1. Show that the distance between points is unchanged by rotation; that is $\|p_1 - p_2\| = \|Rp_1 - Rp_2\|$.

Solution: The squared distance can be computed using the dot product, and the dot product $x \cdot y$ is equal to the matrix product $x^T y$:

$$\begin{aligned}
\|Rp_1 - Rp_2\|^2 &= (Rp_1 - Rp_2) \cdot (Rp_1 - Rp_2) \\
&= (Rp_1) \cdot (Rp_1) - 2(Rp_1) \cdot (Rp_2) + (Rp_2) \cdot (Rp_2) \\
&= (Rp_1)^T (Rp_1) - 2(Rp_1)^T (Rp_2) + (Rp_2)^T (Rp_2) \\
&= p_1^T R^T R p_1 - 2p_1^T R^T R p_2 + p_2^T R^T R p_2 \\
&= p_1^T p_1 - 2p_1^T p_2 + p_2^T p_2 \\
&= \|p_1 - p_2\|^2
\end{aligned}$$

2. Suppose that A is a 2×2 matrix where $A^T A = I$ and $\det A = 1$. Show that there exists a unique θ such that

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Solution: For a square matrix, if $A^T A = I$ then it must be the case that $A^T = A^{-1}$ because

$$\begin{aligned}
A^T A &= I \\
A^T A A^{-1} &= I A^{-1} \\
A^T I &= A^{-1} \\
A^T &= A^{-1}
\end{aligned}$$

Furthermore, $A^T A = I = AA^T$ because $A^{-1} A = I = AA^{-1}$.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then

$$A^T A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a^2 + b^2 & ac + bd \\ ac + bd & c^2 + d^2 \end{bmatrix}$$

Consider the upper-left matrix entries equal to 1:

$$a^2 + c^2 = 1 \tag{1}$$
$$a^2 + b^2 = 1 \tag{2}$$

From (1) and (2) we can deduce that $b = \pm c$. Consider the case where $b = -c$; substitute into any of the matrix entries equal to 0 to deduce that $a = d$. Therefore, A has the form

$$A = \begin{bmatrix} a & -c \\ c & a \end{bmatrix}$$

The determinant of A is $\det(A) = 1 = a^2 + c^2$. Therefore there exists some value θ such that $a = \cos\theta$ and $c = \sin\theta$ because $\cos^2\theta + \sin^2\theta = 1$.

3. (a) Consider all pairs of rotations and translations along the three principle axes, i.e., $R_x R_x$, $R_x R_y$, ..., $R_x T_x$, ..., $T_z T_z$. Which pairs commute?

Solution: $R_x T_x = T_x R_x$, $R_y T_y = T_y R_y$, $R_z T_z = T_z R_z$, $T_x T_y = T_y T_x$, $T_x T_z = T_z T_x$, $T_y T_z = T_z T_y$.

(b) Given your answer to (a), what other representations are there for the Denavit-Hartenberg transformation?

Solution: The Denavit-Hartenberg (DH) transformation matrix is $R_{z,\theta} T_{z,d} T_{x,a} R_{x,\alpha}$. Because $R_{z,\theta} T_{z,d}$ and $T_{x,a} R_{x,\alpha}$ both commute, there are four equivalent ways to express the DH matrix:

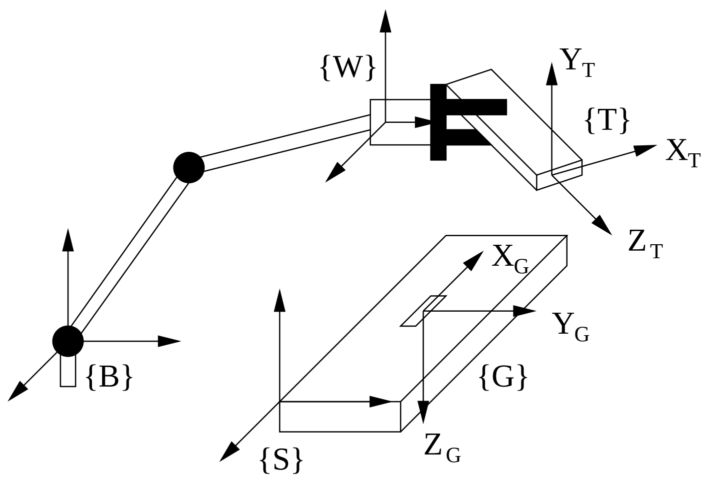
$$\begin{gathered} R_{z,\theta} T_{z,d} T_{x,a} R_{x,\alpha} \\ T_{z,d} R_{z,\theta} T_{x,a} R_{x,\alpha} \\ R_{z,\theta} T_{z,d} R_{x,\alpha} T_{x,a} \\ T_{z,d} R_{z,\theta} R_{x,\alpha} T_{x,a} \end{gathered}$$

