EECS4421Z: Introduction to Robotics
Sample Midterm

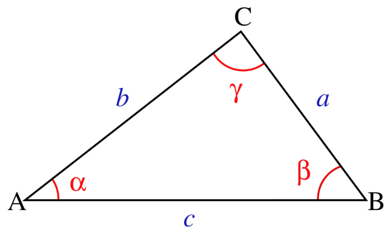
Name: ____________________

Student Number: ____________________

Instructions

1. You have 110 minutes to complete the exam.
2. Write your answers clearly and succintly in the space provided on the question sheets; use the back of the page and the extra blank page if you need additional space for your answer.
3. A non-programmable calculator is allowed but should not be necessary. No other aids are allowed.

Law of cosines:



$c^2 = a^2 + b^2 - 2ab\cos\gamma$

Trigonometric identities:

$\sin(-\theta) = -\sin(\theta)$
$\cos(-\theta) = \cos(\theta)$

$\sin(180 - \theta) = \sin(\theta)$
$\cos(180 - \theta) = -\cos(\theta)$

$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$\cos^2\theta + \sin^2\theta = 1$

Canonical rotation matrices:

$$R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \qquad R_{y,\theta} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \qquad R_{z,\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotation about a unit axis $[k_x\ k_y\ k_z]^T$:

$$R_{k,\theta} = \begin{bmatrix} k_x^2 v_\theta + c_\theta & k_x k_y v_\theta - k_z s_\theta & k_x k_z v_\theta + k_y s_\theta \\ k_x k_y v_\theta + k_z s_\theta & k_y^2 v_\theta + c_\theta & k_y k_z v_\theta - k_x s_\theta \\ k_x k_z v_\theta - k_y s_\theta & k_y k_z v_\theta + k_x s_\theta & k_z^2 v_\theta + c_\theta \end{bmatrix}$$

where $c_\theta = \cos\theta$, $s_\theta = \sin\theta$, and $v_\theta = 1 - \cos\theta$.

Homogeneous translation matrix for a translation of $[x\ y\ z]^T$:

$$D = \begin{bmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

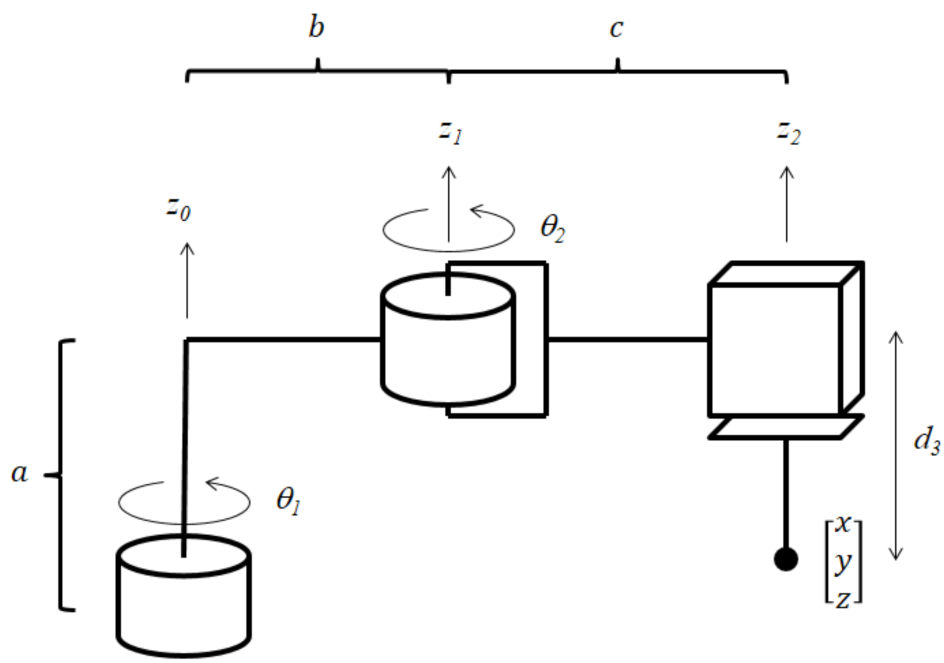
Denavit-Hartenberg transformation:

$$
\begin{aligned}
T_i^{i-1} &= R_z(\theta_i) D_z(d_i) D_x(a_i) R_x(\alpha_i) \\
&= \begin{bmatrix} \cos(\theta_i) & -\sin(\theta_i)\cos(\alpha_i) & \sin(\theta_i)\sin(\alpha_i) & a_i \cos(\theta_i) \\ \sin(\theta_i) & \cos(\theta_i)\cos(\alpha_i) & -\cos(\theta_i)\sin(\alpha_i) & a_i \sin(\theta_i) \\ 0 & \sin(\alpha_i) & \cos(\alpha_i) & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

