EECS 2001N : Introduction to the Theory of Computation

Suprakash Datta
Office: LAS 3043

Course page: `http://www.eecs.yorku.ca/course/2001N`
Also on Moodle

Regular Languages: Definition

- The language recognized by a finite automaton (DFA or NFA) M is denoted by $L(M)$
- A regular language is a language for which there exists a recognizing finite automaton
- We study properties of regular languages to understand finite automata

Important Questions

- Given the description of a finite automaton $M = (Q, \Sigma, \delta, q, F)$, what is the language $L(M)$ that it recognizes?
- In general, what kind of languages can be recognized by finite automata? (What are the regular languages?)
- It is easiest to define regular languages RECURSIVELY

Towards a Recursive Definition of Regular Languages

Recall: a language is a set of words over some alphabet
Base case:

- The empty language is regular (WHY?)
- Every set $\{a\}$, $a \in \Sigma$ is a regular language
- Later: We will show that every finite language is regular

Towards a Recursive Definition of Regular Languages - 2

Need rules to build up bigger languages

- The union of two regular languages is regular
- This needs proof
- If we can run two automata "in parallel", we can decide if a word belongs to the union
- We will present a somewhat complicated proof (Theorem 2.3.1 in the text) that simulates the idea above

The Union of Two Regular Languages is Regular

Suppose $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ and $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ accept languages L_1, L_2

Proof idea: track the state of both automata

- We will construct a DFA M that accepts $L_1 \cup L_2$. So,

$$\forall w \in \Sigma^*, M \text{ accepts } w \Leftrightarrow M_1 \text{ accepts } w \text{ or } M_2 \text{ accepts } w$$

- Define $M = (Q_3, \Sigma, \delta_3, q_3, F_3)$ by
  - $Q_3 = Q_1 \times Q_2 = \{(r_1, r_2) | r_1 \in Q_1, r_2 \in Q_2\}$
  - $\delta_3((r_1, r_2), a) = (\delta_1(r_1, a), \delta_2(r_2, a))$
  - $q_3 = (q_1, q_2)$
  - $F_3 = \{(r_1, r_2) | r_1 \in F_1 \text{ or } r_2 \in F_2\}$
- We can complete the proof by induction on the length of w

Towards a Recursive Definition of Regular Languages - 3

Need rules to build up bigger languages

- The complement of a regular language is regular

- The proof idea is straightforward: Take the DFA that recognizes the language and make all non-accepting states accepting and vice versa

The Complement of a Regular Language is Regular: Proof

Take the DFA M that recognizes the language and construct M' that is identical to M except that all non-accepting states in M are accepting in M' and vice versa.

- We show that $w \in \overline{L}$ if and only if M' accepts w
- Since the set of states, the initial state and the transition function of M and M' are identical, the sequence of states $(r_0, r_1, \ldots, r_n)$ that M goes through on input w is identical to the sequence of states M' goes through on input w
- Now consider 2 cases:
  - $w \in L$
  - $w \notin L$ (i.e., $w \in \overline{L}$)

Towards a Recursive Definition of Regular Languages - 4

Define concatenation of languages: $L_1 \cdot L_2 = \{xy | x \in L_1, y \in L_2\}$
Example: $\{a, b\} \cdot \{0, 11\} = \{a0, a11, b0, b11\}$
Caveat: If any of the 4 elements are missing, the set is not $L_1 \cdot L_2$!

- Another rule to build up bigger languages: The concatenation of two regular languages is regular
- Terminology: regular languages are **closed** under concatenation (and also closed under union from the prior result)
- This also needs proof

Proving the Concatenation Theorem

- Given the two languages, we "know" DFA M_1, M_2 that recognize the two languages
- If a word $w \in L_1 \cdot L_2$ then $w = w_1 w_2$ such that w_1 is accepted by M_1 and w_2 is accepted by M_2
- Problem: given a string w, how does the automaton know where the part accepted by M_1 stops and the part accepted by M_2 substring starts?

