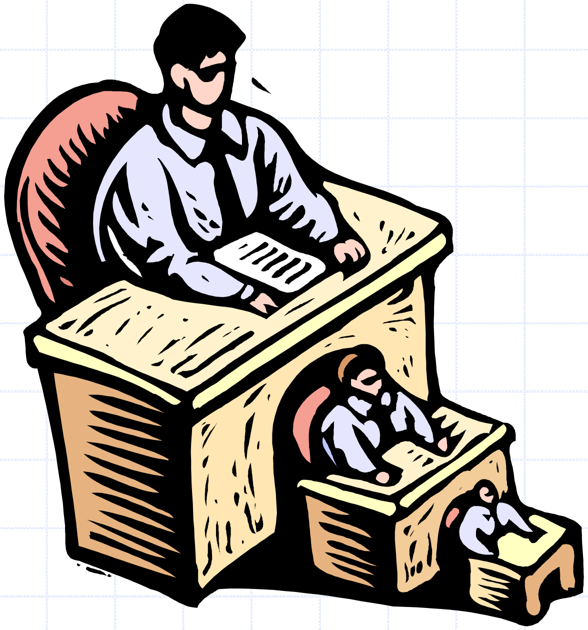
Presentation for use with the textbook Data Structures and Algorithms in Java, 6th edition, by M. T. Goodrich, R. Tamassia, and M. H. Goldwasser, Wiley, 2014

Recursion



The Recursion Pattern

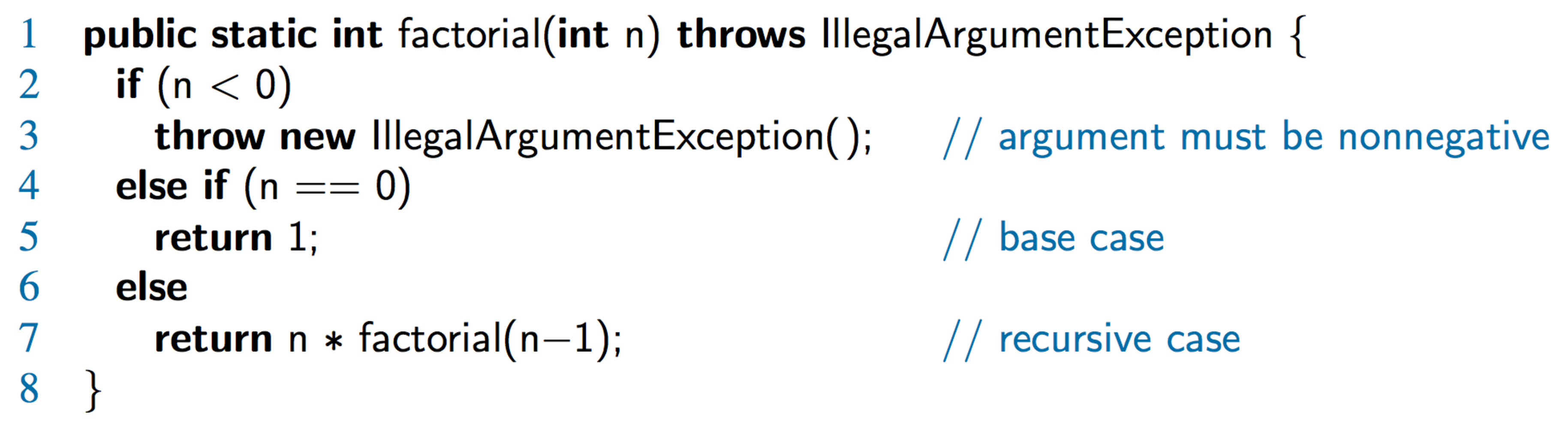
- Recursion: when a method calls itself
- Classic example – the factorial function:

$$n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-1) \cdot n$$

- Recursive definition:

$$f(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot f(n-1) & else \end{cases}$$

- As a Java method:



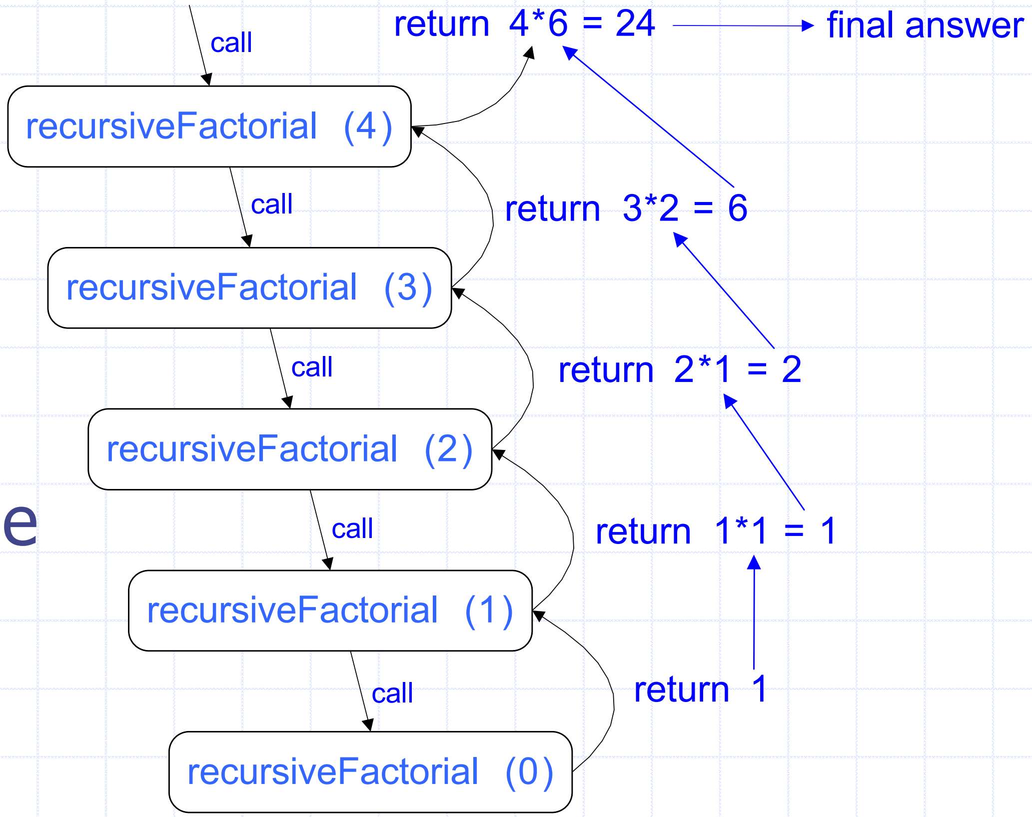
```
public static int factorial(int n) throws IllegalArgumentException {
  if (n < 0)
    throw new IllegalArgumentException( );    // argument must be nonnegative
  else if (n == 0)
    return 1;                                 // base case
  else
    return n * factorial(n−1);                // recursive case
}
```

