EECS 2001N : Introduction to the Theory of Computation

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Course page: `http://www.eecs.yorku.ca/course/2001N`
Also on Moodle

The Big Question

Note that

- NFA can solve every problem that DFA can (DFA are also NFA)
- Can DFA solve every problem that NFA can?
- In other words: Are NFA more powerful than DFA?

The Surprising Answer

We will prove that

- every NFA is equivalent to a DFA (with upto exponentially more states)
- Non-determinism does not help FA's to recognize more languages!
- NFAs recognize regular languages
- Corollary: DFAs and NFAs can be used interchangeably to solve problems or study properties of regular languages

Terminology: ϵ-closure

- Let $N = (Q, \Sigma, \delta, q_0, F)$ be any NFA
- Consider any set $R \subseteq Q$
- Define $E(R) = \{q | q \text{ can be reached from a state in } R \text{ by following 0 or more } \epsilon\text{-transitions}\}$
- $E(R)$ is the ϵ- closure of R under ϵ-transitions

Equivalence of DFA, NFA

- Statement: For all languages $L \subseteq \Sigma^*$,
  $L = L(N)$ for some NFA N if and only if $L = L(M)$ for some DFA M

- One direction is easy:
  A DFA M is also a NFA N. So N does not have to be "constructed" from M

- The other direction: Construct M from N

Equivalence of DFA, NFA - A Special Case

Given $N = (Q, \Sigma, \delta, q_0, F)$, construct $M = (Q', \Sigma, \delta', q_0', F')$ so that for any $w \in \Sigma^*$, M accepts w if and only if N accepts w.
First a special case: Assume that NFA N has no ϵ-transitions

- Need to keep track of each subset of Q
- So $Q' = \mathcal{P}(Q), q_0' = \{q_0\}$
- $\delta'(R, a) = \cup(\delta(r, a))$ over all $r \in R, R \in Q'$
- $F' = \{R \in Q' | R \text{ contains an accept state of } F\}$

Next: let us assume that ϵ-transitions are used in N

Equivalence of DFA, NFA - The General Case

- $Q' = \mathcal{P}(Q)$
- $q_0' = E(\{q_0\})$
- for all $R \in Q'$ and $a \in \Sigma$
  $\delta'(R, a) = \{q \in Q | q \in E(\delta(r, a)) \text{ for some } r \in R\}$
- $F' = \{R \in Q' | R \text{ contains an accept state of } N\}$

Why This Construction Works...

for any string $w \in \Sigma^*$,

- can argue informally that w is accepted by N iff w is accepted by M
- Can prove using induction on the number of steps of computation

Closure: Revisiting Old Terminology

A set is defined to be closed under an operation if that operation on members of the set always produces a member of the same set. E.g.:

- The integers are closed under addition, multiplication
- The integers are not closed under division
- Σ^* is closed under concatenation
- A set can be defined by closure – Σ^* is called the (Kleene) closure of Σ under concatenation.

New Terminology: Regular Operations

The regular operations are:

- Union
- Concatenation
- Star (Kleene Closure): For a language A, define $A^* = \{w_1 w_2 w_3 \ldots w_k | k \geq 0, \text{ and each } w_i \in A\}$

Want to prove that regular languages are closed under regular operations

Proving Closure under Regular Operations

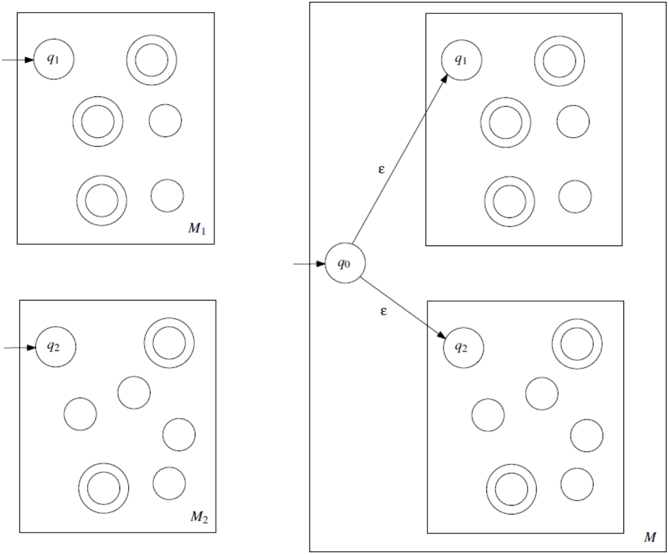
We showed that regular languages are closed under:

