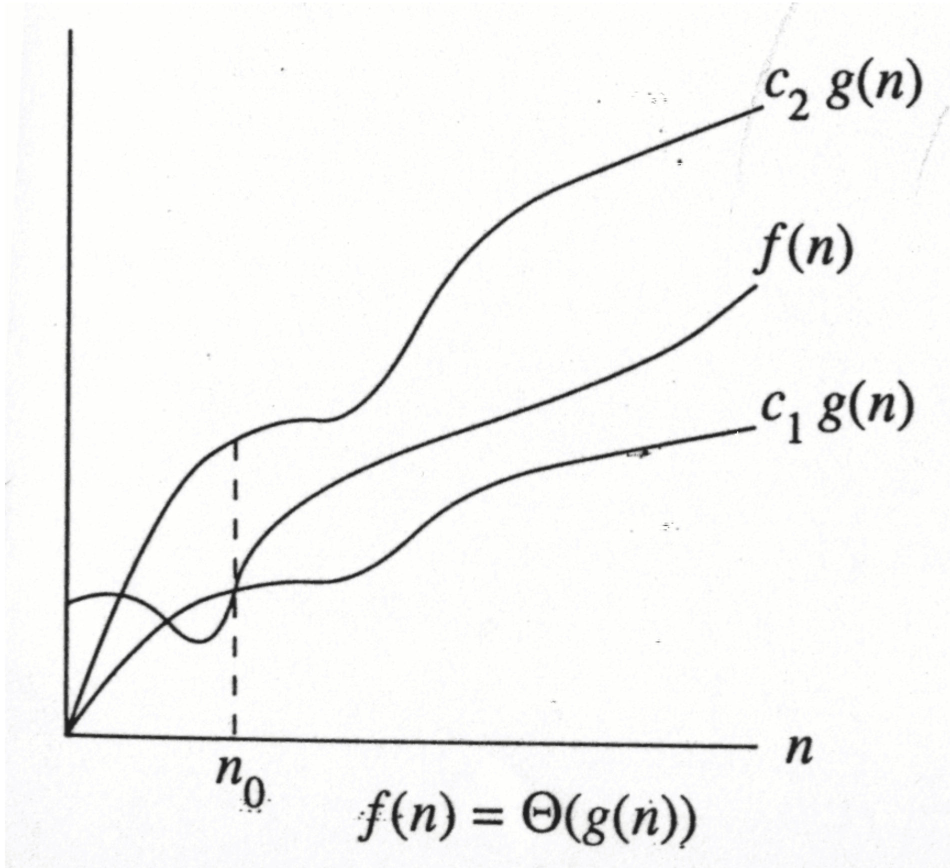
- The growth rate of f(N) is *greater than or equal to* the growth rate of g(N).

- g(N) is a lower bound on f(N).

Big-Omega: Examples

- Let $f(N) = 2N^2$. Then
  - $f(N)$ is $\Omega(N)$ (not tight)
  - $f(N)$ is $\Omega(N^2)$ (best answer)

Big-Theta



- The growth rate of f(N) *is the same as* the growth rate of g(N)
- f(N) is Θ(g(N)) iff f(N) is O(g(N)) and f(N) is Ω(g(N))

Big-Theta: Example

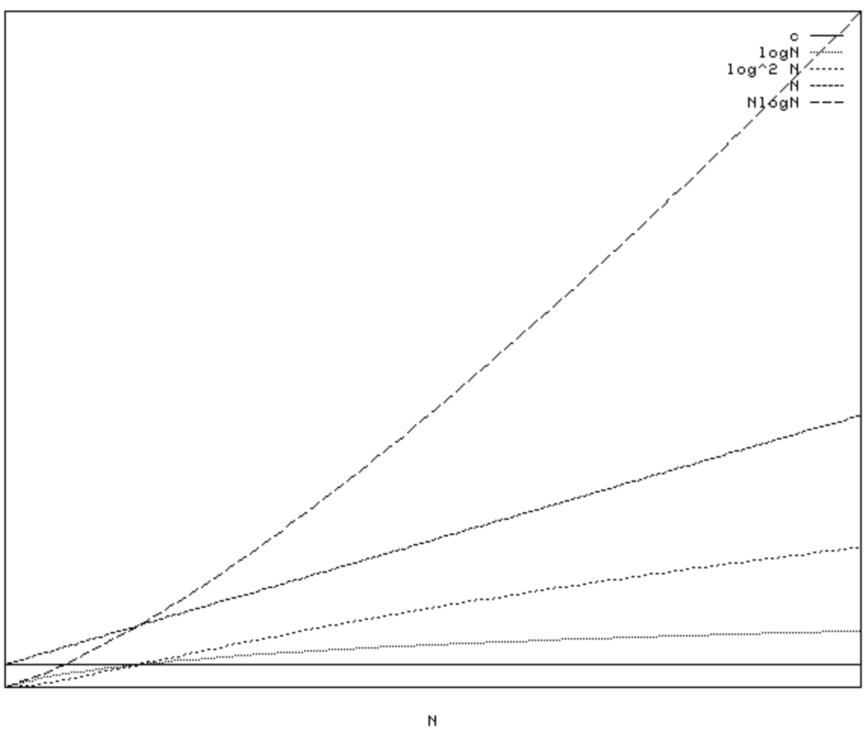
- Let $f(N) = N^2$, $g(N) = 2N^2$
  - Since f(N) is O(g(N)) and f(N) is Ω(g(N)),
    $f(N) = \Theta(g(N))$.