Content of a Recursive Method

- Base case(s)
  - Values of the input variables for which we perform no recursive calls are called base cases (there should be at least one base case).
  - Every possible chain of recursive calls must eventually reach a base case.
- Recursive calls
  - Calls to the current method.
  - Each recursive call should be defined so that it makes progress towards a base case.

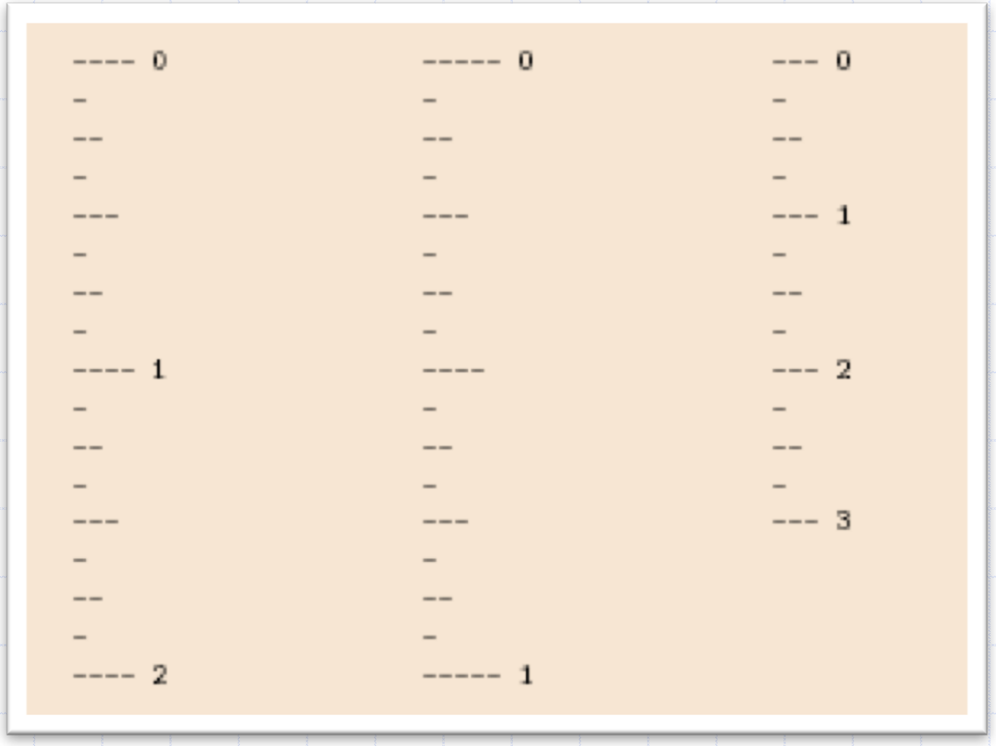
Visualizing Recursion

- Recursion trace
  - A box for each recursive call
  - An arrow from each caller to callee
  - An arrow from each callee to caller showing return value
- Example



Example: English Ruler

- Print the ticks and numbers like an English ruler:



```
---- 0              ----- 0              --- 0
-                   -                    -
--                  --                   --
-                   -                    -
---                 ---                  --- 1
-                   -                    -
--                  --                   --
-                   -                    -
---- 1              ----                 --- 2
-                   -                    -
--                  --                   --
-                   -                    -
---                 ---                  --- 3
-                   -
--                  --
-                   -
---- 2              ----- 1
```

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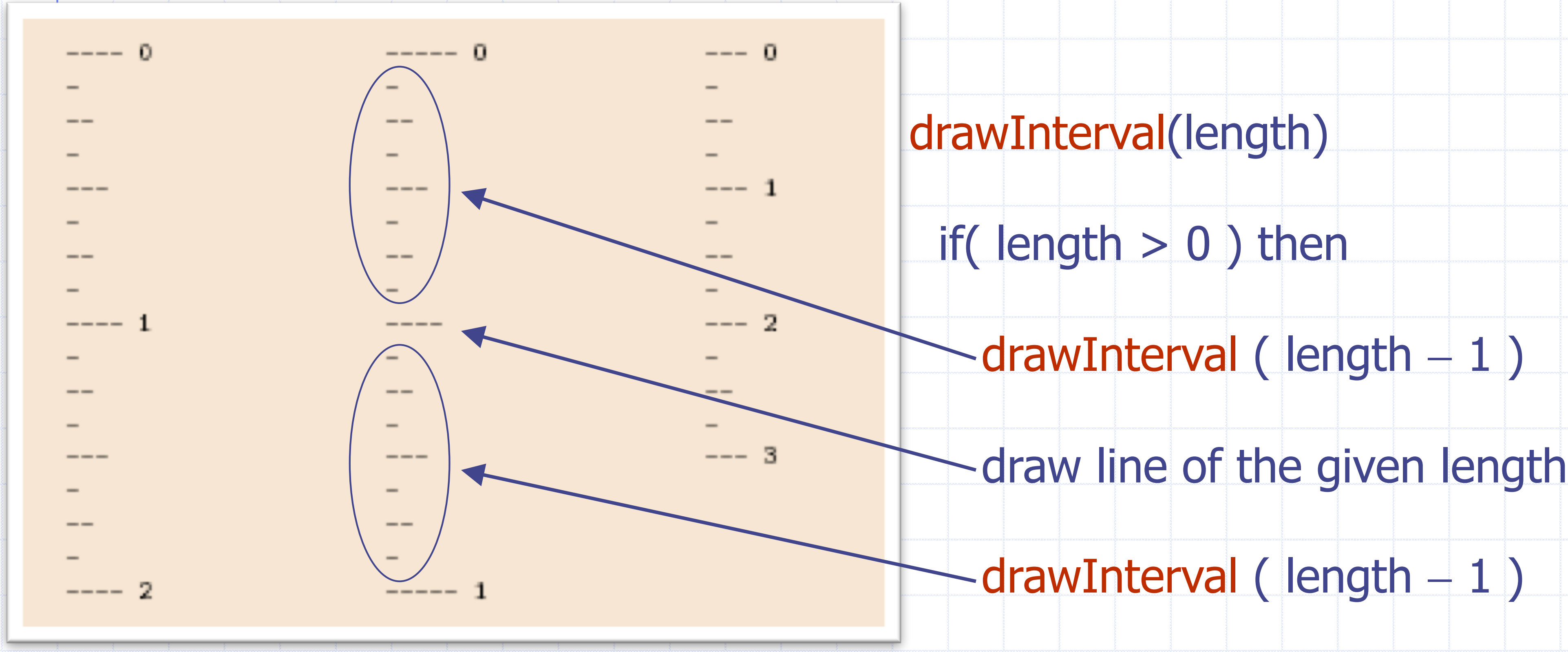
Using Recursion

