EECS4421Z: Introduction to Robotics
Midterm (Version 2)

Instructor: Dr. Burton Ma

Thu Feb 15, 2018

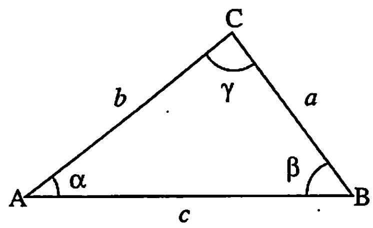
Name: ______________________________

Student Number: ______________________________

Instructions

1. You have 110 minutes to complete the exam.
2. Write your answers clearly and succintly in the space provided on the question sheets; use the back of the page and the extra blank page if you need additional space for your answer.
3. A non-programmable calculator is allowed but should not be necessary. No other aids are allowed.

Law of cosines:



$c^2 = a^2 + b^2 - 2ab\cos\gamma$

Trigonometric identities:

$\sin(-\theta) = -\sin(\theta)$
$\cos(-\theta) = \cos(\theta)$

$\sin(180 - \theta) = \sin(\theta)$
$\cos(180 - \theta) = -\cos(\theta)$

$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$\cos^2\theta + \sin^2\theta = 1$

Canonical rotation matrices:

$$R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \qquad R_{y,\theta} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \qquad R_{z,\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotation about a unit axis $[k_x \; k_y \; k_z]^T$:

$$R_{k,\theta} = \begin{bmatrix} k_x^2 v_\theta + c_\theta & k_x k_y v_\theta - k_z s_\theta & k_x k_z v_\theta + k_y s_\theta \\ k_x k_y v_\theta + k_z s_\theta & k_y^2 v_\theta + c_\theta & k_y k_z v_\theta - k_x s_\theta \\ k_x k_z v_\theta - k_y s_\theta & k_y k_z v_\theta + k_x s_\theta & k_z^2 v_\theta + c_\theta \end{bmatrix}$$

where $c_\theta = \cos\theta$, $s_\theta = \sin\theta$, and $v_\theta = 1 - \cos\theta$.

Homogeneous translation matrix for a translation of $[x \; y \; z]^T$:

$$D = \begin{bmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

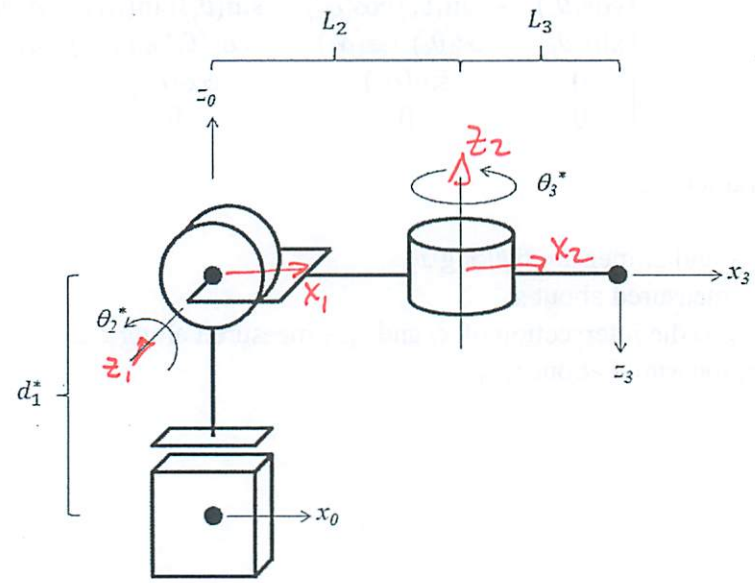
Denavit-Hartenberg transformation:

$$
\begin{aligned}
T_i^{i-1} &= R_z(\theta_i) D_z(d_i) D_x(a_i) R_x(\alpha_i) \\
&= \begin{bmatrix} \cos(\theta_i) & -\sin(\theta_i)\cos(\alpha_i) & \sin(\theta_i)\sin(\alpha_i) & a_i\cos(\theta_i) \\ \sin(\theta_i) & \cos(\theta_i)\cos(\alpha_i) & -\cos(\theta_i)\sin(\alpha_i) & a_i\sin(\theta_i) \\ 0 & \sin(\alpha_i) & \cos(\alpha_i) & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