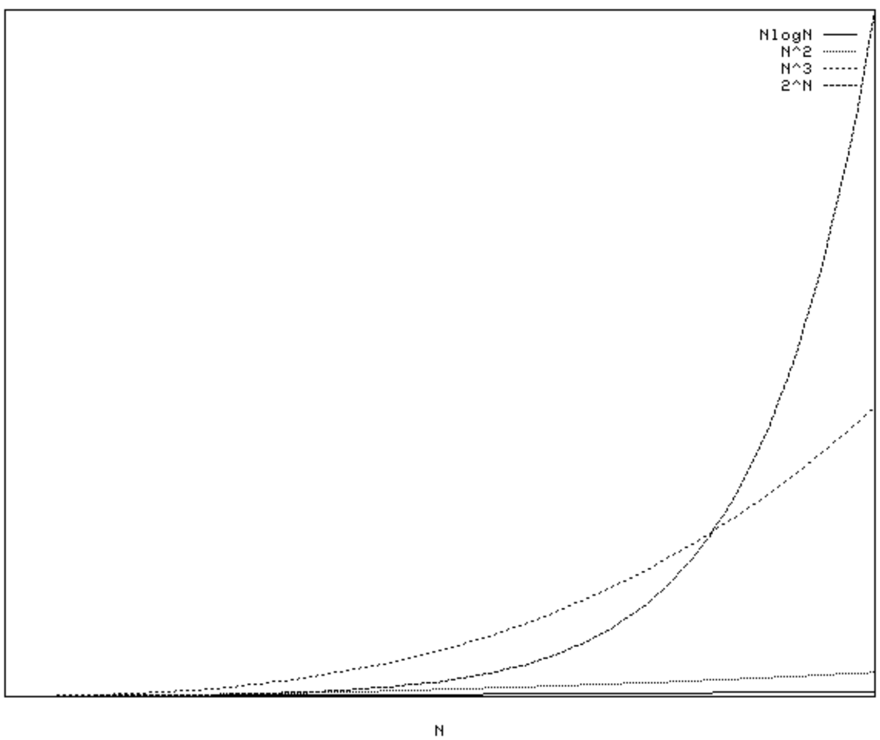
- $c_1 = 1$, $n_1 = 0$
- $c_2 = ½$, $n_2 = 0$