Denavit-Hartenberg parameters:

a_i: distance between z_{i-1} and z_i measured along x_i
α_i: angle from z_{i-1} to z_i measured about x_i
d_i: distance between o_{i-1} to the intersection of x_i and z_{i-1} measured along z_{i-1}
θ_i: angle from x_{i-1} to x_i measured about z_{i-1}

1. Consider the following RRR robot:



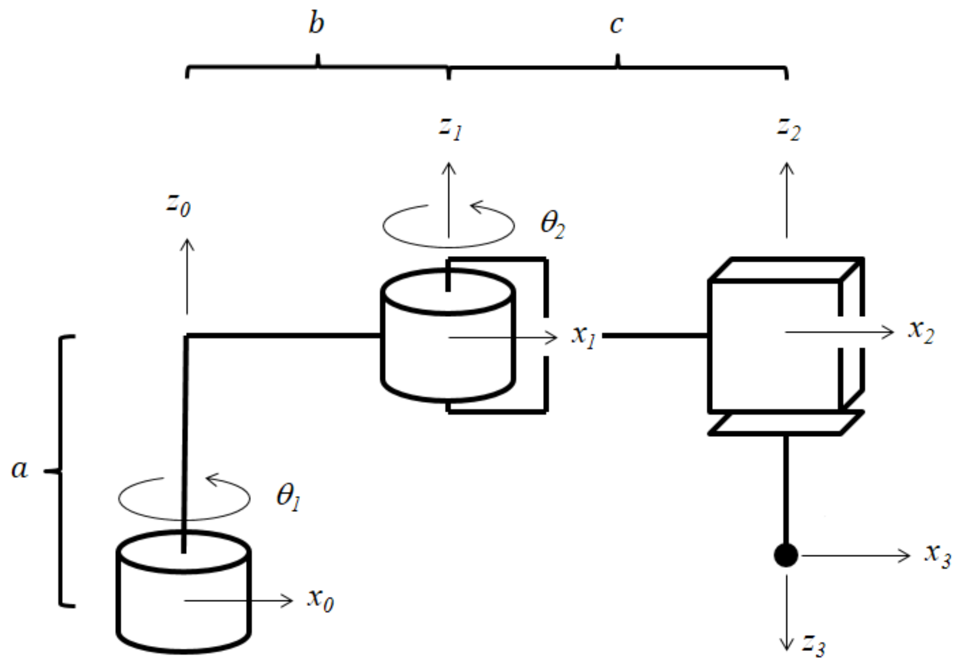
The robot is shown in a position where all of the joint angles are zero degrees and d_3 is some positive value. For the purposes of this question, assume that there are no limits on the joint parameters.

(a) Draw coordinate frames $\{0\}$, $\{1\}$, $\{2\}$, and $\{3\}$ that are suitable for use with the Denavit-Hartenberg convention.

(b) Fill in the Denavit-Hartenberg parameters using your coodinate frame placement.

