EECS 3101 A: Design and Analysis of Algorithms

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Course page: `http://www.eecs.yorku.ca/course/3101A`
Also on Moodle

Recurrences from Divide and Conquer algorithms

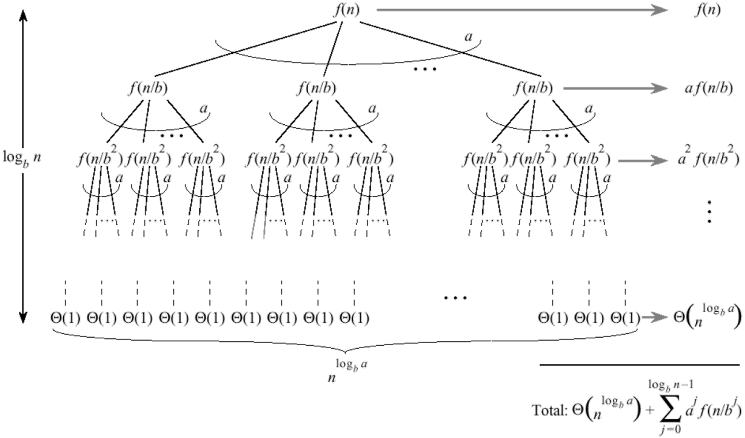
$$T(n) = \begin{cases} \text{time to solve trivial problem} & \text{if } n = 1 \\ aT(n/b) + \text{time to divide} + \text{combine} & \text{if } n > 1 \end{cases}$$

E.g. Merge sort: $T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ 2T(n/2) + \Theta(n) & \text{if } n > 1 \end{cases}$

The Master Theorem

- The idea is to solve a class of recurrences that have the form
$$T(n) = aT(n/b) + f(n)$$
- $a \geq 1$ and $b > 1$, and f is asymptotically positive!
- Abstractly speaking, T(n) is the runtime for an algorithm and we know that
  - a subproblems of size n/b are solved recursively, each in time $T(n/b)$
  - $f(n)$ is the cost of dividing the problem and combining the results. In merge-sort $a = b = 2$, $f(n) = \Theta(n)$

The Master Theorem – 2



Split problem into a parts at $\log_b n$ levels. There are $a^{\log_b n} = n^{\log_b a}$ leaves

The Master Theorem – 3

$$\begin{aligned}
T(n) &= f(n) + aT\left(\frac{n}{b}\right) \\
&= f(n) + af\left(\frac{n}{b}\right) + a^2 T\left(\frac{n}{b^2}\right) \\
&= f(n) + af\left(\frac{n}{b}\right) + a^2 f\left(\frac{n}{b^2}\right) + a^3 T\left(\frac{n}{b^3}\right) \\
&= \ldots \\
&= f(n) + af\left(\frac{n}{b}\right) + \ldots + a^{\log_b n - 1} f\left(\frac{n}{b^{\log_b n - 1}}\right) \\
&\quad + a^{\log_b n} T(1)
\end{aligned}$$

Thus

$$T(n) = \sum_{j=0}^{\log_b n - 1} a^j f\left(\frac{n}{b^j}\right) + \Theta(n^{\log_b a})$$

The Master Theorem: Intuition

$$T(n) = \sum_{j=0}^{\log_b n - 1} a^j f\left(\frac{n}{b^j}\right) + \Theta(n^{\log_b a})$$

- The first term is a division/recombination cost (totaled across all levels of the tree)
- The second term is the cost of doing all $n^{\log_b a}$ subproblems of size 1 (total of all work pushed to leaves)
- Three common cases:
  1. Running time dominated by cost at leaves
  2. Running time evenly distributed throughout the tree
  3. Running time dominated by cost at root

The Master Theorem: Intuition - 2

$$T(n) = \sum_{j=0}^{\log_b n - 1} a^j f\left(\frac{n}{b^j}\right) + \Theta(n^{\log_b a})$$

- Consequently, to solve the recurrence, we need only to characterize the dominant term
- In each case compare $f(n)$ with $O(n^{\log_b a})$

The Master Theorem: Case 1

$$T(n) = \sum_{j=0}^{\log_b n-1} a^j f\left(\frac{n}{b^j}\right) + \Theta(n^{\log_b a})$$

