EECS 2001N: Introduction to the Theory of Computation

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Course page: http://www.eecs.yorku.ca/course/2001N
Also on Moodle

Recap: Countably Infinite Sets

- A set S is infinite if there exists a surjective function $f : S \to \mathbb{N}$: "The set S has at least as many elements as $\mathbb{N}$"
- A set S is countable if there exists a surjective function $f : \mathbb{N} \to S$: "The set S has at most as many elements as $\mathbb{N}$"
- A set S is countably infinite if there exists a bijective function $f : S \to \mathbb{N}$: "The sets $\mathbb{N}$ and S are of the same cardinality"

Countably Infinite Languages

- Let $\Sigma = \{0\}$. Then Σ^* is countable
  $f : \mathbb{N} \to \Sigma^*$, $f(i) = a^{i-1}$
- Let Σ be a finite alphabet. Then Σ^* is countable
  Idea: We list Σ^* in increasing order of length and for strings of the same length we list them in lexicographic order
  E.g.: $\{0, 1\} = \{\epsilon, 0, 1, 00, 01, 10, 11, 000, \ldots\}$
  Then each finite length string gets a unique finite label
- IMPORTANT: Set of all Turing machines T is countable:
  Idea: Every TM can be encoded as a string over some Σ. There is a surjective map from Σ^* to T.

Countably Infinite Languages - 2

- We just argued that the set of all Turing machines T is countable
- What about the set of all languages (problems)?
  We have argued before that this set is $\mathcal{P}(\Sigma^*)$
- We will show next that $\mathcal{P}(\Sigma^*)$ and some other sets (e.g., $\mathbb{R}, \mathcal{P}(\mathbb{N})$) are not countable!

$\mathcal{P}(\Sigma^*)$ is not Countable

Claim: There is no surjection $f : \mathbb{N} \to \mathcal{P}(\Sigma^*)$
Proof by contradiction. Assume there is a surjection f.

- $f(1), f(2), \ldots$ are all infinite bit strings in $\{0,1\}^{\mathbb{N}}$
- Define the infinite string $y = y_1 y_2 \ldots$ by $y_j = \mathrm{NOT(j\text{-}th\ bit\ of}\ f(j))$
- On the one hand $y \in \{0,1\}^{\mathbb{N}}$, but on the other hand: for every $j \in \mathbb{N}$ we know that $f(j) \neq y$ because $f(j)$ and y differ in the j-th bit
- f cannot be a surjection: $\{0,1\}^{\mathbb{N}}$ is uncountable.

Diagonalization

$$
\begin{array}{ll}
s_1 & = 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \ldots \\
s_2 & = 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1 \ldots \\
s_3 & = 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0 \ldots \\
s_4 & = 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1 \ldots \\
s_5 & = 1\ 1\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1 \ldots \\
s_6 & = 0\ 0\ 1\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 0 \ldots \\
s_7 & = 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 0 \ldots \\
s_8 & = 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1 \ldots \\
s_9 & = 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ 0 \ldots \\
s_{10} & = 1\ 1\ 0\ 1\ 1\ 1\ 0\ 0\ 1\ 0\ 1 \ldots \\
s_{11} & = 1\ 1\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 0 \ldots \\
\vdots & \quad \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \ddots
\end{array}
$$

$$s = 1\ 0\ 1\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 1 \ldots$$

- Look at the bit string on the diagonal of this table: $s_d = 0100\ldots$
- The negation of s_d, given by $s = 1011\ldots$, does not appear in the table

Diagonalization: Recap

- We looked at a very innovative technique for proving that a set S is uncountable
- It is a proof by contradiction and starts off by assuming S is countable
- The argument does not (and should not) assume any specific ordering of the set S
- Rather it says: "Give me any enumeration/listing (or labeling with $\mathbb{N}$, or bijection with $\mathbb{N}$), and I will construct an element that is not listed/enumerated/labeled..., and that is a contradiction"

More Diagonalization: $\mathcal{P}(\mathbb{N})$ is not countable

- The set $\mathcal{P}(\mathbb{N})$ contains all the subsets of $\{1, 2, \ldots\}$
- Each subset $X \subseteq \mathbb{N}$ can be identified by an infinite string of bits $x_1 x_2 \ldots$ such that $x_j = 1$ iff $j \in X$
- There is a bijection between $\mathcal{P}(\mathbb{N})$ and $\{0, 1\}^{\mathbb{N}}$ - each bit string represents a unique subset of $\mathbb{N}$ and each subset of $\mathbb{N}$ corresponds to a unique bit string
- We could stop here and invoke the last slide, but let us rework the proof in the last slide
- Proof by contradiction: Assume $\mathcal{P}(\mathbb{N})$ countable. Hence there must exist a surjection f from $\mathbb{N}$ to the set of infinite bit strings $\{0, 1\}^{\mathbb{N}}$, or
  "There is a list of all infinite bit strings"
- Make the exact same diagonalization argument

More Diagonalization: $\mathbb{R}$ is not countable

- Will use diagonalization to prove $R' = [0, 1)$ is uncountable
- Let f be a function $\mathbb{N} \to R'$. So $f(1), f(2), \ldots$ are all infinite digit strings (padded with zeroes if required), and let $f(i)_j$ be the j-th bit of $f(i)$
- Define the infinite string of digits $y = y_1 y_2 \ldots$ by

$$\begin{aligned} y_j &= f(i)_i + 1 \text{ if } f(i)_i < 8 \\ &= 7 \text{ if } f(i)_i \geq 8 \end{aligned}$$

- Invoke diagonalization to get a contradiction
- So $R' \subset \mathbb{R}$ is not countable, and therefore $\mathbb{R}$ is not countable