EECS 1028 M: Discrete Mathematics for Engineers

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Course page: `http://www.eecs.yorku.ca/course/1028`
Also on Moodle

Sequences and Summation

Ch 2.4 in the text

- Finite or infinite
  Notation: $a_n, n \in \mathbb{N}$
- Calculus – limits of infinite sequences (proving existence, evaluation ...)
- The sum of a sequence is called a series
- Examples:
  - Arithmetic progression (sequence): 1, 4, 7, 10, ...
  - Arithmetic series: $1 + 4 + 7 + 10 + ... + (1 + 3(n - 1))$
  - Geometric progression (sequence) 3, 6, 12, 24, 48, ...
  - Geometric series: $3 + 6 + 12 + 24 + 48 + ... + 3 * 2^{n-1}$

Specifying Sequences

- Enumeration:
  - $1, 4, 7, 10, 13, \ldots$ Arithmetic sequence
  - $1, 3, 9, 27, 81, \ldots$ Geometric sequence
- Explicit formulas for each term:
  - $a_n = a + (n-1)b$ Arithmetic sequence
  - $a_n = ar^{n-1}$ Geometric sequence
  - $a_n = n^2$
- Recursively:
  - $a_1 = 1$ and for $n > 1$, $a_n = a_{n-1} + 3$ Arithmetic sequence
  - $a_1 = 1$ and for $n > 1$, $a_n = 3a_{n-1}$ Geometric sequence
  - $a_0 = a_1 = 1$ and for $n > 1$, $a_n = a_{n-1} + a_{n-2}$ Fibonacci sequence

Summation

- $S_n = a_1 + a_2 + a_3 + a_4 + \ldots + a_n,\ n \in \mathbb{N}$

- Consider the sequence $S_1, S_2, S_3, \ldots, S_n$, where
  $S_i = a_1 + a_2 + \ldots + a_i$

- In general we would like to evaluate sums of series – useful in algorithm analysis.
  e.g. what is the total time spent in a nested loop?

Sigma Notation

- Notation: $S_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + a_4 + \ldots + a_n$
- Simple rules for manipulation
  - $S_n = \sum_{i=1}^{n} ca_i = c \sum_{i=1}^{n} a_i$
  - $S_n = \sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$
- $S_n = \sum_{i=1}^{n} 1 = n$
- Related: $\prod_{i=1}^{n} a_i = a_1 \cdot a_2 \cdot \ldots \cdot a_n$

Sum of Series

- Arithmetic series
  e.g. $S_n = 1 + 2 + \ldots + n$ (occurs frequently in the analysis of running time of simple for loops)
  general form $S_n = \sum_{i=0}^{n} t_i$, where $t_i = a + bi$.

- Geometric series
  e.g. $S_n = 1 + 2 + 2^2 + 2^3 + \ldots + 2^n$
  general form $S_n = \sum_{i=0}^{n} t_i$, where $t_i = ar^i$.

- More general series (not either of the above)
  $S_n = 1^2 + 2^2 + 3^2 + 4^2 + \ldots + n^2$

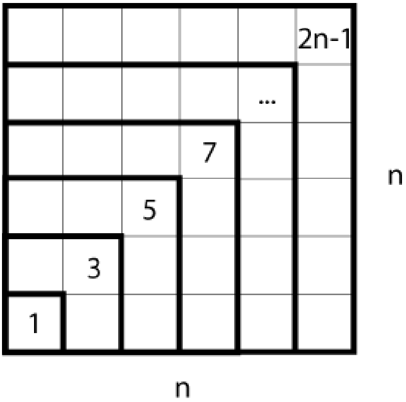
Sum of Arithmetic Series - A Special Case

$S_n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

$$\begin{aligned} S_n &= \sum_{i=1}^{n} i = \sum_{i=1}^{n} (n - i + 1) \text{ adding} \\ 2S_n &= \sum_{i=1}^{n} i + \sum_{i=1}^{n} (n - i + 1) \\ &= \sum_{i=1}^{n} (i + n - i + 1) \\ &= \sum_{i=1}^{n} (n + 1) \\ &= n(n + 1) \end{aligned}$$