Link	a_i	α_i	d_i	θ_i
1				
2				
3				

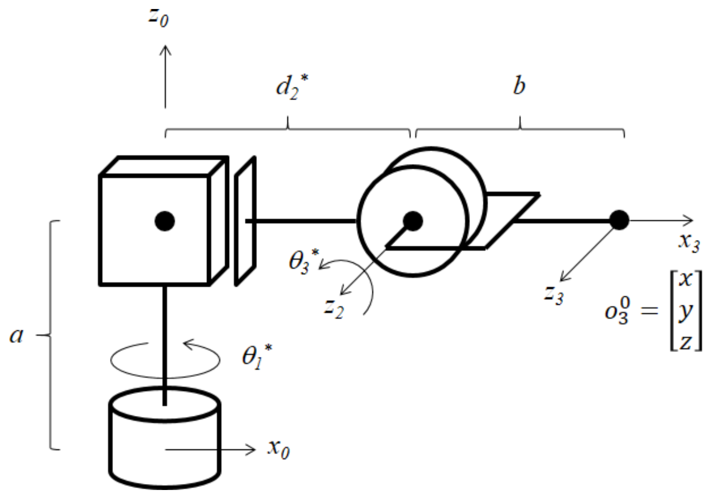
(c) Solve for the forward kinematics of the arm; that is, given joint parameters θ_1^*, θ_2^*, and d_3^*, find the orientation and position of frame $\{3\}$ as a homogeneous matrix T_3^0. If your solution involves a sequence of matrix multiplications then you do not need to perform the actual multiplications. Do not use the Denavit-Hartenberg convention to obtain your answer.



Link	a_i	α_i	d_i	θ_i
1	b	0	a	θ_1^*
2	c	0	0	θ_2^*
3	0	180	$-d_3^*$	0

$$T_3^0 = R_{z,\theta_1^*} D_{z,a} D_{x,b} R_{z,\theta_2^*} D_{x,c} R_{x,180} D_{z,d_3^*}$$

2. Consider the RPR arm shown in Figure 2.



The prismatic joint of the robot has length $5 \leq d_2^* \leq 10$. The range of joint 1 is $0° \leq \theta_1^* < 360°$ and the range of joint 3 is $-170° \leq \theta_3^* \leq 170°$ The exact values of a and b are not important, but assume that $a > b$ and $b < 5$.

(a) For which values of θ_3^* does there exist a unique solution to the inverse kinematics problem?

(b) Given o_3^0 solve the inverse kinematics problem (i.e., given values for x, y, and z solve for the values of θ_1^*, d_2^*, and θ_3^*).

(a) This question was phrased poorly. The intent was to ask for which values of θ_3^* is the solution to the inverse kinematics problem always unique for which the answer is:

$-10 < \theta_3^* < 10$ and $\theta_3^* = \pm 90$

(b) First find the two possible solutions for θ_3^*:

$$\begin{aligned}
\theta_3^* &= \arcsin\left(\frac{z-a}{b}\right) \text{ for } -90 <= \theta_3^* <= 90 \\
\theta_3^* &= 180 - \arcsin\left(\frac{z-a}{b}\right) \text{ for } 90 < \theta_3^* <= 170, -90 > \theta_3^* >= -170
\end{aligned}$$

There is only a single solution for θ_1^* because d_3^* cannot be negative:

$$\theta_1^* = \text{atan2}\,(y, x)$$

Using both solutions for θ_3^*:

$$\begin{aligned}
\sqrt{x^2+y^2} &= d_2^* + b\cos\theta_3^* \\
\therefore d_2^* &= \sqrt{x^2+y^2} - b\cos\theta_3^*
\end{aligned}$$

3. Given a 3×3 rotation matrix R that transforms 3-dimensional points, prove that the transpose R^T is equal to the inverse R^{-1}.

$$R = \begin{bmatrix} | & | & | \\ x_1^0 & y_1^0 & z_1^0 \\ | & | & | \end{bmatrix}$$

$$R^T = \begin{bmatrix} - & x_1^{0T} & - \\ - & y_1^{0T} & - \\ - & z_1^{0T} & - \end{bmatrix}$$

$$\begin{aligned} R^T R &= \begin{bmatrix} - & x_1^{0T} & - \\ - & y_1^{0T} & - \\ - & z_1^{0T} & - \end{bmatrix} \begin{bmatrix} | & | & | \\ x_1^0 & y_1^0 & z_1^0 \\ | & | & | \end{bmatrix} \\ &= \begin{bmatrix} x_1^0 \cdot x_1^0 & x_1^0 \cdot y_1^0 & x_1^0 \cdot z_1^0 \\ y_1^0 \cdot x_1^0 & y_1^0 \cdot y_1^0 & y_1^0 \cdot z_1^0 \\ z_1^0 \cdot x_1^0 & z_1^0 \cdot y_1^0 & z_1^0 \cdot z_1^0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Similarly, you can show that:

$$RR^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore, $R^T = R^{-1}$.