EECS 1028 M: Discrete Mathematics for Engineers

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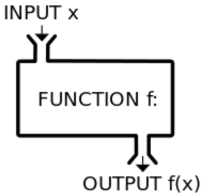
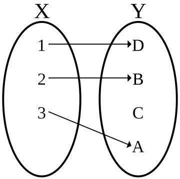
Course page: `http://www.eecs.yorku.ca/course/1028`
Also on Moodle

Functions (Ch 2.3 of the text)

3 ways to think of functions



- Black-box



- Map from one set to another
- Explicit formula $f(n) = n^2$

Functions - More Definitions and Examples

- Describe a family of transformations of inputs
  "A function from A to B is an assignment of exactly one element of B to each element of A."
  Notation: $f : A \to B, f(a) = b$
- A: Domain, B: Co-domain
- $\mathrm{range}(f) = \{y | y = f(x) \text{ for some } x \in A\} \subseteq B$
- Compare to Java: `int floor (float real){ ... }`

Notes:

- f must be **defined** for every $a \in A$
- $f(a)$ must be **unique** for every $a \in A$

Functions - Examples

- Some functions:
  - $A = B = \mathbb{Z}, f(x) = x + 10$
  - $A = B = \mathbb{Z}, f(x) = x^2$
- Examples of transformations that are not functions:
  - $A = B = \mathbb{R}, f(x) = 1/x$
  - $A = B = \mathbb{R}, f(x) = \pm\sqrt{x}$
- Functions may have more than one input, e.g., $f : A \times B \rightarrow C, f(a, b) = c$

Representations of Functions

- A formula or Java code
- A graph
- A lookup table
- An ordered list (for functions on integers)
- A set of ordered pairs

Functions - operations

Let $f_1 : A \to B, f_2 : A \to B$ be functions

- Addition: $(f_1 + f_2)(x) = f_1(x) + f_2(x)$
  e.g., If $A = B = \mathbb{R}, f_1(x) = 2x, f_2(x) = x^2 + 1$, then
  $(f_1 + f_2)(x) = x^2 + 2x + 1$

- Multiplication by a scalar: $(c \cdot f_1)(x) = c \cdot f_1(x)$
  e.g., If $A = B = \mathbb{R}, f_1(x) = 2x$, then $(3 \cdot f_1)(x) = 6x$

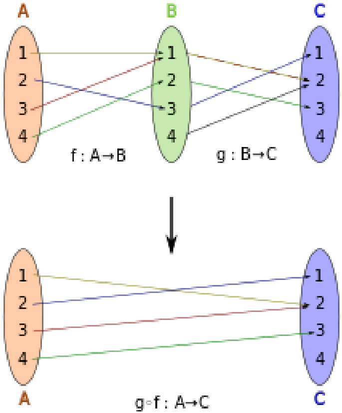
- Multiplication of functions: $(f_1 \cdot f_2)(x) = f_1(x) \cdot f_2(x)$
  e.g., If $A = B = \mathbb{R}, f_1(x) = 2x, f_2(x) = x^2 + 1$, then
  $(f_1 \cdot f_2)(x) = 2x(x^2 + 1)$

Function Composition

Let $f : A \to B, g : B \to C$ be functions
Then, $g \circ f : A \to C, (g \circ f)(x) = g(f(x))$.

E.g., $A = B = \mathbb{R}$,
$f(x) = 2x$,
$g(x) = x^2 + 1$

- $f(g(x)) = 2(x^2 + 1)$
- $g(f(x)) = (2x)^2 + 1$

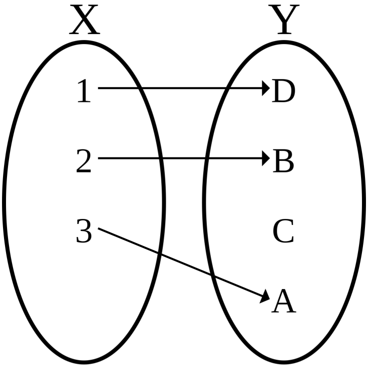


Types of Functions

- Injective (One-to-one)
- Surjective (Onto)
- Bijective (1-1 correspondence): Both injective and surjective.