drawInterval(length)

Input: length of a ‘tick’

Output: ruler with tick of the given length in the middle and smaller rulers on either side

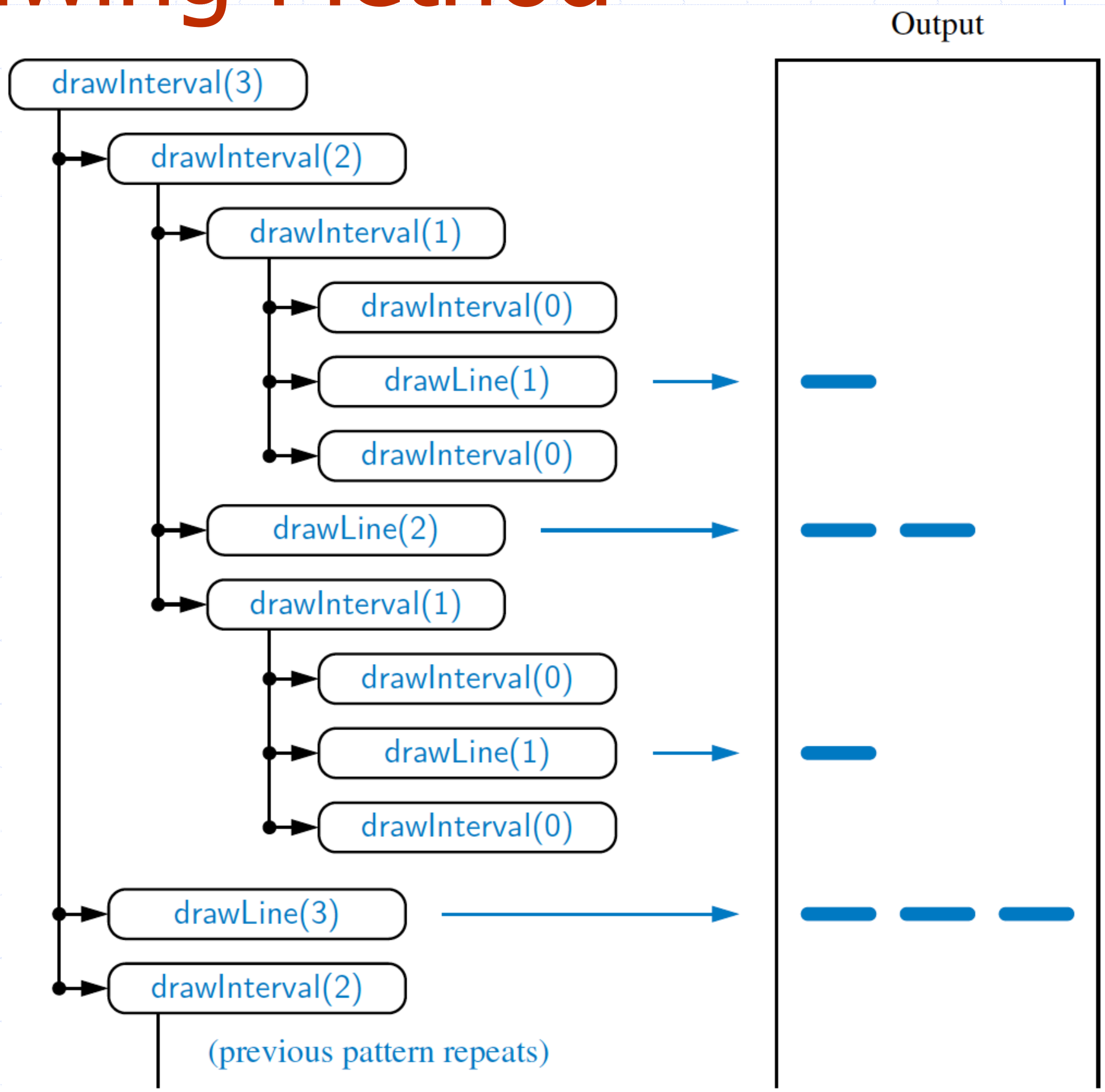


```
---- 0              ----- 0             --- 0
-                   -                   -
--                  --                  --
-                   -                   -
---                 ---                 --- 1
-                   -                   -
--                  --                  --
-                   -                   -
---- 1              ----                --- 2
-                   -                   -
--                  --                  --
-                   -                   -
---                 ---                 --- 3
-                   -
--                  --
-                   -
---- 2              ----- 1
```

```
drawInterval(length)
  if( length > 0 ) then
    drawInterval ( length – 1 )
    draw line of the given length
    drawInterval ( length – 1 )
```