Denavit-Hartenberg parameters:

a_i: distance between z_{i-1} and z_i measured along x_i
α_i: angle from z_{i-1} to z_i measured about x_i
d_i: distance between o_{i-1} to the intersection of x_i and z_{i-1} measured along z_{i-1}
θ_i: angle from x_{i-1} to x_i measured about z_{i-1}

1. Consider the following RRR robot:



The robot is shown in a position where all of the joint angles are zero degrees. For the purposes of this question, assume that there are no limits on the joint angles and that $d_1^* > 0$.

(a) (2 points) Draw coordinate frames $\{1\}$ and $\{2\}$ that are suitable for use with the Denavit-Hartenberg convention.

(b) (6 points) Fill in the Denavit-Hartenberg parameters using your coodinate frame placement.

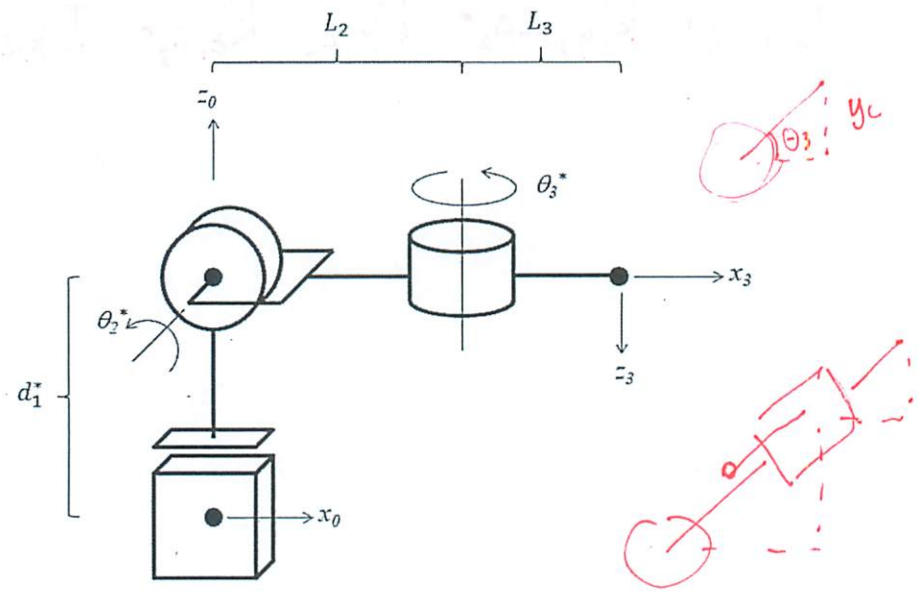
Link	a_i	α_i	d_i	θ_i
1	0	90 (−90)	d_1^*	0 (180)
2	L_2	−90 (90)	0	θ_2^* ($180 + \theta_2^*$)
3	L_3	180 (0)	0	θ_3^* ($180 - \theta_3^*$)

(c) (12 points) Solve for the forward kinematics of the arm; that is, given joint angles d_1^*, θ_2^*, and θ_3^*, find the orientation and position of frame $\{3\}$ as a homogeneous matrix T_3^0. If your solution involves a sequence of matrix multiplications then you do not need to perform the actual multiplications. Do not use the Denavit-Hartenberg convention to obtain your answer.

Use this page for Question 1.

$D_{z, d_1^*} \; R_{y, -\theta_2^*} \; D_{x, L_2} \; R_{z, \theta_3^*} \; D_{x, L_3} \; R_{x, 180}$

2. Consider the PRR arm shown in Figure 2.



The robot is shown in a position where all of the joint angles are zero degrees.

Joint 1 has a range of $d_1^* >= 0$.
Joint 2 has a range of $0 <= \theta_2^* < 180$ degrees.
Joint 3 has a range of $0 <= \theta_3^* < 360$ degrees.
You may assume that the link dimensions L_2, and L_3 are always greater than zero, and that $L_2 > L_3$.