Exercise: $S_n = \sum_{i=1}^{n}(2i - 1) = n^2$

Prove this using the sum on the previous slide



from: http://www.9math.com/book/sum-first-n-odd-natural-numbers

Sum of Arithmetic Series - The general case

$S_n = \sum_{i=1}^{n}(a+bi) = \frac{1}{2}\left(2a + b(n-1)\right)n$

$$
\begin{aligned}
S_n = \sum_{i=0}^{n-1}(a+bi) &= a\sum_{i=0}^{n-1} 1 + b\sum_{i=0}^{n-1} i \\
&= an + bn(n-1)/2 \\
&= \frac{1}{2}\left(2an + bn(n-1)\right) \\
&= n * \frac{1}{2}\left(2a + b(n-1)\right) \\
&= n * \left(\frac{a + a + b(n-1)}{2}\right)
\end{aligned}
$$

number of terms * average of first and last terms

Sum of Geometric Series - A special case

Consider the sequence $1, r, r^2, \ldots, r^{n-1},\ r \neq 1$

$$
\begin{aligned}
S_n &= 1 + r + r^2 + \ldots + r^{n-1} \qquad (1) \\
rS_n &= r + r^2 + r^3 + \ldots + r^{n-1} + r^n \qquad (2)
\end{aligned}
$$

Subtracting (2) from (1)

$$
\begin{aligned}
(1-r)S_n &= 1 - r^n, \text{ or} \\
S_n &= \frac{1-r^n}{1-r}
\end{aligned}
$$

If $|r| < 1$, we can find the infinite sum (because it converges)

$$S = \lim_{n \to \infty} S_n = \frac{1}{1-r}$$

Sum of Geometric Series - The general case

Consider the sequence $a, ar, ar^2, \ldots, ar^{n-1},\ r \neq 1$

$$\begin{aligned} S_n &= \sum_{i=0}^{n-1} ar^i \\ &= a \sum_{i=0}^{n-1} r^i \\ &= \frac{a(1-r^n)}{1-r} \end{aligned}$$

If $|r| < 1$, we can again find the infinite sum

$$S = \lim_{n \to \infty} S_n = \frac{a}{1-r}$$

Some Observations

Verify these:

- For any $a \in \mathbb{R}$, the constant progression $a, a, a, \ldots$ is both arithmetic and geometric
- If a, b, c are in arithmetic progression, $b = \frac{a+c}{2}$, i.e., the arithmetic mean of a and c
- If a, b, c are in geometric progression, $b = \sqrt{ac}$, i.e., the geometric mean of a and c

Challenge problem:

Suppose that $a_1, a_2, \ldots, a_{100}$ form an arithmetic progression and $a_2 + a_3 + \ldots + a_{99} = 196$.
Find $\sum_{i=1}^{100} a_i$.

More General Series

More difficult to prove.

- $S_n = \sum_{i=1}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1)$
  Proved by Induction
- $S_n = \sum_{i=1}^{n} i^3 = \frac{1}{4}n^2(n+1)^2$
  Proved by Induction
- $S = \sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}$
  Proved using Calculus

Caveats

- Need to be very careful with infinite series – can get nonsense results with divergent (i.e., non-converging) series
- In general, tools from calculus are needed to know whether an infinite series sum exists.
- There are instances where the infinite series sum is much easier to compute and manipulate, e.g. geometric series with $|r| < 1$.

Problems with Recursively Defined Sequences

- Suppose a sequence is defined recursively as $a_1 = 1; a_{n+1} - a_n = 3^n$ Find the value of a_9.
- Find the sum of the infinite geometric series whose terms are defined as $a_n = \frac{2^n}{3^{n+1}}$.
- Let a_n be a sequence of numbers defined by the recurrence relation $a_1 = 1; a_{n+1}/a_n = 2^n$. Find $\log_2 a_{100}$.

More Problems

- Prove that $0.99999\ldots = 1$
- Solve the equation $1 + 4 + 7 + ... + x = 925$
- Let $a, b, c, d \in \mathbb{R}$ be a sequence such that a, b, c form an arithmetic progression and b, c, d form a geometric progression. If $a + d = 37$ and $b + c = 36$, find d.