- Complementation (Theorem 2.6.4)
- Union

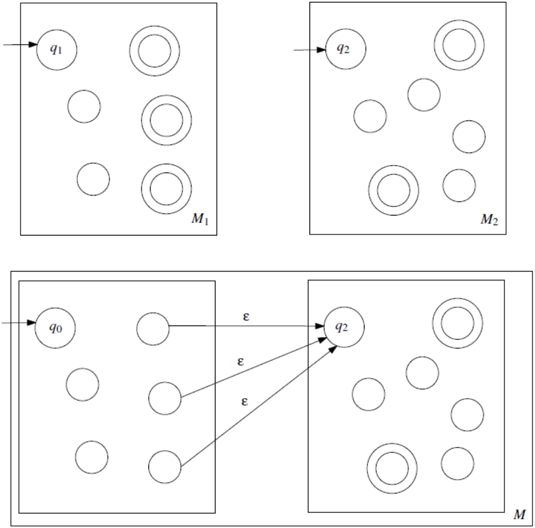
We got stuck at concatenation, and introduced nondeterminism
Next, we show closure under

- Union (easier proof)
- Concatenation
- Star (Kleene Closure)

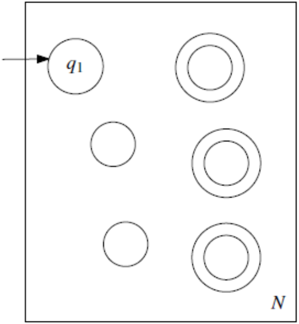
Proving Closure Under Union

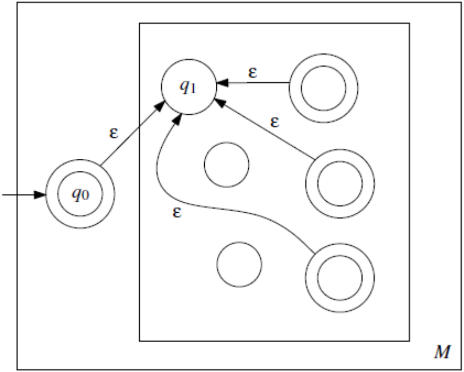


Proving Closure Under Concatenation



Proving Closure Under Kleene Star





Incorrect reasoning about RL

- Since $L_1 = \{w|w = a^n, n \in \mathbb{N}\}$, $L_2 = \{w|w = b^n, n \in \mathbb{N}\}$ are regular, therefore $L_1 \cdot L_2 = \{w|w = a^n b^n, n \in \mathbb{N}\}$ is regular

- If L_1 is a regular language, then $L_2 = \{w^R|w \in L_1\}$ is regular, and therefore $L_1 \cdot L_2 = \{ww^R|w \in L_1\}$ is regular

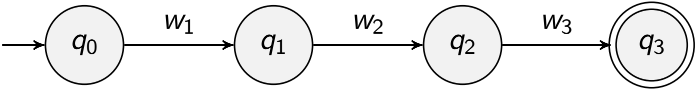
Putting it all together

A recursive definition for regular languages

- $\emptyset$, $\{\epsilon\}$ and $\{a\}$ for any symbol $a \in \Sigma$ are regular languages
- If L_1 and L_2 are regular languages, then $L_1 \cup L_2$, $L_1 L_2$ and L_1^* are regular languages.
- Nothing is a regular language unless it is obtained from the above two clauses.

Every Finite Language is Recognized by a NFA

- Given a word $w = w_1 w_2 \ldots w_k$ there is a NFA that recognizes $\{w\}$. Example of $w = w_1 w_2 w_3$

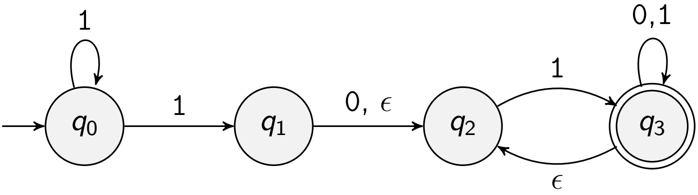


- Use the union construction on languages containing single words...