Recursive Drawing Method

- The drawing method is based on the following recursive definition
- An interval with a central tick length L ≥1 consists of:
  - An interval with a central tick length L–1
  - An single tick of length L
  - An interval with a central tick length L–1



A Recursive Method for Drawing Ticks on an English Ruler

```
/** Draws an English ruler for the given number of inches and major tick length. */
public static void drawRuler(int nInches, int majorLength) {
  drawLine(majorLength, 0);                  // draw inch 0 line and label
  for (int j = 1; j <= nInches; j++) {
    drawInterval(majorLength - 1);           // draw interior ticks for inch
    drawLine(majorLength, j);                // draw inch j line and label
  }
}
private static void drawInterval(int centralLength) {
  if (centralLength >= 1) {                  // otherwise, do nothing
    drawInterval(centralLength - 1);         // recursively draw top interval
    drawLine(centralLength);                 // draw center tick line (without label)
    drawInterval(centralLength - 1);         // recursively draw bottom interval
  }
}
private static void drawLine(int tickLength, int tickLabel) {
  for (int j = 0; j < tickLength; j++)
    System.out.print("-");
  if (tickLabel >= 0)
    System.out.print(" " + tickLabel);
  System.out.print("\n");
}
/** Draws a line with the given tick length (but no label). */
private static void drawLine(int tickLength) {
  drawLine(tickLength, -1);
}
```

Note the two recursive calls

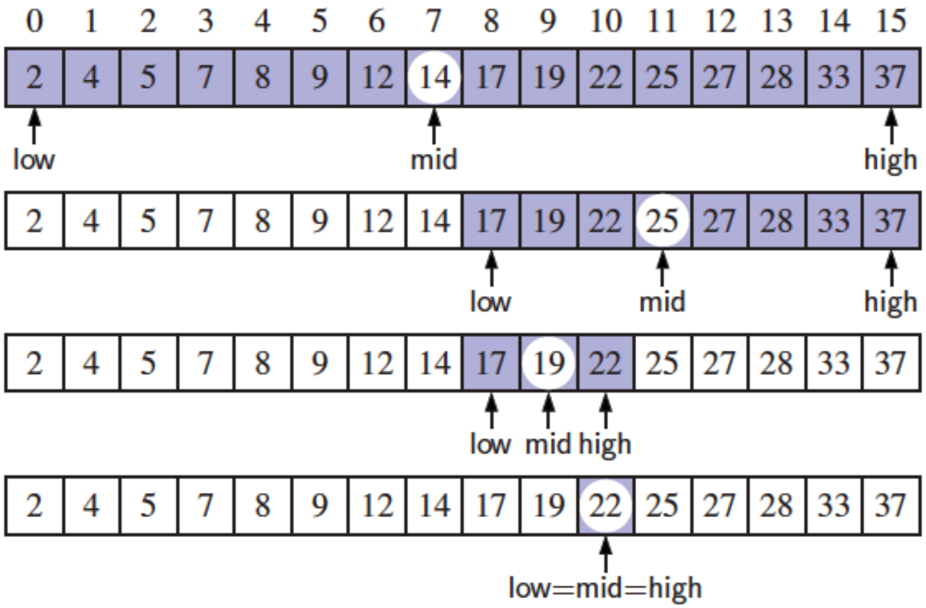
Binary Search

Search for an integer in an ordered list

```
/**
 * Returns true if the target value is found in the indicated portion of the data array.
 * This search only considers the array portion from data[low] to data[high] inclusive.
 */
public static boolean binarySearch(int[ ] data, int target, int low, int high) {
  if (low > high)
    return false;                                              // interval empty; no match
  else {
    int mid = (low + high) / 2;
    if (target == data[mid])
      return true;                                             // found a match
    else if (target < data[mid])
      return binarySearch(data, target, low, mid − 1);         // recur left of the middle
    else
      return binarySearch(data, target, mid + 1, high);        // recur right of the middle
  }
}
```

Visualizing Binary Search

- We consider three cases:
  - If the target equals data[mid], then we have found the target.
  - If target < data[mid], then we recur on the first half of the sequence.
  - If target > data[mid], then we recur on the second half of the sequence.



0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	4	5	7	8	9	12	14	17	19	22	25	27	28	33	37
2	4	5	7	8	9	12	14	17	19	22	25	27	28	33	37
2	4	5	7	8	9	12	14	17	19	22	25	27	28	33	37
2	4	5	7	8	9	12	14	17	19	22	25	27	28	33	37

Analyzing Binary Search

- Runs in O(log n) time.
  - The remaining portion of the list is of size high – low + 1
  - After one comparison, this becomes one of the following:

$$(\mathrm{mid}-1)-\mathrm{low}+1=\left\lfloor\frac{\mathrm{low}+\mathrm{high}}{2}\right\rfloor-\mathrm{low}\leq\frac{\mathrm{high}-\mathrm{low}+1}{2}$$

$$\mathrm{high}-(\mathrm{mid}+1)+1=\mathrm{high}-\left\lfloor\frac{\mathrm{low}+\mathrm{high}}{2}\right\rfloor\leq\frac{\mathrm{high}-\mathrm{low}+1}{2}.$$

  - Thus, each recursive call divides the search region in half; hence, there can be at most log n levels

