EECS 1028 M: Discrete Mathematics for Engineers

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Course page: `http://www.eecs.yorku.ca/course/1028`
Also on Moodle

Why Study Logic?

A formal mathematical "language" for precise reasoning.

- Start with propositions.
- Add other constructs like negation, conjunction, disjunction, implication etc.
- All of these are based on ideas we use daily to reason about things.
- Later: A more expressive language – Predicate logic

Propositions

- Declarative sentence.
- Must be either True or False.
- Examples of propositions:
  - York University is in Toronto
  - York University is in downtown Toronto
  - All students at York are Computer Science majors
- Examples of statements that are not propositions:
  - Do you like this class?
  - There are n students in this class.

Propositions - 2

- Truth value: True or False
- Variables: $p, q, r, s, \ldots$
- Negation: $\neg p$ (In English, "not p")
- Truth tables – enumerative definition of propositions

p	$\neg p$
T	F
F	T

Negating Propositions

$\neg p$: Literally, "it is not the case that p is true"

- p: "it rained more than 20 inches in Toronto last month"
- q: "John has many iPads"
- Page 12, Q10 (a) r: "the election is decided"

Practice: Questions 1-7 page 12.

Combining Propositions

Purpose: express more complex statements

- Conjunction, Disjunction
- Exclusive OR (XOR)
- Conditionals, Biconditionals
- Logical Equivalence

Conjunctions and Disjunctions

Purpose: combine statements using OR and AND

- Conjunction (AND): $p \wedge q$ [“p and q”]
- Disjunction (OR): $p \vee q$ [“p or q”]
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p	q	$p \wedge q$	$p \vee q$
F	F	F	F
F	T	F	T
T	F	F	T
T	T	T	T

Examples

Q11, page 13
p: It is below freezing
q: It is snowing

- It is below freezing and snowing
- It is below freezing but not snowing
- It is either snowing or below freezing (or both)

Exclusive OR

Notation: $p \oplus q$

- TRUE if p and q have different truth values, FALSE otherwise
- Colloquially, we often use OR ambiguously –
  - "an entree comes with soup or salad" implies XOR, but
  - "students can take MATH XXXX if they have taken MATH 2320 or MATH 1019" usually means the normal OR (so a student who has taken both is still eligible for MATH XXXX).

Conditionals

Notation: $p \rightarrow q$ ["if p then q"]
p: hypothesis, q: conclusion
Examples:

- "If you turn in a homework late, (then) it will not be graded"
- If you get 100% in this course, (then) you will get an A+"

$\boxed{\text{A conditional is a proposition}}$

- Tricky question: Is $p \rightarrow q$ TRUE if p is FALSE?
- Think of "If you get 100% in this course, you will get an A+" as a promise – is the promise violated if someone gets 50% and does not receive an A+?

Q: Similarities with `if(...) then ...` statement in programming?

Conditionals - Truth Table

$p \to q$: When is it False?

Q17, pg 14:

- If $1 + 1 = 3$ then $2 + 2 = 4$
- If $1 + 1 = 3$ then $2 + 2 = 5$
- If $1 + 1 = 2$ then $2 + 2 = 4$
- If $1 + 1 = 2$ then $2 + 2 = 5$ FALSE

p	q	$p \to q$	$\neg p \vee q$
F	F	T	T
F	T	T	T
T	F	F	F
T	T	T	T

English Statements to Conditionals (pg 6)

$p \to q$ may be expressed as

- A sufficient condition for q is p
- q whenever p
- q unless $\neg p$
- Difficult: A necessary condition for p is q
  if p happened, q must have happened, i.e., p cannot happen if we do not have q.
- p only if q: not the same as p if q! Same as the previous point, if p happened, q must have happened

Logical Equivalence

$p \to q$ and $\neg p \vee q$ have the truth table:
Does that make them equal? equivalent?

- $p \to q$ and $\neg p \vee q$ are **logically** equivalent
- Truth tables are the simplest way to prove such facts.
- We will learn other ways later.

Biconditionals

Notation: $p \leftrightarrow q$ ["if and only if"]

- True if p, q have same truth values, false otherwise.
- Can also be defined as $(p \to q) \wedge (q \to p)$
- Example: Q16(c) "1+1=3 if and only if monkeys can fly".
- Q: How is this related to XOR?

p	q	$p \leftrightarrow q$	$p \oplus q$
F	F	T	F
F	T	F	T
T	F	F	T
T	T	T	F

Contrapositive

Contrapositive of $p \to q$ is $\neg q \to \neg p$

- E.g. The contrapositive of "If you get 100% in this course, you will get an A+" is "If you do not get an A+ in this course, you did not get 100%".
- Any conditional and its contrapositive are logically equivalent (have the same truth table).