(a) (22 points) Given $o_c = [x_c\ y_c\ z_c]^T$, the location of the origin of frame $\{3\}$ relative to frame $\{0\}$, solve for the inverse kinematics of arm.

(b) (3 points) Are there any reachable points o_c that have more than one solution for the inverse kinematics problem? If you answer "yes" provide an example of the multiple solutions. If you answer "no" briefly explain why every reachable point has a unique solution.

Use this page for Question 2 if necessary.

(a) y_c depends only on θ_3^*

$$y_c = L_3 \sin \theta_3^*$$

$$\theta_3^* = \begin{cases} \operatorname{asin}\left(\frac{y_c}{L_3}\right) \text{ or } 180 - \operatorname{asin}\left(\frac{y_c}{L_3}\right) & \text{if } y_c \geq 0 \\ 180 - \operatorname{asin}\left(\frac{y_c}{L_3}\right) \text{ or } 360 + \operatorname{asin}\left(\frac{y_c}{L_3}\right) & \text{if } y_c < 0 \end{cases}$$

(x_c, z_c) are the coordinates on a circle of radius $r = (L_2 + L_3 \cos\theta_3^*)$ centered at $(0, d_3^*)$

$$x_c = (L_2 + L_3 \cos\theta_3^*) \cos\theta_2^*$$

$$\theta_2^* = \operatorname{acos}\left(\frac{x_c}{L_2 + \underline{L_3 \cos\theta_3^*}}\right)$$

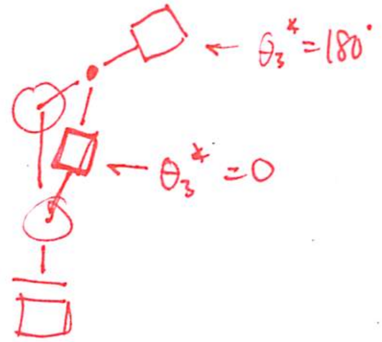
(acos is ok here because $0 \leq \theta_2^* < 180$)

may have 2 solutions for θ_2^*

$$z_c = d_3^* + (L_2 + L_3 \cos\theta_3^*) \sin\theta_2^*$$

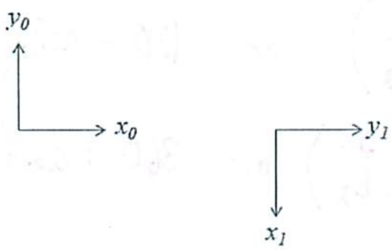
$$d_3^* = z_c - \underbrace{(L_2 + L_3 \cos\theta_3^*) \sin\theta_2^*}_{\text{may have 2 solutions for } d_3^*}$$

(b) Yes. Consider $\theta_3^* = 0$ or $180°$



3. (5 points) Suppose that you have two coordinate frames $\{0\}$ and $\{1\}$ and you know T_1^0 the pose of frame $\{1\}$ relative to $\{0\}$. Suppose that you have some rigid transformation U_0^0 defined in frame $\{0\}$; what is U_0^0 expressed in frame $\{1\}$?

For example, consider the two frames shown below:



Here, $T_1^0 = \begin{bmatrix} 0 & 1 & 4 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (note: T_1^0 transforms 2D homogeneous points $\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$). Suppose U_0^0 is the translation $U_0^0 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Then U_0^0 expressed in frame $\{1\}$ is $U_1^1 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

Hint: U_0^0 transforms a point p^0 in frame $\{0\}$ to a new point q^0 in frame $\{0\}$ such that $q^0 = U_0^0 p^0$.

$$q^0 = U_0^0 p^0 \quad (1)$$

$$q^1 = U_1^1 p^1 \quad (2)$$

$$p^0 = T_1^0 p^1 \quad \text{subs. into } (1)$$

$$q^0 = U_0^0 T_1^0 p^1 = T_1^0 q^1$$

$$\therefore \; q^1 = \underbrace{(T_1^0)^{-1} U_0^0 T_1^0}_{U_1^1} p^1 \quad \text{equate to } (2)$$

Use this page for Question 3 if necessary.