Linear Recursion

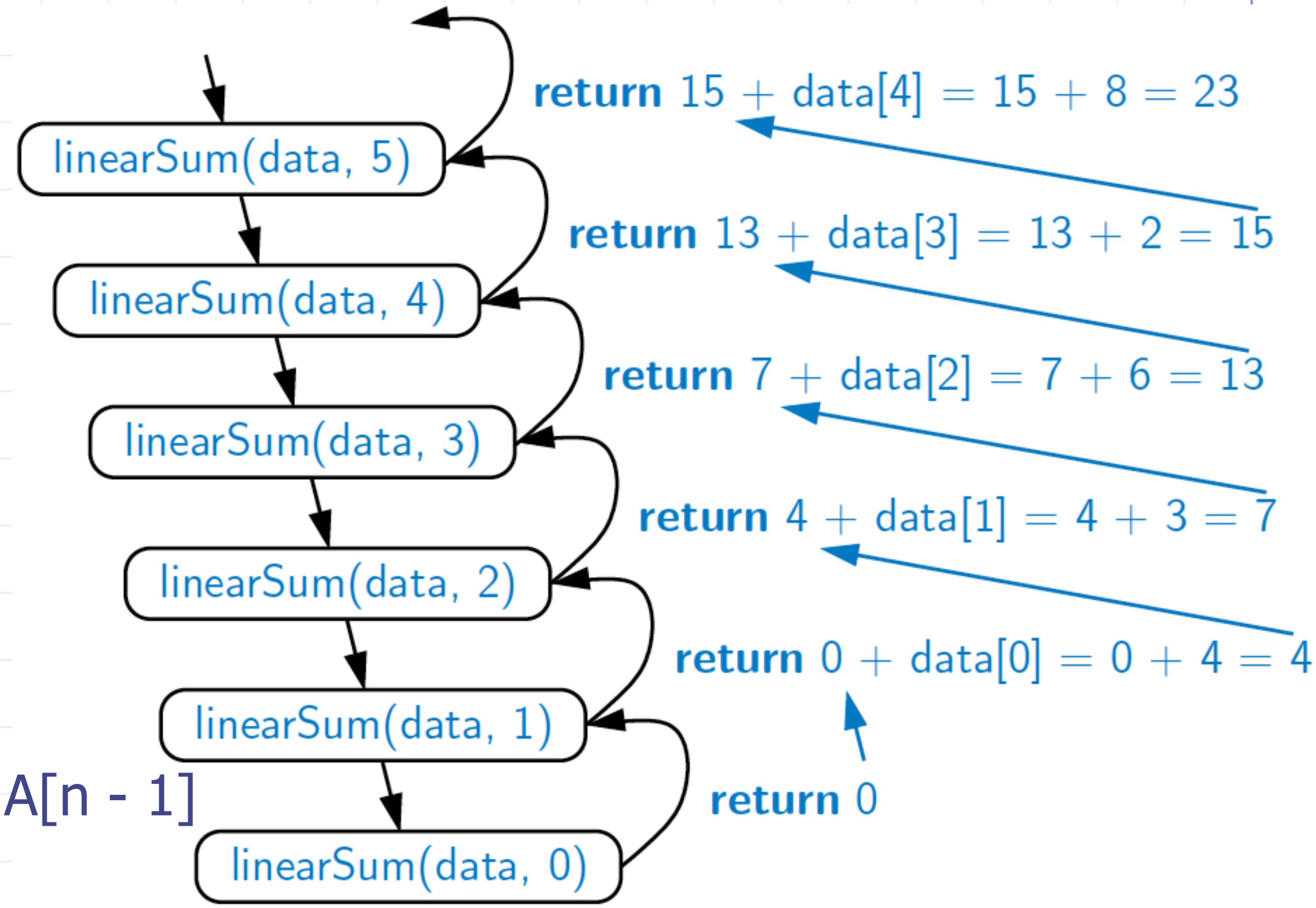
- Test for base cases
    - Begin by testing for a set of base cases (there should be at least one).
    - Every possible chain of recursive calls must eventually reach a base case, and the handling of each base case should not use recursion.
- Recur once
    - Perform a single recursive call
    - This step may have a test that decides which of several possible recursive calls to make, but it should ultimately make just one of these calls
    - Define each possible recursive call so that it makes progress towards a base case.

Example of Linear Recursion

```
Algorithm linearSum(A, n):
Input:
  Array, A, of integers
  Integer n such that
    0 ≤ n ≤ |A|
Output:
    Sum of the first n
    integers in A

if n = 0 then
  return 0
else
  return
linearSum(A, n - 1) + A[n - 1]
```

Recursion trace of linearSum(data, 5) called on array data = [4, 3, 6, 2, 8]



linearSum(data, 5)

linearSum(data, 4)

linearSum(data, 3)

linearSum(data, 2)

linearSum(data, 1)

linearSum(data, 0)

return $15 + \text{data}[4] = 15 + 8 = 23$

return $13 + \text{data}[3] = 13 + 2 = 15$

return $7 + \text{data}[2] = 7 + 6 = 13$

return $4 + \text{data}[1] = 4 + 3 = 7$

return $0 + \text{data}[0] = 0 + 4 = 4$

return 0

Reversing an Array

```
Algorithm reverseArray(A, i, j):
Input: An array A and nonnegative integer
   indices i and j
Output: The reversal of the elements in A
   starting at index i and ending at

if i < j then
      Swap A[i] and A[j]
      reverseArray(A, i + 1, j – 1)
return
```

Defining Arguments for Recursion

- In creating recursive methods, it is important to define the methods in ways that facilitate recursion.
- This sometimes requires we define additional parameters that are passed to the method.
- For example, we defined the array reversal method as reverseArray(A, i, j), not reverseArray(A)

```
/** Reverses the contents of subarray data[low] through data[high] inclusive. */
public static void reverseArray(int[ ] data, int low, int high) {
  if (low < high) {                              // if at least two elements in subarray
    int temp = data[low];                        // swap data[low] and data[high]
    data[low] = data[high];
    data[high] = temp;
    reverseArray(data, low + 1, high - 1);       // recur on the rest
  }
}
```

Computing Powers

- The power function, p(x,n)=x^n, can be defined recursively:

$$p(x,n) = \begin{cases} 1 & \text{if } n = 0 \\ x \cdot p(x, n-1) & \text{else} \end{cases}$$

- This leads to an power function that runs in O(n) time (for we make n recursive calls)
- We can do better than this, however