4. (a) Suppose you have a frame $\{A\}$ and a frame $\{B\}$. The 4×4 homogeneous matrix T_B^A, where the upper-left 3×3 sub-matrix is a rotation matrix, has three distinct interpretations. What are the interpretations?

Solution:

1. the orientation of frame $\{B\}$ expressed in terms of frame $\{A\}$
2. transformation of a point from frame $\{B\}$ to frame $\{A\}$
3. an operator that takes a point and produces a new point in the same frame

(b) Consider the figure shown below. The pose of the tool relative to the wrist, T_T^W, is not known. By limping the arm joints, the tool tip can be inserted into the socket, or goal, at location T_G^S. In this calibration configuration, frames $\{G\}$ and $\{T\}$ are coincident, and the pose of the wrist relative to the base, T_W^B, can be retrieved from the robot. Assuming T_S^B and T_G^S are known, give the transform equation to compute the unknown pose of the tool, T_T^W.



Solution: $(T_W^B)^{-1} T_S^B T_G^S$

5. (a) What is the forward kinematics problem for a robotic arm?

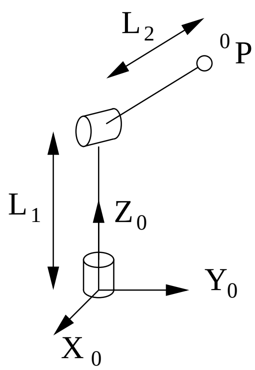
   Solution: Given the joint variables, find the pose of the end effector frame expressed in the base frame.

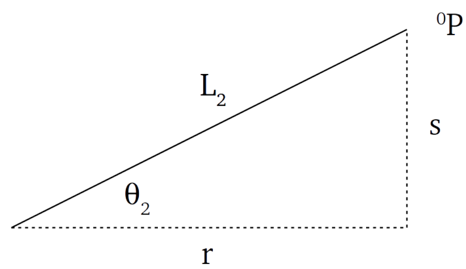
   (b) What is the inverse kinematics problem for a robotic arm?

   Solution: Given the pose of the end effector frame expressed in the base frame, find the value of the joint variables.

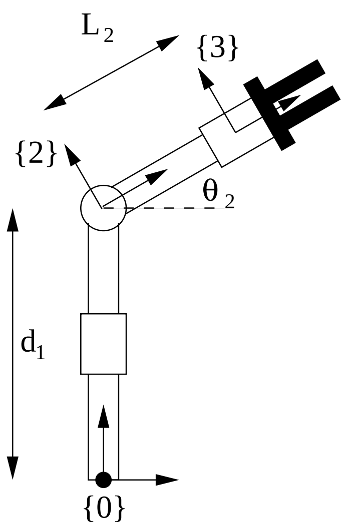
   (c) Consider the RR robot (shown below), that is similar to a robot made up of the waist and shoulder joints of the A150 robot. Given a point ${}^0P = \begin{bmatrix} x & y & z \end{bmatrix}^T$ known to be in the workspace of the robot what are the joint angles θ_1 and θ_2? Assume that θ_1 is measured from X_0 and θ_2 is measured from the horizon.

