EECS 2001A : Introduction to the Theory of Computation

Suprakash Datta

Course page: `http://www.eecs.yorku.ca/course/2001`
Also on Moodle

Review of Some Mathematics

- Sets
- Functions
- Graphs
- Recursive definitions
- Logic

Next

Review of sets

Sets

- Representations
- Notation: $\in, \subseteq, \subset$
- Common sets: natural numbers ($\mathbb{N}$), integers ($\mathbb{Z}$), real numbers ($\mathbb{R}$), positive real numbers $\mathbb{R}^+ = \{x|x \in \mathbb{R}, x > 0\}$
- Universal set
- Custom defined: $L_{prime} = \{x|x \in \mathbb{N}, x \text{ is prime}\}$
- Operations: Union ($A \cup B$), Intersection ($A \cap B$), Complement (A^c)
- Cardinality: $|A|$

Sets - Examples and Properties

- Cartesian Product: $A \times B = \{(a,b) | a \in A, b \in B\}$
- If $\Sigma = \{0,1\}$, then $\Sigma \times \Sigma = \{(0,0),(0,1),(1,0),(1,1)\}$
- Power set: set of all subsets of a set A. $\mathcal{P}(A) = \{S | S \subseteq A\}$
  - If $A = \{x,y\}$, then $\mathcal{P}(A) = \{\{\},\{x\},\{y\},\{x,y\}\}$
  - For any set A, $A \in \mathcal{P}(A)$ and $\emptyset \in \mathcal{P}(A)$
  - For finite sets, $|\mathcal{P}(A)| = 2^{|A|}$, $|A \times B| = |A| * |B|$

Next

Review of functions

Functions

- $f : A \rightarrow C$, for all $a \in A, f(a) \in C$
- $f : A \times B \rightarrow C$, for all $a \in A, b \in B, f(a, b) \in C$
- Examples:
  $f : \mathbb{N} \rightarrow \mathbb{N},\ f(a) = 3a + 1$
  $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N},\ f(a, b) = 3a + 2b + ab + 1$
- Types: injective (one-one), surjective (onto), bijective (1:1 correspondence)

Functions: Representation

- Formula: $f : \mathbb{N} \rightarrow \mathbb{N}$, $f(a) = 3a + 1$
- Table: $g : \{a, b\} \times \{0, 1\} \rightarrow \{a, b\}$

	0	1
a	a	b
b	b	a

- List (f): $\{(1, 4), (2, 7), (3, 10), (4, 13), \ldots\}$
  Each pair: first element is the input, second element is the output
- List (g): $\{(a, 0, a), (a, 1, b), (b, 0, b), (b, 1, a)\}$
  Each triple: first 2 elements are inputs, last element is the output

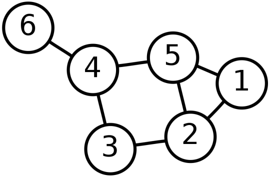
Next

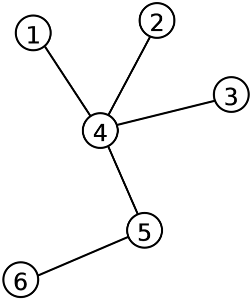
Review of graphs

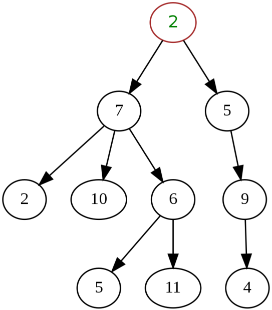
Graphs

- Nodes and Edges, weights
- Undirected, directed
- Cycles, trees
- Connected
- New: Self-loops

Examples







Next

Introduction to recursive definitions

Recursive definitions - 1

1. Sequences, E.g., Fibonacci sequence: $f_i, i \in \mathbb{N}$
   - $f_1 = f_2 = 1$
   - for $n > 2$, $f_n = f_{n-1} + f_{n-2}$
2. Structures, E.g., Binary trees
   - an empty tree is a binary tree
   - a node pointing to two binary trees, one its left child and the other one its right child, is a binary tree
3. Sets, Example: Even natural numbers N_e.
   - $2 \in N_e$
   - $\forall n \in N_e, n + 2 \in N_e$
   - No other numbers are in N_e

Recursive definitions - 2

Recursively defined sets of binary strings:

- Example 1: The set of palindromic strings P
  - $\epsilon \in P$
  - $0 \in P, 1 \in P$
  - $\forall x \in P, 0x0 \in P, 1x1 \in P$
  - No other strings are in P
- Example 2: The set E of all binary strings with an equal number of zeroes and ones.
  - $\epsilon \in E$
  - for every x, y in E, $0x1y$ and $1x0y$ are both in E
  - nothing else is in E.

Recursive definitions - Exercises

- Recursively define the following:
  - The set of odd natural numbers
  - The sequence of powers of 3 $(1, 3, 9, 27, 81, \ldots)$
  - The set of all strings over $\{0, 1\}$ that have exactly one zero
- What set L does the following definition produce?
  $a \in L$; for any $x \in L$, ax, bx, xb are in L. Nothing is in L unless it can be obtained by the previous statements
- Prove that Example 2 on the previous slide is correct

Next

Review of Logic

Logic Review - 1

- Boolean Logic: The only 'truth values' are True, False
- Operations: $\vee, \wedge, \neg$
- Quantifiers: $\forall, \exists$
- statement: Suppose $x \in \mathbb{Z}, y \in \mathbb{Z}$, then $\forall x \exists y (y > x)$
  "for any integer, there exists a larger integer"
- Logical equivalence: $a \rightarrow b$ "is the same as" (is logically equivalent to) $\neg a \vee b$
- Bidirectional Implication: $a \leftrightarrow b$ is logically equivalent to $(a \rightarrow b) \wedge (b \rightarrow a)$

Logic Review - 2

Contrapositive and converse:

- the contrapositive of $a \rightarrow b$ is $\neg b \rightarrow \neg a$

- the converse of $a \rightarrow b$ is $b \rightarrow a$

- Any statement is logically equivalent to its contrapositive, but not to its converse.

Logic Review - 3

- Predicate Logic: each variable must be quantified:
  $\forall x(x < y)$ is meaningless
  (Abuse of notation: $\forall x \in \mathbb{R}\ldots$)
- Implication or AND/OR?
  - "All parrots like chilies"
    $\forall x(Parrot(x) \wedge LikeChilies(x))$
    vs
    $\forall x(Parrot(x) \rightarrow LikeChilies(x))$
  - "Some parrots like chilies"
    $\exists x(Parrot(x) \wedge LikeChilies(x))$
    vs
    $\exists x(Parrot(x) \rightarrow LikeChilies(x))$

Logic Review - 4

Subtleties of quantifiers

- Negation of statements: $\neg(a \rightarrow b) = ?$
  $\neg(\forall x \exists y (y > x)) \equiv \exists x \forall y (y \leq x)$
  LHS: negation of "for every integer, there exists a larger integer",
  RHS: "there exists an integer that is larger than every integer"

- $\forall x \exists y P(x, y)$ is not the same as $\exists y \forall x P(y, x)$
  Consider $P(y, x) : x \leq y$.
  $\forall x \exists y (x \leq y)$ is TRUE over $\mathbb{Z}$ (set $y = x + 1$)
  $\exists y \forall x (x \leq y)$ is FALSE over $\mathbb{Z}$ (there is no largest number in $\mathbb{Z}$)

Understanding which Quantifier to Use

$\exists$ or $\forall$?

- Some people like basketball
- People are careless when it comes to social distancing
- Humans are mortal
- Discrete Math problems are hard