EECS 1028 M: Discrete Mathematics for Engineers

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Course page: `http://www.eecs.yorku.ca/course/1028`
Also on Moodle

What is Predicate Logic?

Generalizes Propositional Logic to make statements about sets

What is a Predicate?

A predicate is a proposition that is a function of one or more variables
E.g.:For an integer x, $Even(x)$: x is an even number. So $Even(1)$ is false, $Even(2)$ is true, and so on.

Examples of predicates:

- Domain ASCII characters - IsAlpha(x) : TRUE iff x is an alphabetical character.
- Domain floating point numbers - IsInt(x): TRUE iff x is an integer.
- Domain integers: Prime(x) - TRUE if x is prime, FALSE otherwise.

The domain of the variable(s) must ALWAYS be specified

Predicates

- Ways of defining predicates:
  Domain $\mathbb{R}$
  - Explicit: $Positive(x)$ is True iff x is positive
  - Implicit: $x > 0$
- Recall: $Positive(2), Positive(-3)$ are predicates, but $Positive(2/0), Positive(x)$ is not.
  Q:Is $Positive(x^2)$ a predicate?
- The purpose of Predicate Logic is to make general statements like "all birds can fly". Need some more constructs to make such statements

Quantifiers

describes the values of a variable that make the predicate true.

- Existential quantifier: $\exists x P(x)$ – "$P(x)$ is true for some x in the domain" or "there exists x such that $P(x)$ is TRUE".
- Universal quantifier: $\forall x P(x)$ – "$P(x)$ is true for all x in the domain"
- Either is meaningless if the domain is not known/specified.
- Examples (domain $\mathbb{R}$, uses implicit predicates):
  - $\forall x(x^2 \geq 0)$
  - $\exists x(x > 1)$
  - Quantifiers with restricted domain $(\forall x > 1)(x^2 > x)$

Using Quantifiers

Domain integers:

- The cube of all negative integers is negative.
  $\forall x(x < 0) \rightarrow (x^3 < 0)$

- Expressing sums :

$$\forall n \left(\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \right)$$

Scope of Quantifiers

- $\exists\forall$ have higher precedence than operators from Propositional Logic; so $\forall x P(x) \lor Q(x)$ is not logically equivalent to $\forall x(P(x) \lor Q(x))$

- $\exists x(P(x) \land Q(x)) \lor (\forall x R(x))$
  Say $P(x)$: x is odd, $Q(x)$: x is divisible by 3,
  $R(x) : (x = 0) \lor (2x > x)$

Conditionals

Defined similarly as before. Examples:

- Domain: set of all birds. $Parrots(x)$ is a predicate that is true iff x is a parrot. $Fly(x)$ is a predicate that is true iff x can fly.
  All parrots can fly: $(\forall x)Parrot(x) \to Fly(x)$
- Definition of injective functions: $f : A \to B$,
  $(\forall x_1)(\forall x_2)x_1 \neq x_2 \to f(x_1) \neq f(x_2)$.
- Expressing "There are at exactly 2 real square roots of any natural number n, namely $\sqrt{n}$, $-\sqrt{n}$."
  Define the predicate $Sqrt(a, b)$: "b is a square root of a".
  Then the statement is (Domain of n is $\mathbb{N}$, y_1, y_2, z is $\mathbb{R}$)
  $\forall n \exists y_1 \exists y_2 (Sqrt(n, y_1) \wedge Sqrt(n, y_2) \wedge (y_1 \neq y_2)$
  $\wedge (\forall z) Sqrt(n, z) \to (z = y_1) \vee (z = y_2))$

When to use Conditionals

- Suppose the domain is the set of all birds.
  "All parrots can fly": $(\forall x) Parrot(x) \rightarrow Fly(x)$ is correct;
  $(\forall x) Parrot(x) \wedge Fly(x)$ is incorrect
- More tricky: Domain: all York students. $CySec(x)$ is a predicate that is true iff x is a Cybersecurity major. $Takes1019(x)$ is a predicate that is true iff x takes EECS 1019.
  "Some Cybersecurity major at York takes EECS 1019":
  $\exists x(CySec(x) \rightarrow Takes1019(x))$?
  $\exists x(CySec(x) \wedge Takes1019(x))$?
- Related confusing usage:
  $(\forall x \in S)P(x)$ – really means $(\forall x)(x \in S \rightarrow P(x))$
  $(\exists x \in S)P(x)$ – really means $(\exists x)(x \in S \wedge P(x))$

Mathematical Properties Stated using Predicate Logic

- A function is surjective. Consider $f : A \to B$.
  $(\forall x) x \in B \to (\exists y)(y \in A) \wedge (f(y) = x)$
- A set S has a maximum (largest element). Let the domain be S.
  $(\exists y)(\forall x) x \leq y$
- A set $S \subseteq \mathbb{N}$ is finite. We need to use the following property of the natural numbers: a subset S is finite iff it has a maximum. Let the domain be S.
  $(\exists y)(\forall x) x \leq y$
- A set $S \subseteq \mathbb{N}$ is infinite. Negate the above (the domain is S):
  $(\forall y)(\exists x) x > y$

Logical Equivalence

- Logical Equivalence: $P \equiv Q$ iff they have same truth value no matter which domain is used and no matter which predicates are assigned to predicate variables

Negating Predicate Logic Statements

Examples:

- "There is no student who can ..."
- "Not all professors are bad"
- "There is no Toronto Raptor that can dunk like Vince Carter"

Rules:

- $\neg \forall x P(x) \equiv \exists x \neg P(x)$ Why?
- $\neg \exists x P(x) \equiv \forall x \neg P(x)$ Why?

