EECS 2001A : Introduction to the Theory of Computation

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Course page: `http://www.eecs.yorku.ca/course/2001`
Also on Moodle

Reasoning about Undecidable Problems

Questions:

- Q: How do we know there are undecidable problems?
  A: Through a counting argument: there are more languages than Turing machines and so there are languages than Turing machines. Thus some languages cannot be decidable

- Q: How do we show that a specific problem is undecidable?
  A: Through a very novel argument!

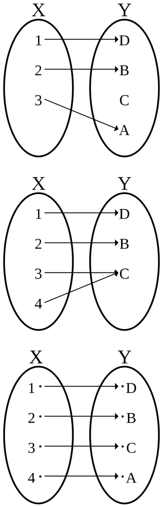
What is Counting

- Elementary view: Labeling with natural numbers
- This is the same as "listing" the numbers – as $a_1, a_2, \ldots$
- More advanced view: Correspondence with a set (often $\{1, 2, \ldots, k\}, k \in \mathbb{N}$

Relationship with Functions

Types of functions $f : X \to Y$:

- f is one-to-one (injective) if every $x \in X$ has a unique image $f(x)$, i.e., if $f(x) = f(y)$ then $x = y$
- f is onto (surjective) if every $z \in Y$ is 'hit' by $f()$, i.e., if $z \in Y$ then there is an $x \in X$ such that $f(x) = z$
- f is a 1:1 correspondence (bijection) between X and Y if it is both one-to-one and onto

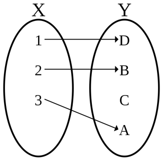


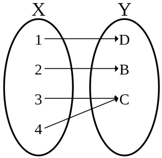
Relationship with Functions - 2

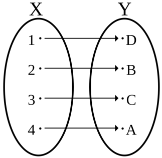
If X, Y are **finite** sets, and $f : X \to Y$ is:

- f is one-to-one (injective): X has no more elements than Y, i.e., $|X| \leq |Y|$
- f is onto (surjective): X has at least as many elements as Y, i.e., $|X| \geq |Y|$
- f is a 1:1 correspondence (bijection): X has exactly as many elements as Y, i.e., $|X| = |Y|$

Q: Do these hold for infinite sets as well?







Infinite Sets

- Our intuition breaks down for infinite sets!
- Example: Consider $A = \mathbb{N}$, $B = \{2, 4, 6, 8, \ldots\}$ (the set of positive even numbers), and $f : A \to B, f(n) = 2n$
  - Note that f is a bijection, so intuitively, $|A| = |B|$
  - Now note that $B \subset A$ (B is a proper subset of A)
- What went wrong?

Cardinality of Sets

- Intuitively, "number of elements"
- Intuition not useful for infinite sets
- New definition is needed
- A set S has k elements if and only if there exists a bijection between S and $\{1, 2, \ldots, k\}$
  S and $\{1, 2, \ldots, k\}$ have the *same* cardinality.
- If there is a surjection possible from $\{1, 2, \ldots, n\}$ to S, then $n \geq |S|$
- We can generalize this way of comparing the sizes of sets to infinite ones

Refining the Notion of Infinite Sets

- A set S is infinite if there exists a surjective function $f : S \to \mathbb{N}$: "The set S has at least as many elements as $\mathbb{N}$"

- A set S is countable if there exists a surjective function $f : \mathbb{N} \to S$: "The set S has at most as many elements as $\mathbb{N}$"

- A set S is countably infinite if there exists a bijective function $f : S \to \mathbb{N}$: "The sets $\mathbb{N}$ and S are of the same cardinality"

Counterintuitive facts

- Previously given example: Consider $A = \mathbb{N}$, $B = \{2, 4, 6, 8, \ldots\}$ (the set of positive even numbers), and $f : A \to B, f(n) = 2n$, f is a bijection, so A, B have the same cardinality
- A proper subset of $\mathbb{N}$ has the same cardinality as $\mathbb{N}$!
- Same holds for odd natural numbers
- What about the integers?

Counting the Number of Languages over $\{0, 1\}$

- Suppose we consider only words of size $k \in \mathbb{N}$
- There are 2^k such words
- The number of possible languages are 2^{2^k} - because each word can be part of the language or not, so 2 choices for each of 2^k words
- If k is allowed to be unbounded, then the number of languages is infinite
- The number of possible Java programs is also infinite
- How can we show that there are more problems than Java programs?

Cardinality of Integers

- Clearly $\mathbb{N} \subset \mathbb{Z}$; in fact $\mathbb{N}$ is "about half of" $\mathbb{Z}$
- Can we get a bijection from $\mathbb{N}$ to $\mathbb{Z}$? How?
- So we have to handle zero and the negative integers. Suppose we label 0 with 1. How to we handle the negative numbers?
- Idea: use the fact that the set of odd and even natural numbers are each in bijection with $\mathbb{N}$
-

...	7	5	3	1	2	4	6	...
...	-3	-2	-1	0	1	2	3	...

Q: What about the non-integer numbers?

Cardinality of Rational Numbers

- There are many more positive rational numbers than natural numbers
- Between any two successive integers $n, n+1$, there are an infinite number of rationals (e.g., consider the set of numbers of the form $n + \frac{1}{k}$, where $k = 2, 3, 4, \ldots$)
- We have to be very creative in labeling the rationals

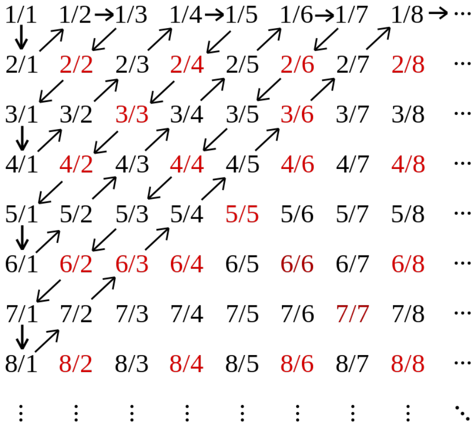
The Rational Numbers are Countably Infinite

- Let us first deal with the positive rationals $\mathbb{Q}^+$
- Claim: There is an surjection f from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{Q}^+$
- Proof: Let f map $(m, n) \in \mathbb{N} \times \mathbb{N}$ to $\frac{m}{n} \in \mathbb{Q}^+$
- Every element of $\mathbb{Q}^+$ can be put in the form $\frac{m}{n}$ by definition of $\mathbb{Q}$
- $\frac{m}{n} = \frac{2m}{2n} = \frac{3m}{3n} = \ldots$, so f is a many-one mapping
- So it is enough to prove that $\mathbb{N} \times \mathbb{N}$ is countably infinite (Why?)

The Rational Numbers are Countably Infinite - 2

Claim: $\mathbb{N} \times \mathbb{N}$ is countably infinite

Proof: Use Cantor numbering



The Rational Numbers are Countably Infinite - 3

- So we showed that $\mathbb{Q}^+$ is countable. Next we argue that the positive integers have a bijection with the positive rationals, the negative integers to the negative rationals and zero maps to zero. So there is a bijection between $\mathbb{Q}$ and $\mathbb{Z}$, and thus with $\mathbb{N}$
- Note that the ordering of $\mathbb{Q}$ is not in increasing order or decreasing order of value
- In proofs, you CANNOT assume that an ordering has to be in increasing or decreasing order
- So cannot use ideas like "between any two rational numbers x, y, there exists a rational number $0.5(x + y)$" to prove uncountability of $\mathbb{Q}$