   Solution: Projecting ${}^0P$ into the X_0Y_0 plane yields $\theta_1 = \text{atan2}(y, x)$ and $\theta_2 = \text{atan2}(s, r)$ where $r = \sqrt{x^2 + y^2}$ and $s = z - L_1$
   A second possible solution is $\theta_1 = \text{atan2}(y, x) + 180$ and $\theta_2 = 180 - \text{atan2}(s, r)$





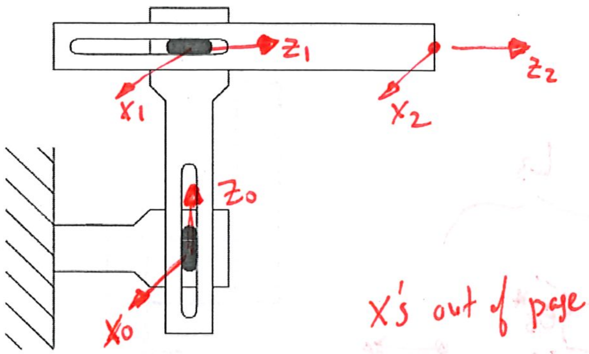
(d) Consider the robotic arm shown below made up of a prismatic joint (moving vertically) and a revolute joint (positive rotation counter-clockwise in the page). Derive the matrix T_3^0. Assume that the frames shown indicate the X and Y axes of the frames. Do not use the Denavit-Hartenberg convention to obtain a solution; the manipulator is simple enough that you should be able to derive a solution using basic linear algebra.

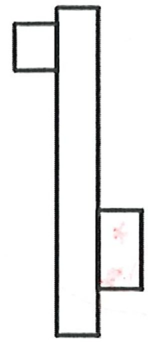


Solution: Using the moving frame convention $T_3^0 = D_{y,d_1} R_{z,\theta_2} D_{x,L_2}$

6. (a) Consider the figure shown below of a PP robot (left: view of the side of the robot, right: view of the front of the robot). Derive the forward kinematics of the robot using the DH-convention; choose your own variables for the missing dimensions.

Solution:





link	a	α	d	θ
1	0	-90	d_1	0
2	0	0	d_2	0

$$T_2^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & d_2 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) Given the end effector position (say at the end of link ~~3~~ 2), solve for the inverse kinematics of the robot.

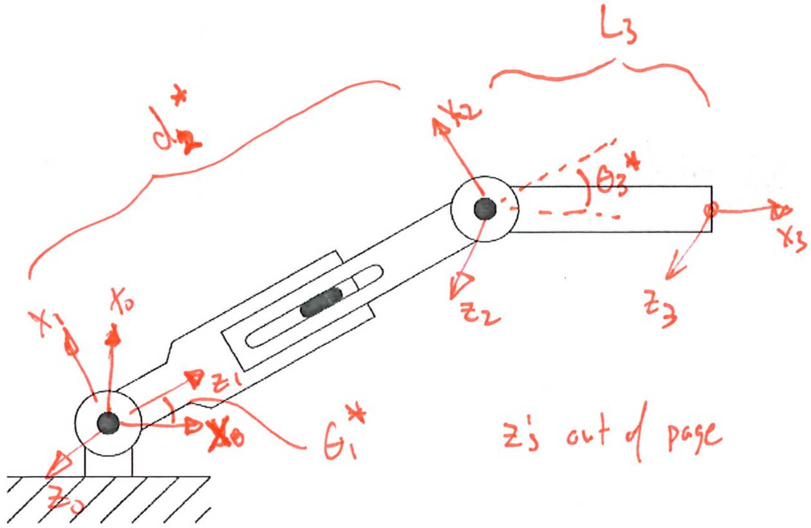
Solution: If the end-effector frame position is given by $d_2^0 = [x\ y\ z]^T$ then the solution to the inverse kinematics problem is simply $d_1 = z$ and $d_2 = y$.

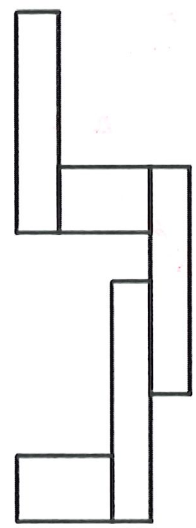
7. (a) Consider the figure shown below of a RPR robot (left: view of the side of the robot, right: view of the top of the robot). Derive the forward kinematics of the robot using the DH-convention; choose your own variables for the missing dimensions.