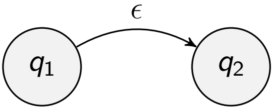
$\boxed{\text{We need a new idea!}}$

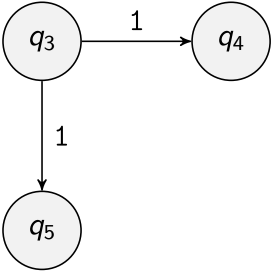
Nondeterminism

- Nondeterministic machines are capable of being lucky, no matter how small the probability
- Alternatively, it can "magically" make the right choices
- As mentioned before, nondeterminism cannot be implemented
- For any (sub)string w, the nondeterministic machine can be in a set of possible states
- If any if the final states is an accepting state, then the machine accepts the string
- "The automaton processes the input in a parallel fashion. Its computational path is no longer a line, but a tree." (Sipser)

Nondeterministic Finite Automata (NFA)

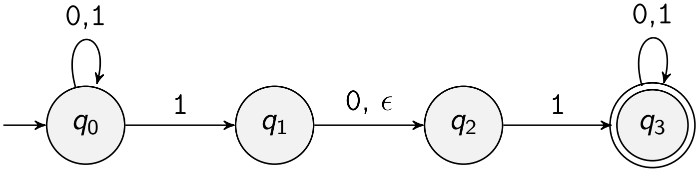
A NFA **may** have transition rules/possibilities like

- 

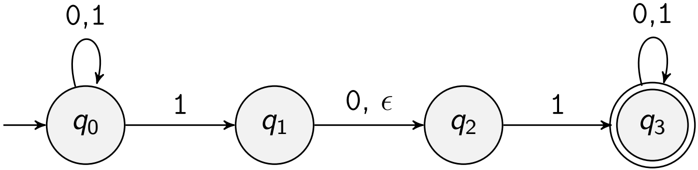
- 

Nondeterministic Finite Automata (NFA) - 2

What does this NFA do?



NFA: Tracing Examples



- 1: May be in states q_0, q_1, q_2! None of those are accepting states, so reject
- 01: May be in states q_0, q_1, q_2! None of those are accepting states, so reject
- 0110: It can reach state q_3, hence accept!
  $(q_0 \to q_0 \to q_1 \to q_2 \to q_3 \to q_3)$
- the fact that there are non-accepting paths is of no consequence

NFA Drawing Conventions

- All transitions need **not** be present
- All but one state **must** be drawn
- Unlabeled transitions are assumed to go to a reject state (not drawn) from which the automaton **cannot** escape

NFA: Formal Definition

A NFA M is defined by a 5-tuple $M = (Q, \Sigma, \delta, q_0, F)$, with

- Q: finite set of states
- Σ: finite alphabet
- δ: transition function $\delta : Q \times \Sigma_\epsilon \to \mathcal{P}(Q)$
- $q_0 \in Q$: start state
- $F \subseteq Q$: set of accepting states

NFA: More on the Transition function

- The function $\delta : Q \times \Sigma_\epsilon \rightarrow \mathcal{P}(Q)$ is the crucial difference between DFA, NFA
- It means: "When reading symbol 'a' while in state q, the machine can go to one of the states in $\delta(q, a) \subseteq Q$"
- The ϵ in $\Sigma_\epsilon = \Sigma \cup \{\epsilon\}$ takes care of the empty string transitions

NFA: recognizing languages

- Informal idea: Given a language, the NFA recognizes it, i.e., it accepts every string in the language, and rejects every string not in the language
- Formally: A NFA $M = (Q, \Sigma, \delta, q, F)$ accepts a string/word $w = w_1 \ldots w_n$ if and only if we can rewrite w as $y_1 \ldots y_m$ with $y_i \in \Sigma_\epsilon$ and there is a sequence $r_0, \ldots, r_m$ of states in Q such that:
  - $r_0 = q_0$
  - $r_{i+1} \in \delta(r_i, y_{i+1})$ for all $i = 0, 1, \ldots, m$–1
  - $r_m \in F$

NFA: Exercises - 1

Give NFAs with the specified number of states that recognize the following languages over the alphabet $\Sigma = \{0, 1\}$:

- $\{w | w \text{ ends with } 00\}$, three states
- $\{0\}$; two states
- $\{w | w \text{ contains even number of zeroes, or exactly two ones}\}$, six states
- $\{0^n | n \in \{0, 1, 2, \ldots\}\}$, one state