- $f(n) = O(n^{\log_b a-\epsilon})$ for some constant $\epsilon > 0$: $f(n)$ grows polynomially (by factor n^ϵ) slower than $n^{\log_b a}$
- The work at the leaf level dominates
  - Summation of recursion-tree levels: $O(n^{\log_b a})$
  - Cost of all the leaves $\Theta(n^{\log_b a})$
  - Thus, the overall cost is $\boxed{T(n) = \Theta(n^{\log_b a})}$

The Master Theorem: Case 2

$$T(n) = \sum_{j=0}^{\log_b n - 1} a^j f\left(\frac{n}{b^j}\right) + \Theta(n^{\log_b a})$$

- $f(n) = \Theta(n^{\log_b a})$: $f(n)$ and $n^{\log_b a}$ are asymptotically “the same”
- The work is distributed equally throughout the tree
- Total cost: level cost $\times$ number of levels
- Thus, the overall cost is $\boxed{T(n) = \Theta(n^{\log_b a} \lg n)}$

The Master Theorem: Case 3

$$T(n) = \sum_{j=0}^{\log_b n - 1} a^j f\left(\frac{n}{b^j}\right) + \Theta(n^{\log_b a})$$

- $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$: $f(n)$ grows polynomially (by factor n^ϵ) faster than $n^{\log_b a}$
- The work the root dominates
- Inverse of Case 1
- Also need a regularity condition: $\exists c \in (0, 1), \exists n_0 > 0, \forall n > n_0, af(n/b) \leq cf(n)$
- Thus, the overall cost is $\boxed{T(n) = \Theta(f(n))}$

The Master Theorem: Summary

$$T(n) = aT(n/b) + f(n)$$

- $f(n) = O(n^{\log_b a - \epsilon})$, $\epsilon > 0$: $T(n) = \Theta(n^{\log_b a})$
- $f(n) = \Theta(n^{\log_b a})$: $T(n) = \Theta(n^{\log_b a} \lg n)$
- $f(n) = \Omega(n^{\log_b a + \epsilon})$, $\epsilon > 0$: $T(n) = \Theta(f(n))$

Caveat: The master method cannot solve every recurrence of this form; there is a gap between cases 1 and 2, as well as cases 2 and 3

The Master Theorem: Examples

$$T(n) = aT(n/b) + f(n)$$

- Mergesort ($a = 2, b = 2, f(n) = \Theta(n)$). Case 2: $T(n) = \Theta(n \lg n)$
- Binary Search (recursive): $T(n) = T(n/2) + 1$. $a = 1, b = 2, n^{\log_2 1} = 1,\ f(n) = 1 \in \Theta(1)$. Case 2: $T(n) = \Theta(lgn)$
- Artificial example 1: $T(n) = 9T(n/3) + n$. $a = 9, b = 3, f(n) = n \in \Theta(n), n^{\log_3 9} = n^2$, $f(n) = O(n^{2-\epsilon})$ with $\epsilon = 1$ Case 1: $T(n) = \Theta(n^2)$

The Master Theorem: More Examples

$$T(n) = aT(n/b) + f(n)$$

- Artificial example 2: $T(n) = 4T(n/2) + n^3$.
  $a = 4$, $b = 2$, $f(n) \in \Theta(n^3)$, $n^{\log_2 4} = n^2$,
  $f(n) = \Omega(n^{2+\epsilon})$ with $\epsilon = 1$
  Case 3: $T(n) = \Theta(n^3)$ provided the regularity condition holds
  Check: $4f(n/2) \leq cf(n)$ for some $c < 1$

$$\begin{aligned} 4f(n/2) &= 4(n/2)^3 \\ &= n^3/2 \\ &\leq cn^3 \text{ for } c \leq 1/2 \end{aligned}$$

The Master Theorem: Last Example

$$T(n) = aT(n/b) + f(n)$$

- Artificial example 3: $T(n) = 2T(n/2) + n \lg n$.
  $n^{\log_b a} = n^{\log_2 2} = n$, $f(n) = n \lg n$

Neither Case 2 nor Case 3.

Master Theorem – Points to Remember

- We ignore floors and ceilings, because the final answer does not change

- We ignore constants in $T(1)$, $f(n)$

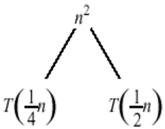
If the Master Theorem fails...