Recursive Squaring

- We can derive a more efficient linearly recursive algorithm by using repeated squaring:

$$p(x,n) = \begin{cases} 1 & \text{if } x = 0 \\ x \cdot p(x,(n-1)/2)^2 & \text{if } x > 0 \text{ is odd} \\ p(x,n/2)^2 & \text{if } x > 0 \text{ is even} \end{cases}$$

- For example,

$$2^4 = 2^{(4/2)2} = (2^{4/2})^2 = (2^2)^2 = 4^2 = 16$$

$$2^5 = 2^{1+(4/2)2} = 2(2^{4/2})^2 = 2(2^2)^2 = 2(4^2) = 32$$

$$2^6 = 2^{(6/2)2} = (2^{6/2})^2 = (2^3)^2 = 8^2 = 64$$

$$2^7 = 2^{1+(6/2)2} = 2(2^{6/2})^2 = 2(2^3)^2 = 2(8^2) = 128$$

Recursive Squaring Method

```
**Algorithm** Power(x, n):
    **Input:** A number x and integer n = 0
    **Output:** The value $x^n$
  **if** $n = 0$ **then**
      **return** 1
  **if** n is odd **then**
      $y = \text{Power}(x, (n - 1)/ 2)$
      **return** $x \cdot y \cdot y$
  **else**
      $y = \text{Power}(x, n/ 2)$
      **return** $y \cdot y$
```

Analysis

```
Algorithm Power(x, n):
    Input: A number x and integer n = 0
    Output: The value $x^n$
  if n = 0   then
      return 1
  if n is odd then
      y  = Power(x, (n - 1)/ 2)
      return x · y · y
  else
      y = Power(x, n/ 2)
      return y · y
```

Each time we make a recursive call we halve the value of n; hence, we make log n recursive calls. That is, this method runs in O(log n) time.

It is important that we use a variable twice here rather than calling the method twice.

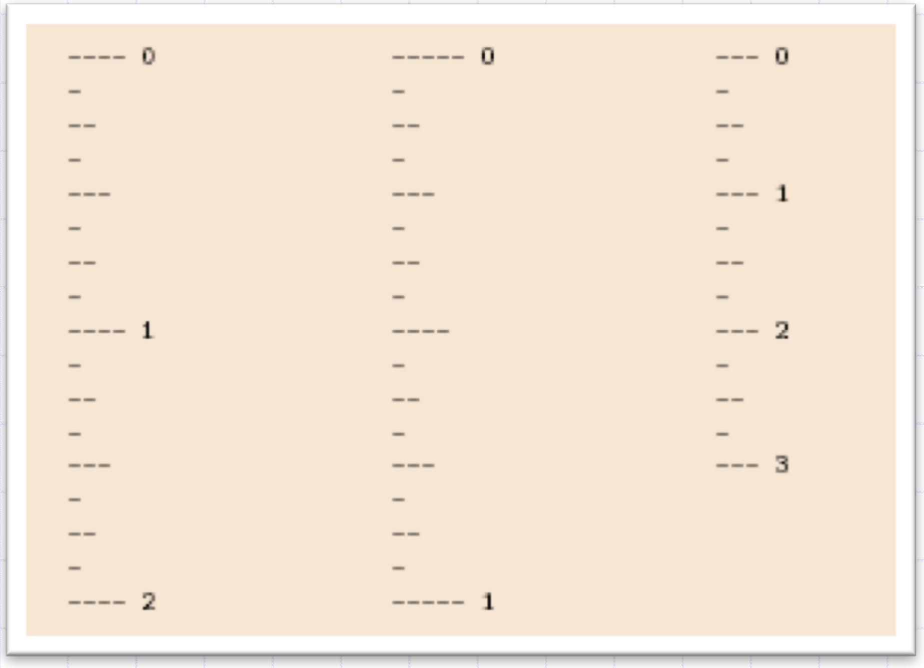
Tail Recursion

- Tail recursion occurs when a linearly recursive method makes its recursive call as its last step.
- The array reversal method is an example.
- Such methods can be easily converted to non-recursive methods (which saves on some resources).
- Example:

```
Algorithm IterativeReverseArray(A, i, j ):
    Input: An array A and nonnegative integer indices i and j
    Output: The reversal of the elements in A starting at
  index i and ending at j
   while i <  j do
      Swap A[i ] and A[ j ]
      i  = i + 1
      j  = j - 1
   return
```

Binary Recursion

- Binary recursion occurs whenever there are **two** recursive calls for each non-base case.
- Example from before: the drawInterval method for drawing ticks on an English ruler.



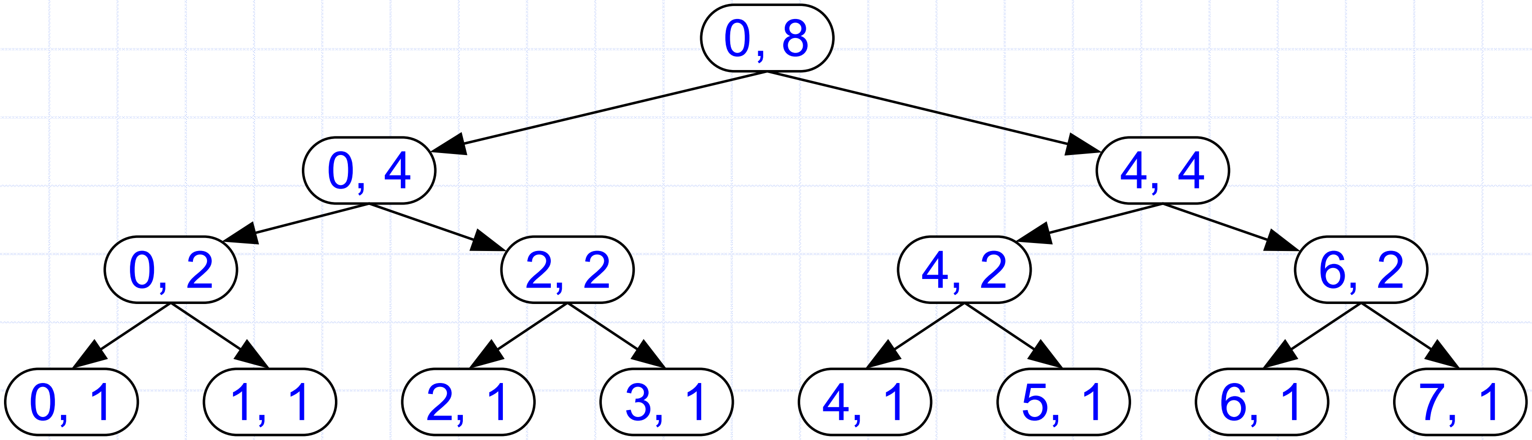
```
---- 0                ----- 0                --- 0
-                     -                      -
--                    --                     --
-                     -                      -
---                   ---                    --- 1
-                     -                      -
--                    --                     --
-                     -                      -
---- 1                ----                   --- 2
-                     -                      -
--                    --                     --
-                     -                      -
---                   ---                    --- 3
-                     -
--                    --
-                     -
---- 2                ----- 1
```