p	q	$p \to q$	$\neg q$	$\neg p$	$\neg q \to \neg p$
F	F	T	T	T	T
F	T	T	F	T	T
T	F	F	T	F	F
T	T	T	F	F	T

Proof using Contrapositive

Prove: If x^2 is even, then x is even

- Proof 1: Using contradiction, seen before.
- Proof 2:
  $x^2 = 2a$ for some integer a. Since 2 is prime, 2 must divide x.
  (Uses knowledge of primes)
- Proof 3:
  if x is not even, then x is odd. Therefore x^2 is odd. This is the contrapositive of the original assertion.
  (Uses only facts about odd and even numbers)

Converse and Inverse

Converse of $p \rightarrow q$ is $q \rightarrow p$

Converse of $p \rightarrow q$ is $\neg p \rightarrow \neg q$

- Converse examples:
  - "If you get 100% in this course, you will get an A+", converse "If you get an A+ in this course, you scored 100%".
  - "If you won the lottery, you are rich", converse "If you are rich, you (must have) won the lottery".
- Neither is logically equivalent to the original conditional

p	q	$p \rightarrow q$	$q \rightarrow p$	$\neg p \rightarrow \neg q$
F	F	T	T	T
F	T	T	F	F
T	F	F	T	T
T	T	T	T	T

Tautology and Logical Equivalence

Tautology: A (compound) proposition that is always TRUE, e.g. $q \vee \neg q$

- Logical equivalence redefined: p, q are logical equivalences (Symbolically $p \equiv q$) if $p \leftrightarrow q$ is a tautology. .
- Intuition: $p \leftrightarrow q$ is true precisely when p, q have the same truth values.

Compound Propositions: Precedence

Example: $p \wedge q \vee r$: Could be interpreted as $(p \wedge q) \vee r$ or $p \wedge (q \vee r)$

- precedence order: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$
  (Overruled by brackets)
- We use this order to compute truth values of compound propositions.

Translating English Sentences to Propositional Logic statements

Pages 14-15:

- I will remember to send you the address only if you send me an email message
- The beach erodes whenever there is a storm
- John will go swimming unless the water is too cold
- Getting elected follows from knowing the right people.

Readings and Notes

- Read pages 1-12.
- Think about the notion of truth tables.
- Master the rationale behind the definition of conditionals.
- Practice translating English sentences to propositional logic statements.

Manipulating Propositions (Sec 1.3)

- Compound propositions can be simplified by using simple rules. Read page 25 - 28.
- Some are obvious, e.g. Identity, Domination, Idempotence, Negation, Double negation, Commutativity, Associativity
- Less obvious: Distributive, De Morgan's laws, Absorption

TABLE 6 Logical Equivalences.

Equivalence	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

Distributive Laws

- $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
  Intuition (not a proof!) - For the LHS to be true: p must be true and q or r must be true. This is the same as saying p and q must be true or p and r must be true.

- $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
  Intuition (less obvious) - For the LHS to be true: p must be true or both q and r must be true. This is the same as saying p or q must be true and p or r must be true.

Proof: use truth tables.

De Morgan's Laws

- $\neg(q \vee r) \equiv \neg q \wedge \neg r$
  Intuition - For the LHS to be true: neither q nor r can be true. This is the same as saying q and r must both be false.

- $\neg(q \wedge r) \equiv \neg q \vee \neg r$
  Intuition - For the LHS to be true: $q \wedge r$ must be false. This is the same as saying that q or r must be false.

Proof: use truth tables.

Negating Conditionals

$\boxed{\text{The negation of } p \rightarrow q \text{ is NOT } \neg p \rightarrow \neg q \text{ or any other conditional}}$

- Easiest to negate the logically equivalent form of $p \rightarrow q$, viz., $\neg p \vee q$.
  So $\neg(p \rightarrow q) \equiv \neg(\neg p \vee q) \equiv p \wedge \neg q$
- Relate to the truth table of $p \rightarrow q$

p	q	$p \rightarrow q$	$\neg(p \rightarrow q)$	$p \wedge \neg q$
F	F	T	F	F
F	T	T	F	F
T	F	F	T	T
T	T	T	F	F

Using the laws

Q: Is $p \rightarrow (p \rightarrow q)$ a tautology?

- Can use truth tables

p	q	$p \rightarrow q$	$p \rightarrow (p \rightarrow q)$
F	F	T	T
F	T	T	T
T	F	F	F
T	T	T	T

- Can write a compound proposition and simplify:

$$\begin{aligned} p \rightarrow (p \rightarrow q) &\equiv \neg p \vee (\neg p \vee q) \\ &\equiv \neg p \vee \neg p \vee q \\ &\equiv \neg p \vee q \end{aligned}$$

This is False when p is True and q is False