ALGORITHM ANALYSIS

EECS 2011

1

Introduction

- What is an algorithm?
  - a clearly specified set of simple instructions to be followed to solve a problem
    - takes a set of values, as input and
    - produces a value, or set of values, as output
  - may be specified
    - in English
    - as a computer program (Java, C, C++, etc.)
    - as a pseudo-code
- Data structures
  - Methods of organizing data
- Program = algorithms + data structures

2

Introduction

- Why need algorithm analysis ?
  - Writing a working program is not good enough.
  - The program may be inefficient!
  - If the program is run on a large data set, then the running time becomes an issue.

Example: Selection Problem

- Given a list of *N* numbers, determine the k^{th} largest, where $k \leq N$.
- Algorithm 1:
  (1) Read *N* numbers into an array
  (2) Sort the array in decreasing order by some simple algorithm
  (3) Return the element in position *k*

Example: Selection Problem (2)

- Algorithm 2:
  (1) Read the first *k* elements into an array and sort them in decreasing order
  (2) Each remaining element is read one by one
    - If smaller than the k^{th} element, then it is ignored
    - Otherwise, it is placed in its correct spot in the array, bumping one element out of the array.
  (3) The element in the k^{th} position is returned as the answer.

Example: Selection Problem (3)

- Which algorithm is better when
  - N = 100 and k = 100?
  - N = 100 and k = 1?

- What happens when N = 1,000,000 and k = 500,000?

- There exist better algorithms than the above two algorithms.

Algorithm Analysis

- We only analyze correct algorithms.

- An algorithm is correct
  - if, for every input instance, it halts with the correct output.

- Incorrect algorithms
  - might not halt at all on some input instances.
  - might halt with output other than the desired answer.

7

Algorithm Analysis (2)

- Analyzing an algorithm
  - Predicting the resources that the algorithm requires.
  - Resources include
    - Memory (space)
    - Computational time (usually most important)
    - Communication bandwidth (in parallel and distributed computing)

8

Algorithm Analysis (3)

- Factors affecting the running time:
  - computer
  - compiler
  - algorithm used
  - input to the algorithm
    - The content of the input affects the running time
    - Typically, the input size (number of items in the input) is the main consideration.
      - sorting problem $\Rightarrow$ the number of items to be sorted
      - multiply two matrices together $\Rightarrow$ the total number of elements in the two matrices
    - And sometimes the input order as well (e.g., sorting algorithms).
- Machine model assumed
  - Instructions are executed one after another, with no concurrent operations $\Rightarrow$ not parallel computers

Analysis Model

- It takes exactly one time unit to do any calculation such as
  - + , -, * , /, %, &, |, &&, ||, etc.
  - comparison
  - assignment
- There is an infinite amount of memory.
- It does not consider the cost associated with page faulting or swapping.
- It does not include I/O costs (which is usually one or more orders of magnitude higher than computation costs).

An Example

```
int sum ( int n ) {
      int partialSum;
1     partialSum = 0;
2     for ( int i = 0; i <= n-1; i++ )
3           partialSum += i*i*i;
4     return partialSum;
}
```

- Lines 1 and 4: one unit each
- Line 3: 4N
- Line 2: 1+(N+1)+N=2N+2
- Total: 6N+4 $\Rightarrow$ O(N)

Running Time Calculations

- Throw away leading constants.
- Throw away low-order terms.
- Compute a Big-Oh running time:
  - An upper bound for running time
  - Never underestimate the running time of a program
  - The program may end earlier, but never later (worst-case running time)

General Rules for Big-Oh: *for* loops

- *for* loops
  - at most the running time of the statements inside the *for* loop (including tests) times the number of iterations.
- Nested *for* loops

```
for (i=0;i<n;i++)
    for (j=0;j<n;j++)
        k++;
```

  - the running time of the statement multiplied by the product of the sizes of all the *for* loops.
  - $O(N^2)$

13

Consecutive Statements

- Consecutive statements

```
for (i=0;i<n;i++)
    a[i]=0;
for (i=0;i<n;i++)
    for (j=0;j<n;j++)
        a[i] += a[j]+i+j;
```

  - These just add.
  - $O(N) + O(N^2) = O(N^2)$

14

if – then – else

- `if C then S1`
  `else S2`
  - never more than the running time of the test plus the larger of the running times of S1 and S2.

```
if (n > 0)
   for ( int i = 0; i < n; i++ )
      sum += i;
else
   System.out.println( "Invalid input" );
```


Strategies

- Analyze from the inside out (loops).
- If there are method calls, analyze these first.
- Recursive methods (later):
  - Could be just a hidden “for” loop ⇒ simple.
  - Solve a recurrence ⇒ more complex.