Typical Growth Rates

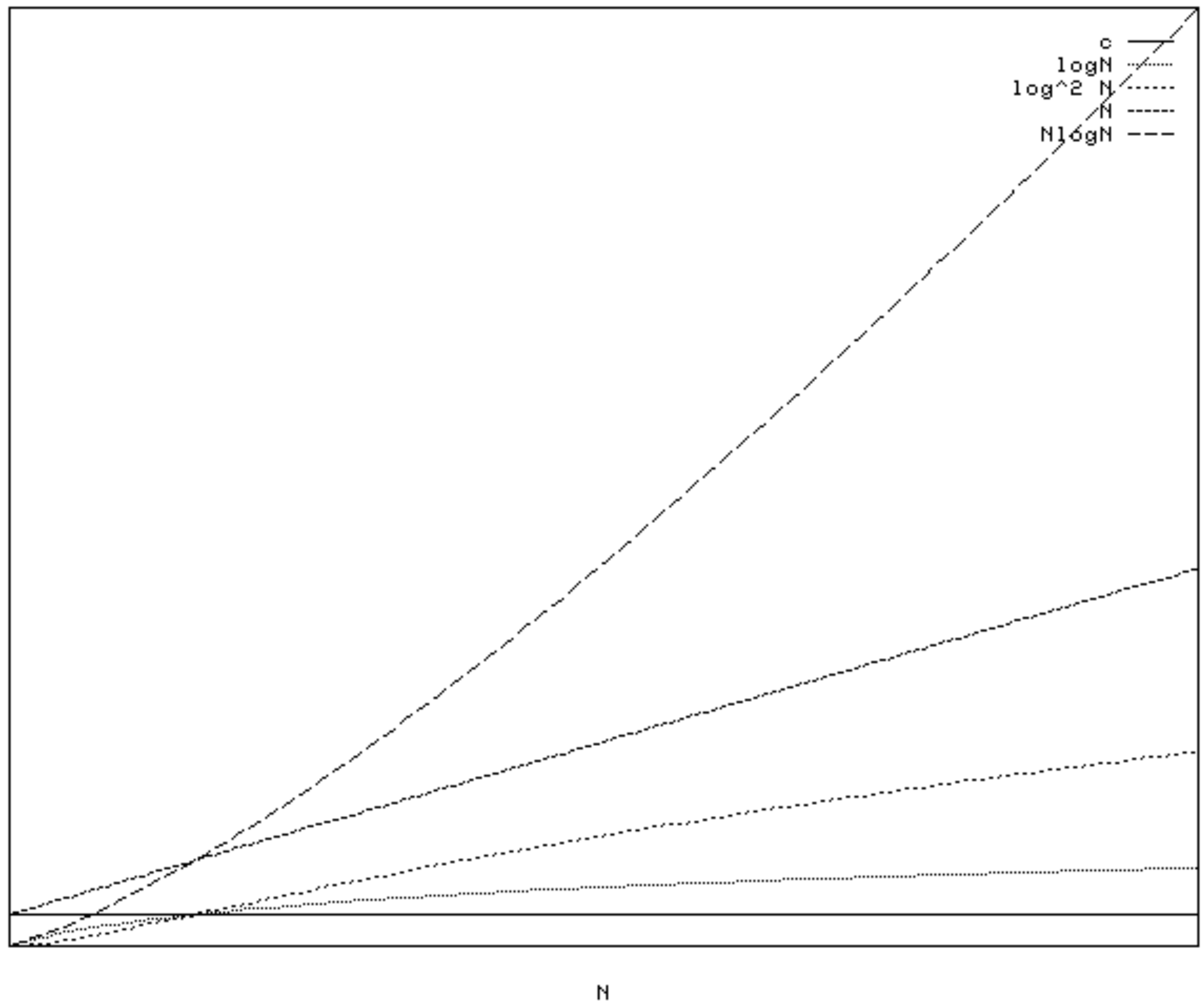
Function	Name
c	Constant
$\log N$	Logarithmic
$\log^2 N$	Log-squared
N	Linear
$N \log N$	
N^2	Quadratic
N^3	Cubic
2^N	Exponential