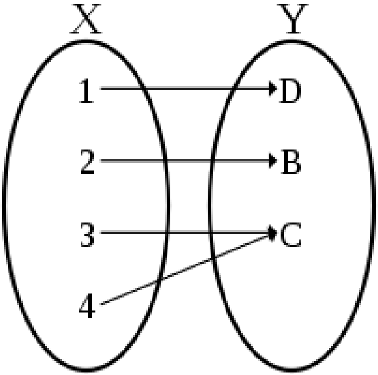
Injective Functions

- "Different elements map to different elements" – or no two elements in the domain map to the same element in the range
- No two arrows point to the same element on the right hand side
- E.g.: $A = B = \mathbb{N}, f(n) = 2n$
- Not injective: $A = B = \mathbb{Z}, f(n) = n^2$



Surjective Functions

- Every elements in the range is mapped to
- Some arrow(s) point to each element on the right hand side
- E.g.: $A = \mathbb{R}, B = \mathbb{N}, f(x) = \lfloor 1 + x^2 \rfloor$
- Not surjective: $A = B = \mathbb{N}, f(n) = n^2$



Surjective Functions – Subtle issue

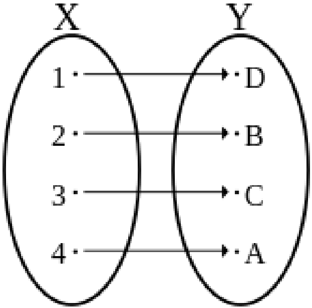
The domain and codomain have a crucial role

- $f : \mathbb{N} \to \mathbb{N}, f(n) = 2n$: NOT surjective
- $f : \mathbb{R} \to \mathbb{R}, f(n) = 2n$: surjective

Bijective Functions

If a function is injective AND surjective we call it bijective

- A bijective function is invertible
- Inverse: $f^{-1}(y) = x$ iff $f(x) = y$
- Note: $f^{-1}(x) \neq 1/f(x)$
- E.g.: $f : \mathbb{R} \to \mathbb{R}, f(x) = x^3 + 2$, $f^{-1}(x) = \sqrt[3]{x-2}$



Proving Injections and Surjections

- Fact: A function is injective if and only if $f(a) \neq f(b)$ whenever $a \neq b$
  Use: Show that if $f(a) = f(b)$ then $a = b$
  E.g., $f : \mathbb{R} \to \mathbb{R}, f(x) = 2x + 1$,
  $f : \mathbb{R} \to \mathbb{R}, f(x) = x^3$
- Fact: A function is surjective if and only if $f : A \to B$, for any $b \in B$, there must exist an $a \in A$, such that $f(a) = b$
  Use: consider an arbitrary $b \in B$. Find $a \in A$ such that $f(a) = b$.
  E.g., $f : \mathbb{R} \to \mathbb{R}, f(x) = 2x + 1$,
  $f : \mathbb{R} \to \mathbb{R}, f(x) = x^3$

Proving Injections and Surjections - 2

- Q12b, c, pg 153: Are these injective: $f, g : \mathbb{Z} \to \mathbb{Z}$, $f(n) = n^2 + 1, g(n) = n^3$?

- Q14 a, b, pg 153: Are these surjective: $f, g : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$, $f(m, n) = 2m - n, g(n) = m^2 - n^2$?

Some Special Functions

- Identity function $\mathcal{I}(x) = x$: valid on every domain
  Note: For every invertible function f, $f \circ f^{-1} = f^{-1} \circ f = \mathcal{I}$
- Reals to Integers: floor, ceiling
- Integers: DecimalToBinary, BinaryToDecimal
- Reals: exponential, log
  $e^x, \ln x$ are inverses. So $e^{\ln x} = \ln e^x = x$ for every x where the functions are defined.

DecimalToBinary, BinaryToDecimal

- DecimalToBinary: E.g., $7 = 1112, 10012 = 9$
  Algorithm DecimalToBinary – steps:
  $n = 7$:
  $b_1 = n \mod 2 = 1, n = n \text{ div } 2 = 3$
  $b_2 = n \mod 2 = 1, n = n \text{ div } 2 = 1$
  $b_3 = n \mod 2 = 1, n = n \text{ div } 2 = 0.$
  RETURN

- BinaryToDecimal: E.g., $n = 10012$
  $n = 1 * 2^3 + 0 * 2^2 + 0 * 2^1 + 1 * 2^0 = 9$

More on Changing Bases

In general need to go through the decimal representation
E.g: Convert 101_7 to base 9

- Other bases to decimal: $101_7 = 1 * 7^2 + 0 * 7^1 + 1 * 7^0 = 50$
- Decimal to Base 9:
  $d_1 = n \mod 9 = 5, n = n \text{ div } 9 = 5$
  $b_2 = n \mod 9 = 5, n = n \text{ div } 9 = 0.$
  STOP
  So $101_7 = 55_9$.
- Changing bases that are powers of 2: Can often use shortcuts.
  - Binary to Octal: $10111101_2 = \boxed{10}\boxed{111}\boxed{101} = 275_8$
  - Binary to Hexadecimal: $10111101_2 = \boxed{1011}\boxed{1101} = BD_{16}$
  - Hexadecimal to Octal: Go through binary, not decimal.