Worst- / Average- / Best-Case

- Worst-case running time of an algorithm:
  - The **longest** running time for **any** input of size ***n***
  - An upper bound on the running time for any input
    ⇒ guarantee that the algorithm will never take longer
  - Example: Sort a set of numbers in increasing order; and the input is in decreasing order
  - The worst case can occur fairly often
    - Example: searching a database for a particular piece of information
- Best-case running time:
  - sort a set of numbers in increasing order; and the input is already in increasing order
- Average-case running time:
  - May be difficult to define what "average" means

17

Example

- Given an array of integers (unsorted), return true if the array contains number 100, and false otherwise.
  - Best case: ?
  - Worst case: ?
  - Average case: ?
- If the array is sorted …
  - Best case: ?
  - Worst case: ?
  - Average case: ?

18

Informal Introduction to O, Ω and Θ

- Given an unsorted array of integers, return true if a number k is in the array and false otherwise.

```
for( i = 0; i < N; i++ )
   if ( k == A[i] )
      return ( true );
return ( false );
```

- Worst-case running time is O(N).
  ⇒The alg has O(N) running time.
- Best-case running time is O(1).
  ⇒ The alg has Ω(1) running time.

- Given an unsorted array of integers, find and return the maximum value stored in the array.

```
max = A[0];
for( i = 1; i < N; i++ )
   if ( max < A[i] )
      max = A[i];
return( max );
```

- Worst-case running time is O(N).
- Best-case running time is O(N).
- ⇒The alg has Θ(N) running time.

19

Running Time of Algorithms

- Bounds are for algorithms, rather than programs.
  - Programs are just implementations of an algorithm.
  - Almost always the details of the program do not affect the bounds.
- Bounds are for algorithms, rather than problems.
  - A problem can be solved with several algorithms, some are more efficient than others.

20

Example: Insertion Sort

1) Initially p = 1
2) Let the first p elements be sorted
3) Insert the $(p+1)^{th}$ element properly in the list so that now p+1 elements are sorted.
4) Increment p and go to step (3)

Insertion Sort: Example

Original	34	8	64	51	32	21	Positions Moved
After $p = 1$	8	34	64	51	32	21	1
After $p = 2$	8	34	64	51	32	21	0
After $p = 3$	8	34	51	64	32	21	1
After $p = 4$	8	32	34	51	64	21	3
After $p = 5$	8	21	32	34	51	64	4

Insertion Sort: Algorithm

```
for (int p=1; p<a.size(); p++)
{
        int tmp=a[p];
        for (j=p; j> 0 && tmp < a[j-1]; j--)
                a[j] = a[j-1];
        a[j] = tmp;
}
```

- Consists of N – 1 passes
- For pass p = 1 through N – 1, ensures that the elements in positions 0 through p are in sorted order
  - elements in positions 0 through p - 1 are already sorted
  - move the element in position p left until its correct place is found among the first p + 1 elements

Example 2

To sort the following numbers in increasing order:

34 8 64 51 32 21

p = 1; tmp = 8;

34 > tmp, so second element a[1] is set to 34: {8, 34}…

We have reached the front of the list. Thus, 1st position a[0] = tmp=8

After 1st pass: 8 34 64 51 32 21

(first 2 elements are sorted)

P = 2; tmp = 64;
34 < 64, so stop at 3rd position and set 3rd position = 64
After 2nd pass: 8 34 64 51 32 21
(first 3 elements are sorted)

P = 3; tmp = 51;
51 < 64, so we have 8 34 64 64 32 21,
34 < 51, so stop at 2nd position, set 3rd position = tmp,
After 3rd pass: 8 34 51 64 32 21
(first 4 elements are sorted)

P = 4; tmp = 32,
32 < 64, so 8 34 51 64 64 21,
32 < 51, so 8 34 51 51 64 21,
next 32 < 34, so 8 34 34, 51 64 21,
next 32 > 8, so stop at 1st position and set 2nd position = 32,
After 4th pass: 8 32 34 51 64 21

P = 5; tmp = 21, . . .
After 5th pass: 8 21 32 34 51 64

25

Analysis: Worst-case Running Time

- What is the worst input?
- Consider a reversed sorted list as input.
- When a[p] is inserted into the sorted sub-array a[0...p-1], we need to compare a[p] with all elements in a[0...p-1] and move each element one position to the right
  $\Rightarrow$ i steps.
- Inner loop is executed p times, for each p = 1, 2, , ..., N-1
  $\Rightarrow$ Overall: 1 + 2 + 3 + . . . + N-1 = … = $O(N^2)$

26

Analysis: Best-case Running Time

- The input is already sorted in the right order.
- When inserting a[p] into the sorted sub-array a[0...p-1], only need to compare a[p] with a[p-1] and there is no data movement
  $\Rightarrow$ O(1)

- For each iteration of the outer for-loop, the inner for-loop terminates after checking the loop condition once
  $\Rightarrow$ O(N) time

- If input is *nearly sorted*, insertion sort runs fast.

Insertion Sort: Summary

```
for (int p=1; p<a.size(); p++)
{
        int tmp=a[p];
        for (j=p; j> 0 && tmp < a[j-1]; j--)
                a[j] = a[j-1];
        a[j] = tmp;
}
```

- $O(N^2)$
- $\Omega(N)$
- Space requirement is O(?)

Next time …

- Growth rates
- O, Ω, Θ, o

- Reading for this lecture: chapter 4