Careful: The negation of "Every Canadian loves Hockey" is NOT "No Canadian loves Hockey"!

Many, many students make this mistake!

Nested Quantifiers

Allows simultaneous quantification of many variables
E.g. – domain $\mathbb{Z}$,

- $\exists x \exists y \exists z (x^2 + y^2 = z^2)$ (Pythagorean triples)
- $\forall n \exists x \exists y \exists z (x^n + y^n = z^n)$ (Fermat's Last Theorem implies this is false)

Domain $\mathbb{R}$:

- $\forall x \forall y \exists z ((x < z < y) \vee (y < z < x))$ Is this true?
- $\forall x \forall y \exists z ((x = y) \vee (x < z < y) \vee (y < z < x))$
- $\forall x \forall y \exists z ((x \neq y) \rightarrow ((x < z < y) \vee (y < z < x)))$

Nested Quantifiers - Similarities with loops

Analogy: An inner quantified variable (inner loop limit) can depend on the outer quantified variable (outer loop index)

- E.g. in $\forall x \exists y (x + y = 0)$ we chose $y = -x$, so for different x we need different y to satisfy the statement.
- $\forall p \exists j Accept(p, j)$ (p, j have different domains) does NOT say that there is a j that will accept all p.

CAUTION: In general, order MATTERS

- $\forall x \exists y (x < y)$: "there is no maximum integer"
- $\exists y \forall x (x < y)$: "there is a maximum integer"

Negation of Nested Quantifiers

Use the same rule as before carefully.

- $\neg\exists x\forall y(x + y = 0)$:

$$\begin{aligned}\neg\exists x\forall y(x + y = 0) &\equiv \forall x\neg\forall y(x + y = 0)\\ &\equiv \forall x\exists y\neg(x + y = 0)\\ &\equiv \forall x\exists y(x + y \neq 0)\end{aligned}$$

- $\neg\forall x\exists y(x < y)$

$$\begin{aligned}\neg\forall x\exists y(x < y) &\equiv \exists x\neg\exists y(x < y)\\ &\equiv \exists x\forall y\neg(x < y)\\ &\equiv \exists x\forall y(x \geq y)\end{aligned}$$

Exercises

Check that

- $\forall x \exists y (x + y = 0)$ is not true over the positive integers.
- $\exists x \forall y (x + y = 0)$ is not true over the integers.
- $\forall x \neq 0 \exists y (y = 1/x)$ is true over the real numbers.

Read 1.4-1.5.
Practice: Q2,8,16,30 (pg 65-67)

Proofs vs Counterexamples

To prove quantified statements of the form

- $\forall x P(x)$: an example (or 10) x for which $P(x)$ is true is/are NOT enough; a proof is needed
- $\exists x P(x)$: an example x for which $P(x)$ is true is enough.

To DISPROVE quantified statements of the form

- $\forall x P(x)$: a COUNTERexample x for which $P(x)$ is false is enough
- $\exists x P(x)$: an example x for which $P(x)$ is false is NOT enough; a proof is needed

Intuition:

Disproving $(\forall x)P(x)$ means proving $\neg(\forall x)P(x) \equiv (\exists x)\neg P(x)$

Inference in Predicate Logic

Most rules are very intuitive

- Universal instantiation – If $\forall x P(x)$ is true, we infer that $P(a)$ is true for any given a
- Universal Modus Ponens: $\forall x P(x) \rightarrow Q(x)$ and $P(a)$ imply $Q(a)$
  Example: If x is odd then x^2 is odd. a is odd. So a^2 is odd.
- Many other rules, see page 76.
  - Again, understanding the rules is crucial, memorizing is not.
  - You should be able to see that the rules make sense and correspond to our intuition about formal reasoning.

Inference Rules

TABLE 2 Rules of Inference for Quantified Statements.

Rule of Inference	*Name*
$\forall x P(x)$ $\therefore P(c)$	Universal instantiation
$P(c)$ for an arbitrary c $\therefore \forall x P(x)$	Universal generalization
$\exists x P(x)$ $\therefore P(c)$ for some element c	Existential instantiation
$P(c)$ for some element c $\therefore \exists x P(x)$	Existential generalization

Commonly used technique: Universal generalization

Examples

- Prove: If x is even, $x + 2$ is even
- Prove: If x^2 is even, x is even
  [Note that the problem is to prove an implication.]
  Proof: if x is not even, x is odd. Therefore x^2 is odd. This is the contrapositive of the original assertion.

Aside: Inference and Planning

- The steps in an inference are useful for planning an action.
  - Example: your professor has assigned reading from an out-of-print book. How do you do it?
  - Example 2: you are participating in the television show "Amazing race". How do you play?
- The steps in an inference are useful for proving assertions from axioms and facts.
  Q: Why is it important for computers to prove theorems?
  - Proving program-correctness
  - Hardware design
  - Data mining
  - Many more

Aside: Inference and Planning - 2

- Sometimes the steps of an inference (proof) are useful. E.g. on Amazon book recommendations are made.
- You can ask why they recommended a certain book to you (reasoning).