Another Binary Recusive Method

- Problem: add all the numbers in an integer array A:

```
Algorithm BinarySum(A, i, n):
    Input: An array A and integers i and n
    Output: The sum of the n integers in A starting at index i
    if n = 1 then
        return A[i]
    return BinarySum(A, i, n/ 2) + BinarySum(A, i + n/ 2, n/ 2)
```

- **Example trace:**



Computing Fibonacci Numbers

- Fibonacci numbers are defined recursively:

$F_0 = 0$
$F_1 = 1$
$F_i = F_{i-1} + F_{i-2}$ for $i > 1$.

- Recursive algorithm (first attempt):

```
Algorithm BinaryFib(k):
    Input: Nonnegative integer k
    Output: The kth Fibonacci number F_k
  if k = 1 then
    return k
  else
    return BinaryFib(k – 1) + BinaryFib(k – 2)
```

Analysis

- Let n_k be the number of recursive calls by BinaryFib(k)
  - $n_0 = 1$
  - $n_1 = 1$
  - $n_2 = n_1 + n_0 + 1 = 1 + 1 + 1 = 3$
  - $n_3 = n_2 + n_1 + 1 = 3 + 1 + 1 = 5$
  - $n_4 = n_3 + n_2 + 1 = 5 + 3 + 1 = 9$
  - $n_5 = n_4 + n_3 + 1 = 9 + 5 + 1 = 15$
  - $n_6 = n_5 + n_4 + 1 = 15 + 9 + 1 = 25$
  - $n_7 = n_6 + n_5 + 1 = 25 + 15 + 1 = 41$
  - $n_8 = n_7 + n_6 + 1 = 41 + 25 + 1 = 67$.
- Note that n_k at least doubles every other time
- That is, $n_k > 2^{k/2}$. It is exponential!

A Better Fibonacci Algorithm

- Use linear recursion instead

```
Algorithm LinearFibonacci(k):
  Input: A nonnegative integer k
  Output: Pair of Fibonacci numbers $(F_k , F_{k-1})$
 if k = 1 then
    return (k, 0)
 else
    (i,  j)  =  LinearFibonacci(k – 1)
    return (i +j, i)
```

- LinearFibonacci makes k–1 recursive calls

Multiple Recursion

- Motivating example:
  - summation puzzles
    - *pot* + *pan* = *bib*
    - *dog* + *cat* = *pig*
    - *boy* + *girl* = *baby*
- Multiple recursion:
  - makes potentially many recursive calls
  - not just one or two

Algorithm for Multiple Recursion

```
Algorithm PuzzleSolve(k,S,U):
 Input: Integer k, sequence S, and set U (universe of elements to
    test)
 Output:  Enumeration of all k-length extensions to S using elements
    in U without repetitions
 for all e  in U do
    Remove e from U      {e is now being used}
    Add e to the end of S
    if k = 1 then
          Test whether S is a configuration that solves the puzzle
          if S solves the puzzle then
               return "Solution found: " S
    else
          PuzzleSolve(k - 1, S,U)
    Add e back to U       {e is now unused}
    Remove e from the end of S
```

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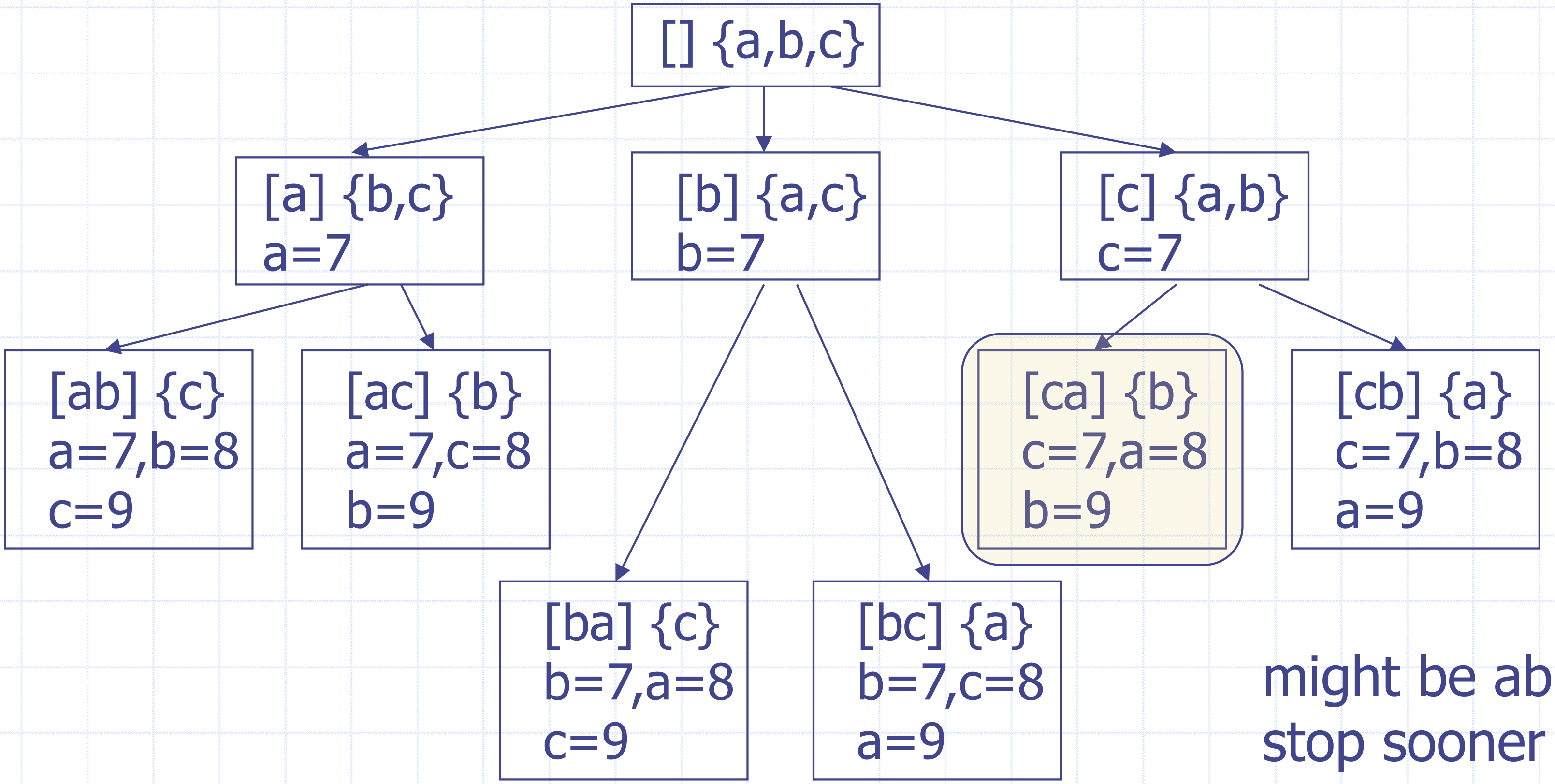
Example