Solution:

link	a	α	d	θ
1	0	90	0	$\theta_1^* + 90$
2	0	-90	d_2^*	0
3	L_3	0	0	$\theta_3^* - 90$

$$T_3^0 = \begin{bmatrix} \cos(\theta_1+\theta_3) & -\sin(\theta_1+\theta_3) & 0 & d_2^*\cos(\theta_1) + L_3\cos(\theta_1+\theta_3) \\ \sin(\theta_1+\theta_3) & \cos(\theta_1+\theta_3) & 0 & d_2^*\sin(\theta_1) + L_3\sin(\theta_1+\theta_3) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$





The matrix T_3^0 should look familiar; it is identical to the result for the RR robot from Day 1 and Day 5 where the length of link 1 is replaced with d_2.

(b) Given the end effector position (say at the end of link ~~4~~ 3), solve for the inverse kinematics of the robot.

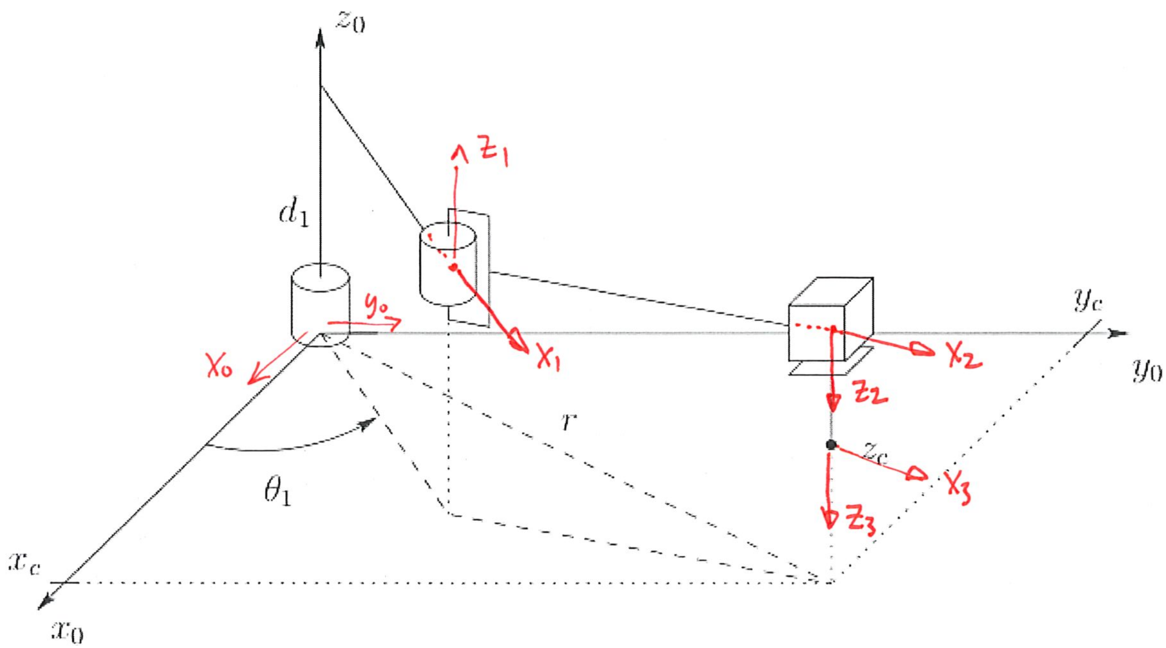
Solution: The inverse kinematic solution can be obtained using the inverse kinematics solution of the RR robot from Day 1.

8. (a) Consider the figure shown below of a SCARA robot. Derive the forward kinematics of the robot using the DH-convention; choose your own variables for the missing dimensions.

Solution:

link	a	α	d	θ
1	L_1	0	d_1	θ_1^*
2	L_2	180	0	θ_2^*
3	0	0	d_3^*	0

$$T_3^0 = \begin{bmatrix} \cos(\theta_1+\theta_2) & \sin(\theta_1+\theta_2) & 0 & L_1\cos(\theta_1)+L_2\cos(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & -\cos(\theta_1+\theta_2) & 0 & L_1\sin(\theta_1)+L_2\sin(\theta_1+\theta_2) \\ 0 & 0 & -1 & d_1-d_3^* \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



(b) Given the wrist center location $[x_c \; y_c \; z_c]^T$, solve for the inverse kinematics of the robot.

Solution: θ_1 and θ_2 can be obtained using the inverse kinematics solution for the planar RR robot (again from Day 1), and $d_3^* = d_1 - z_c$.