Figure 2.1 Typical growth rates

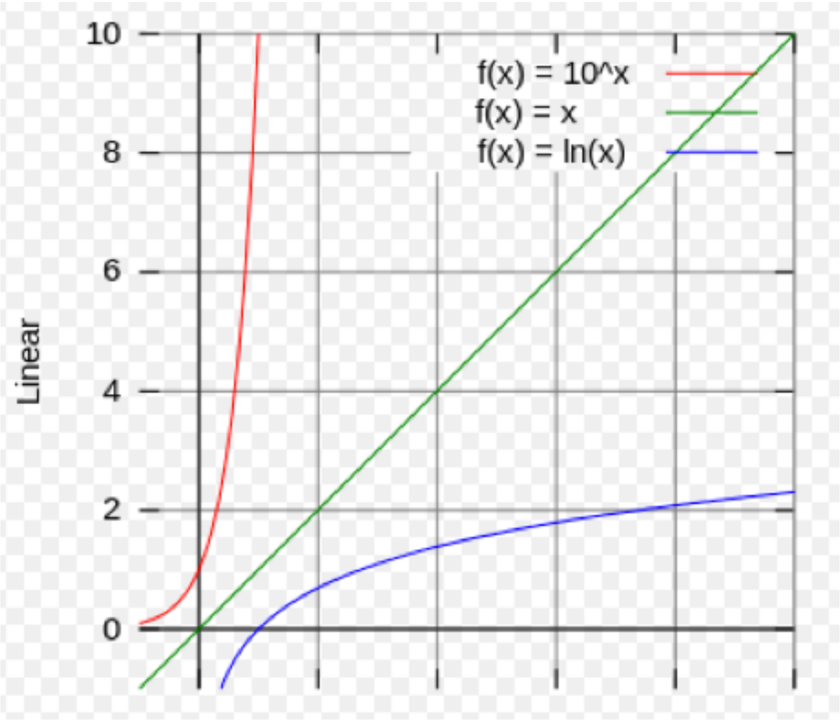


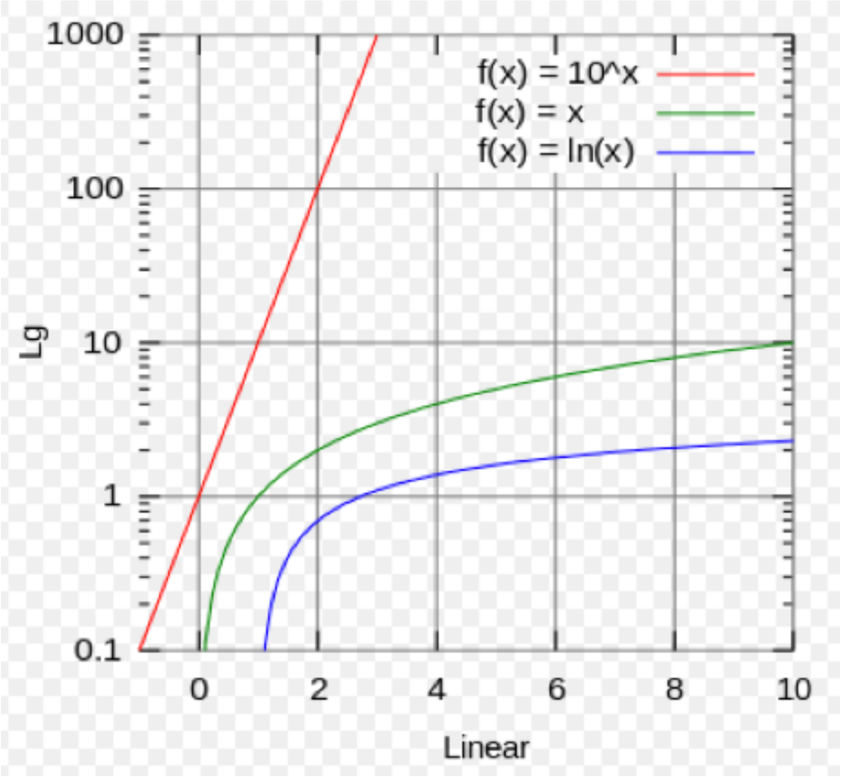


Growth Rates: Linear Scale

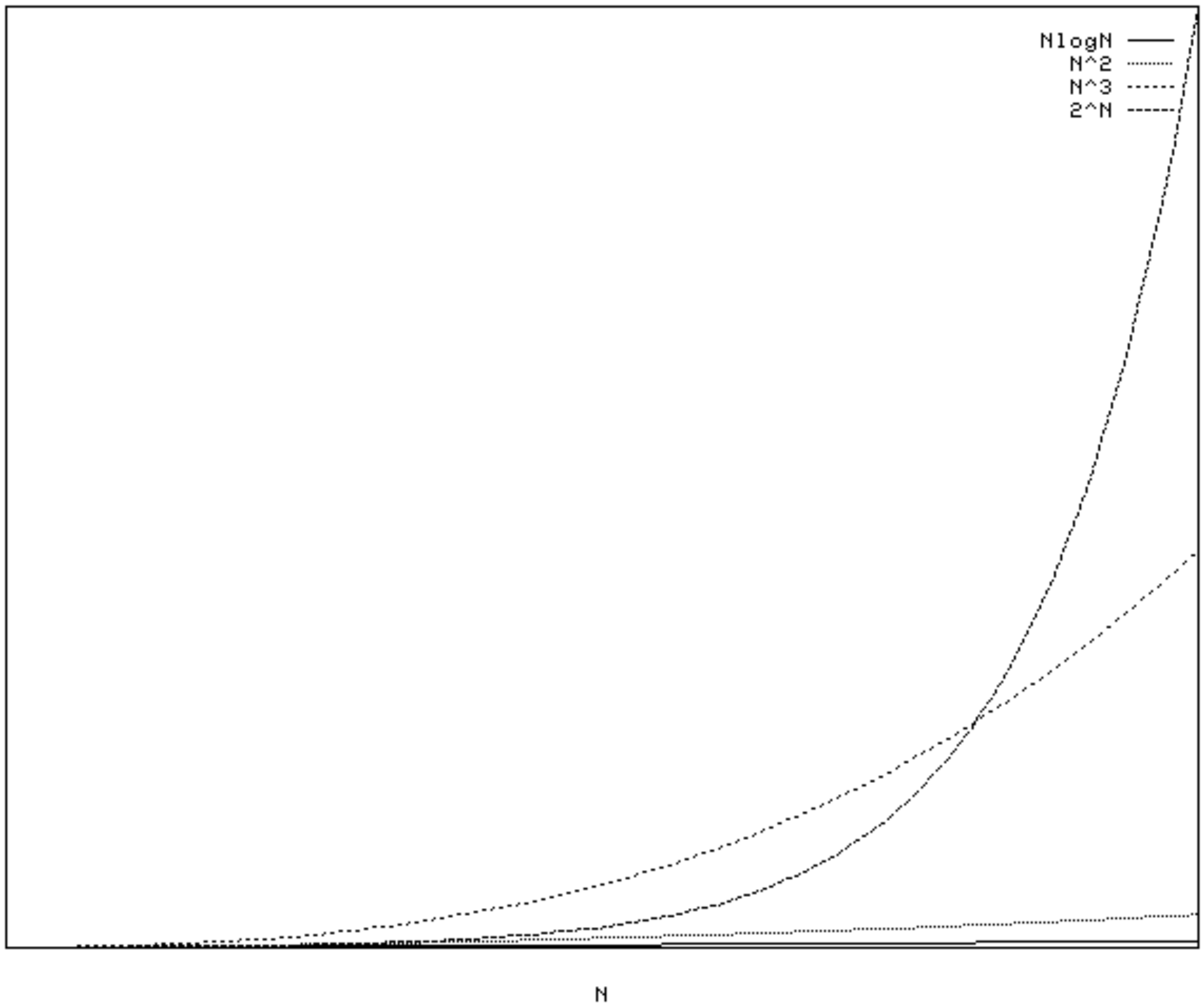


Logarithmic Scale





Growth Rates: Logarithmic Scale



Some More Rules

- If T(N) is a polynomial of degree *k*, then
  T(N) is $\Theta(N^k)$.

- For logarithmic functions,
  $T(\log_m N)$ is $\Theta(\log N)$.

- $\log^k N$ is O(N) for any constant *k*
  (logarithms grow very slowly)

Small-oh

- f(N) is ***o***(g(N)) if

- **$\forall c$**, $\exists n_0$ such that $f(N) < c\, g(N)$ when $N > n_0$

- Less formally, f(N) is ***o***(g(N))
  if f(N) is ***O***(g(N)) and f(N) is not Θ(g(N))

- g(N) grows faster than f(N) for “large” N.

Small-oh: Example

- Let $f(N) = \frac{3}{4} N^2$ and f(N) be ***o***(g(N)).
  - $g(N) = N^2$?
  - $g(N) = N^2 \log N$?
  - $g(N) = N^3$?

Determining Relative Growth Rates of Two Functions

1. Using simple algebra (slide 14)

Example: which function grows faster?

- $f(N) = N \log N$
- $g(N) = N^{1.5}$

2. Using L'Hôpital's rule

Using L' Hôpital's Rule

- L' Hôpital's rule

If $\lim_{n\to\infty} f(N) = \infty$ and $\lim_{n\to\infty} g(N) = \infty$

then $\lim_{n\to\infty} \frac{f(N)}{g(N)} = \lim_{n\to\infty} \frac{f'(N)}{g'(N)}$

- Determine the relative growth rates: compute $\lim_{n\to\infty} \frac{f(N)}{g(N)}$

  - if 0: f(N) is ***o***(g(N))
  - if constant $\neq$ 0: f(N) is Θ(g(N))
  - if ∞: g(N) is ***o***(f(N))
  - limit oscillates: no relation

Summary of Chapter 4

- Given an algorithm, compute its running time in terms of O, Ω, and Θ (if any).
  - Usually the big-Oh running time is enough.
- Given f(n) = 5n + 10, show that f(n) is O(n).
  - Find c and n_0
- Compare the grow rates of 2 functions.
- Order the grow rates of several functions.
  - Use slide 14.
  - Use L’Hôpital’s rule.

Next time …

- Recursion (Chapter 3)

- Reading for this lecture: Chapter 4