cbb + ba = abc

799 + 98 = 997

a,b,c stand for 7,8,9; not necessarily in that order



[] {a,b,c}

[a] {b,c}
a=7

[b] {a,c}
b=7

[c] {a,b}
c=7

[ab] {c}
a=7,b=8
c=9

[ac] {b}
a=7,c=8
b=9

[ca] {b}
c=7,a=8
b=9

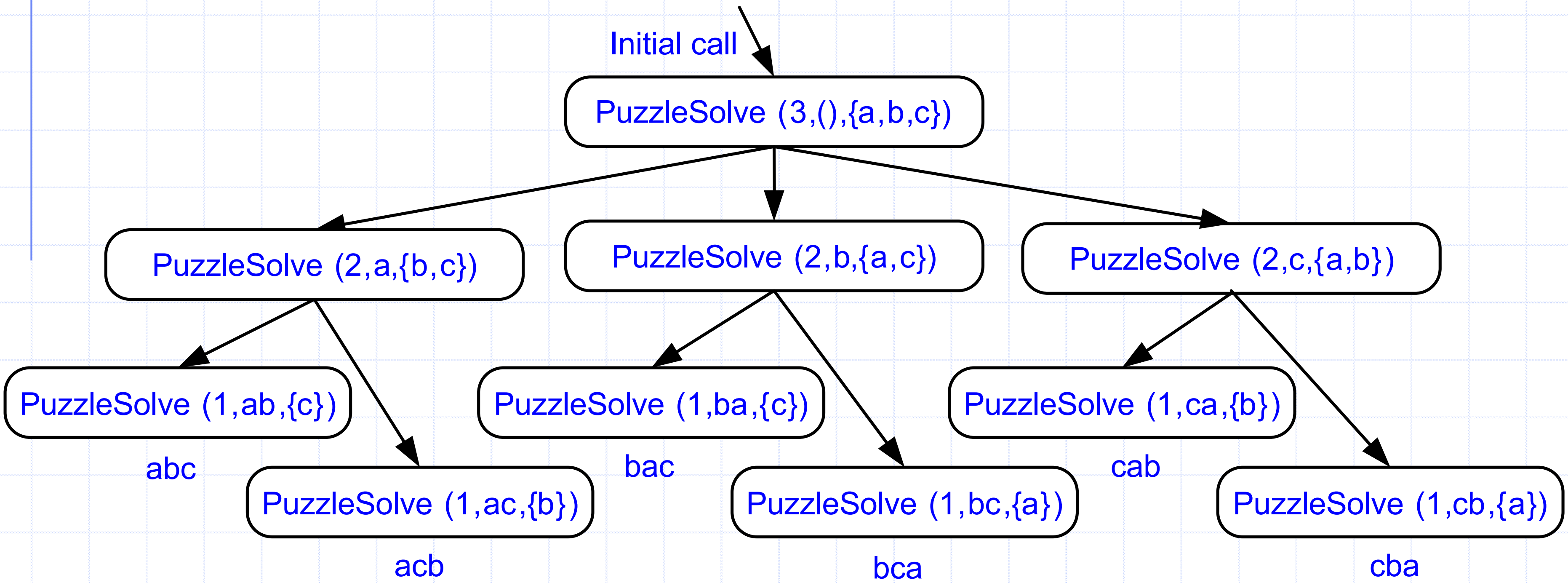
[cb] {a}
c=7,b=8
a=9

[ba] {c}
b=7,a=8
c=9

[bc] {a}
b=7,c=8
a=9

might be able to stop sooner

Visualizing PuzzleSolve



Initial call

PuzzleSolve (3,(),{a,b,c})

PuzzleSolve (2,a,{b,c})

PuzzleSolve (2,b,{a,c})

PuzzleSolve (2,c,{a,b})

PuzzleSolve (1,ab,{c})

abc

PuzzleSolve (1,ac,{b})

acb

PuzzleSolve (1,ba,{c})

bac

PuzzleSolve (1,bc,{a})

bca

PuzzleSolve (1,ca,{b})

cab

PuzzleSolve (1,cb,{a})

cba