# EECS 2001N : Introduction to the Theory of Computation

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Course page: `http://www.eecs.yorku.ca/course/2001N`
Also on Moodle

# The Big Question

- Are there languages that are not regular?
- How can we show a language is NOT regular?

# An Example Problem

$$L = \{0^n 1^n | n \in \mathbb{N}\}$$

- 'Playing around' with FA convinces you that the 'finiteness' of FA is problematic for "all $n \in \mathbb{N}$"
- The problem occurs between the $0^n$ and the $1^n$
- Informal: the memory of a FA is limited by the the number of states $|Q|$

# Proving Non-regularity

- Prove a general statement – **no** DFA exists for a given problem
- Cannot assume an automaton structure or a specific strategy
- Need an argument that holds for **all** DFA's

# Repeating DFA Paths

- Consider an accepting DFA $M$ with size $|Q|$
- On a string of length $p$, $p+1$ states get visited
- For an accepting path $p \geq |Q|$, there must be a $j$ such that the computational path looks like: $q_1, \ldots, q_j, \ldots, q_j, \ldots, q_k$
- The action of the DFA in $q_j, \ldots, q_j$ is always the same. If we repeat (or ignore) the $q_j, \ldots, q_j$ part, the new path will again be an accepting path

# Line of Reasoning

Proof by contradiction:

- Assume that $L$ is regular
- Hence, there is a DFA $M$ that recognizes $L$
- For strings of length $> |Q|$ the DFA $M$ has to 'repeat itself'
- Show that $M$ will accept strings outside $L$
- Conclude that the assumption was wrong, so $L$ is not regular

Note that we use the simple DFA, not the more elaborate (but equivalent) NFA or GNFA

# Pumping Lemma

For every regular language $L$, there is a **pumping length** $p$, such that for any string $s \in L$ and $|s| \geq p$, we can write $s = xyz$ with

1. $xy^iz \in L$ for every $i \in \{0, 1, 2, \ldots\}$
2. $|y| \geq 1$
3. $|xy| \leq p$

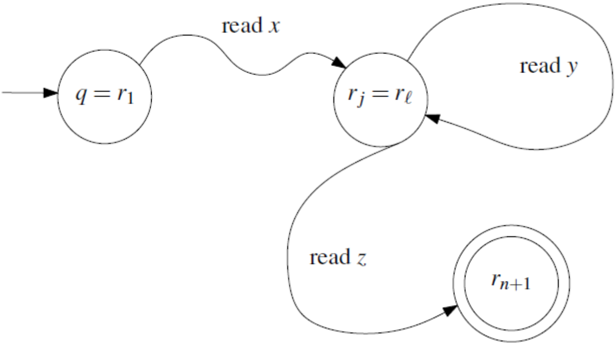
Note that:
(1) implies that $xz \in L$.
(2) says that $y$ cannot be the empty string $\epsilon$.
(3) is not always used

# Formal Proof of the Pumping Lemma

- Let $M = (Q, \Sigma, \delta, q_1, F)$ with $Q = \{q_1, \ldots, q_p\}$
- Let $s = s_1 \ldots s_n \in L(M)$ with $|s| = n \geq p$
- Computational path of $M$ on $s$ is the sequence $r_1, \ldots, r_{n+1}$, with each $r_i \in Q$, and with $r_1 = q_1$, $r_{n+1} \in F$ and $r_{t+1} = \delta(r_t, s_t)$ for $1 \leq t \leq n$
- When DFA $M$ reads $s$, it visits the states $r_1, \ldots, r_{n+1}$, and since $s \in L(M)$, $r_{n+1} \in F$
- Because $n + 1 \geq p + 1 > p$, there exist $j, \ell$ such that $r_j = r_\ell$ (with $j < \ell$ and $\ell \leq p + 1$)

# Formal Proof of the Pumping Lemma

- Let $x = s_1..s_{j-1}$, $y = s_j..s_{k-1}$, and $z = s_k \ldots s_{n+1}$.
  $x$ takes $M$ from $q_1 = r_1$ to $r_j$, $y$ takes $M$ from $r_j$ to $r_j$, and $z$ takes $M$ from $r_j$ to $r_{n+1} \in F$
- As a result: $xy^iz$ takes $M$ from $q_1$ to $r_{n+1} \in F$ $(i \geq 0)$



# Pumping Lemma-based Proof: Example 1

$$B = \{0^n 1^n | n \in \{0, 1, \ldots\}\}$$

- Assume that $B$ defined above is regular
- Let $p$ be the pumping length, and $s = 0^p 1^p \in B$
- P.L.: There exists $s = xyz = 0^p 1^p$, with $xy^i z \in B$ for all $i \geq 0$
- Three options for $y$:
  - $y = 0^k$, hence $xyyz = 0^{p+k} 1^p \notin B$
  - $y = 1^k$ hence $xyyz = 0^p 1^{k+p} \notin B$
  - $y = 0^k 1^m$, hence $xyyz = 0^p 1^m 0^k 1^p \notin B$
- Conclusion: The pumping lemma does not hold, the language $B$ is not regular.

# Pumping Lemma-based Proof: Example 2

$$C = \{ww | w \in \{0,1\}^*\}$$

- Let $p$ be the pumping length, and take $s = 0^p 1 0^p 1 \in B$
- and take $s = 0^p 1 0^p 1$
- Let $s = xyz = 0^p 1 0^p 1$ with condition 3) $|xy| \leq p$
- Only one option: $y = 0^k$, so $xyyz = 0^{p+k} 1 0^p 1 \notin B$
- Without (3) this would have been a pain.

# Intersecting Regular Languages

$$C = \{w | w \in \{0,1\}^*, w \text{ has equal numbers of 0, 1}\}$$

- Using the Pumping Lemma is not easy
- Idea: If $C$ is regular and $F$ is regular, then the intersection $C \cap F$ has to be regular as well
- Assume that $C$ is regular
- Take the regular language $F = \{0^n 1^m | n, m \in \mathbb{N}\}$, then the intersection is $C \cap F = \{0^n 1^n | n \in \mathbb{N}\}$
- But we know that $C \cap F$ is not regular
- Conclusion: $C$ is not regular

# Example of Pumping Down

$$E = \{0^i 1^j | i, j \in \mathbb{N}, i \geq j\}$$

- Problem: 'pumping up' $s = 0^p 1^p$ with $y = 0^k$ gives $xyyz = 0^{p+k}1^p$, $xy^3z = 0^{p+2k}1^p$, which are all in $E$ (hence do not give contradictions)
- Solution: pump down to $xz = 0^{p-k}1^p$
- Overall for $s = xyz = 0^p 1^p$ (recall $|xy| \leq p$): $y = 0^k$, hence $xz = 0^{p-k}1^p \notin E$
- Contradiction: $E$ is not regular

# Pumping Lemma Usage: Summary

- You are given a pumping number
- You choose a string $s$
- There exist $x, y, z$ (satisfying some criteria)
- You choose $i$ in $xy^iz$, and show it violates criterion of set for that $i$
- Now you can conclude that the language is not regular

Note: There are other, less general, less popular techniques for showing a language is not regular