Regular Languages: Exercises

- Prove the following result:
  If L_1 and L_2 are regular languages, then $L_1 \cap \overline{L_2}$ is a regular language too

- Describe the language that is recognized by this NFA:



Another Characterization of Regular Languages

Regular Expressions

- Unix 'grep' command: Global Regular Expression and Print
- Lexical Analyzer Generators (part of compilers)
- Other practical uses in software design
- Will see some examples and then formulate a precise definition
- Finally will obtain another characterization of regular languages!

Examples of Regular Expressions

- $e_1 = a \cup b,\ L(e_1) = \{a, b\}$
- $e_2 = ab \cup ba,\ L(e_2) = \{ab, ba\}$
- $e_3 = a^*,\ L(e_3) = \{a\}^*$
- $e_4 = (a \cup b)^*,\ L(e_4) = \{a, b\}^*$
- $e_5 = (e_m \cdot e_n),\ L(e_5) = L(e_m) \cdot L(e_n)$
- $e_6 = a^* b \cup a^* bb,\ L(e_6) = \{w | w \in \{a, b\}^*$ and w has 0 or more a's followed by 1 or 2 b's$\}$

Regular Expressions: Recursive Definition

Regular Expressions (RE)

- $R = a$, with $a \in \Sigma$: $R \in \mathrm{RE}$
- $R = \epsilon$ (empty expression): $R \in \mathrm{RE}$
- $R = \emptyset$: $R \in \mathrm{RE}$
- $R = (R_1 \cup R_2)$, where $R_1, R_2 \in \mathrm{RE}$: $R \in \mathrm{RE}$
- $R = (R_1 \cdot R_2)$, where $R_1, R_2 \in \mathrm{RE}$: $R \in \mathrm{RE}$
- $R = (R_1^*)$, where $R_1 \in \mathrm{RE}$: $R \in \mathrm{RE}$

Precedence order: $*, \cdot, \cup$

Regular Expressions: Identities (Thm 2.7.4)

- $R_1\emptyset = \emptyset R_1 = \emptyset$
- $R_1\epsilon = \epsilon R_1 = R_1$
- $R_1 \cup \emptyset = \emptyset \cup R_1 = R_1$
- $R_1 \cup R_1 = R_1$
- $R_1 \cup R_2 = R_2 \cup R_1$
- $R_1(R_2 \cup R_3) = R_1R_2 \cup R_1R_3$
- $(R_1 \cup R_2)R_3 = R_1R_3 \cup R_2R_3$
- $R_1(R_2R_3) = (R_1R_2)R_3$
- $\emptyset^* = \epsilon$
- $\epsilon^* = \epsilon$
- $(\epsilon \cup R_1)^* = R_1^*$
- and a few others

Regular Expressions: The Big Result

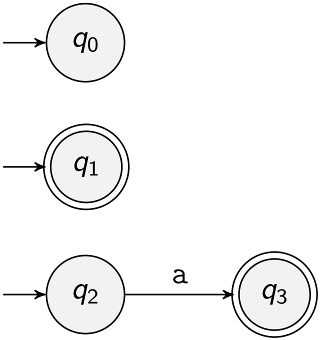
Regular expressions (RE) and Regular Languages are the same set

- Theorem 2.8.1 Let L be a language. Then L is regular if and only if there exists a regular expression that describes L
- Part 1: If a language is described by a regular expression, then it is regular (We will show how to convert a regular expression R into an NFA M such that $L(R) = L(M)$)
- Part 2: If a language is regular, then it can be described by a regular expression

Part 1: RE to RL (NFA construction)

Construction: Use recursive definition

- $R = \emptyset$, $R = \epsilon$
- $R = a$, with $a \in \Sigma$
- $R = (R_1 \cup R_2)$, with R_1 and R_2 regular expressions
- $R = (R_1 \cdot R_2)$, with R_1 and R_2 regular expressions
- $R = (R_1^*)$, with R_1 a regular expression



Last 3 are similar to closure of RL under union, concatenation, star

RE to NFA: Examples

- $R = ab \cup ba$ ($L = \{ab, ba\}$)
- $R = ab(ab)^*$ ($L = \{ab, abab, ababab, \ldots\}$)
- $L = \{w | w = a^m b^n, m < 10, n > 10\}$

Part 2: RL to RE

If a language is regular, then it can be described by a regular expression.

- Proof idea: Sipser's and our texts use different techniques. Each is somewhat complex, but accessible. You can read the proof in Ch 2.8.2. You will not be tested on this part.

- Why is this useful?
  – one use in answering "what language does this NFA accept?"