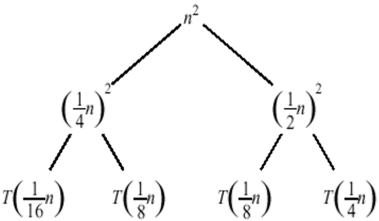
- Recursion tree approach
- Induction

Recursion Tree Method

Example: $T(n) = T(n/4) + T(n/2) + n^2$

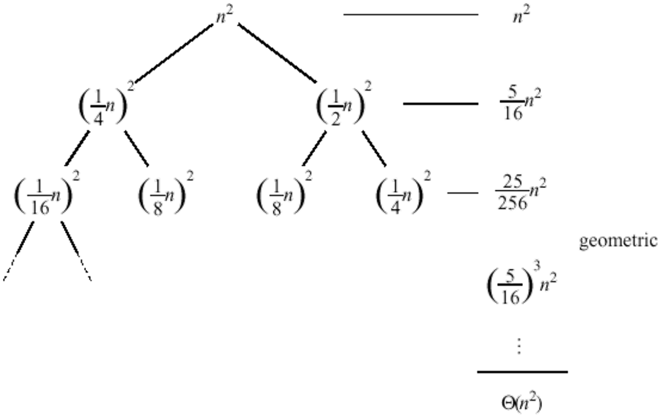
Rule: recursive term creates children, other term attached to node





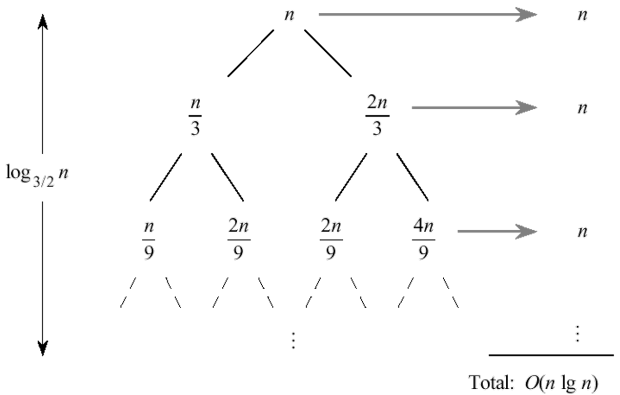
Recursion Tree Method

Example: $T(n) = T(n/4) + T(n/2) + n^2$



Recursion Tree Method – Another Example

Example: $T(n) = T(n/3) + T(2n/3) + n$
Rule: recursive term creates children, other term attached to node



Induction

Example: $T(n) = 4T(n/2) + n$

Attempt 1: $T(n) = O(n^3)$. Assume $T(k) \leq ck^3$ for $k \leq n/2$

$$\begin{aligned} T(n) &= 4T(n/2) + n \\ &\leq 4c(n/2)^3 + n \\ &= cn^3/2 + n \\ &\leq cn^3 \text{ if } n \leq cn^3/2, n \geq n_0 \end{aligned}$$

True if $c = 2, n_0 = 1$

Induction – Tighter Bound

Example: $T(n) = 4T(n/2) + n$

Try to show $T(n) = O(n^2)$
Attempt 1: Assume $T(k) \leq ck^2$ for $k \leq n/2$

$$\begin{aligned} T(n) &= 4T(n/2) + n \\ &\leq 4c(n/2)^2 + n \\ &= cn^2 + n \\ &\not\leq cn^2 \text{ for any } c > 0 \end{aligned}$$

try to strengthen the hypothesis:
$T(n) \leq$ (answer you want) - (something positive)

Induction – Tighter Bound

Example: $T(n) = 4T(n/2) + n$

Try to show $T(n) = O(n^2)$
Attempt 2: Assume $T(k) \leq c_1 k^2 - c_2 k$ for $k < n$

$$
\begin{aligned}
T(n) &= 4T(n/2) + n \\
&\leq 4(c_1(n/2)^2 - c_2(n/2)) + n \\
&= c_1 n^2 - 2c_2 n + n \\
&\leq c_1 n^2 - c_2 n - c_2 n + n \\
&= c_1 n^2 - c_2 n - (c_2 - 1)n \\
&\leq c_1 n^2 - c_2 n \text{ for } c_2 \geq 1
\end{aligned}
$$

Note: c_1 must be chosen to be large enough so that $T(1) \leq c_1 - c_2$.