Probability Review

Why Probabilistic Robotics?

- autonomous mobile robots need to accommodate the uncertainty that exists in the physical world
- sources of uncertainty
  - environment
  - sensors
  - actuation
  - software
  - algorithmic
- probabilistic robotics attempts to represent uncertainty using the calculus of probability theory

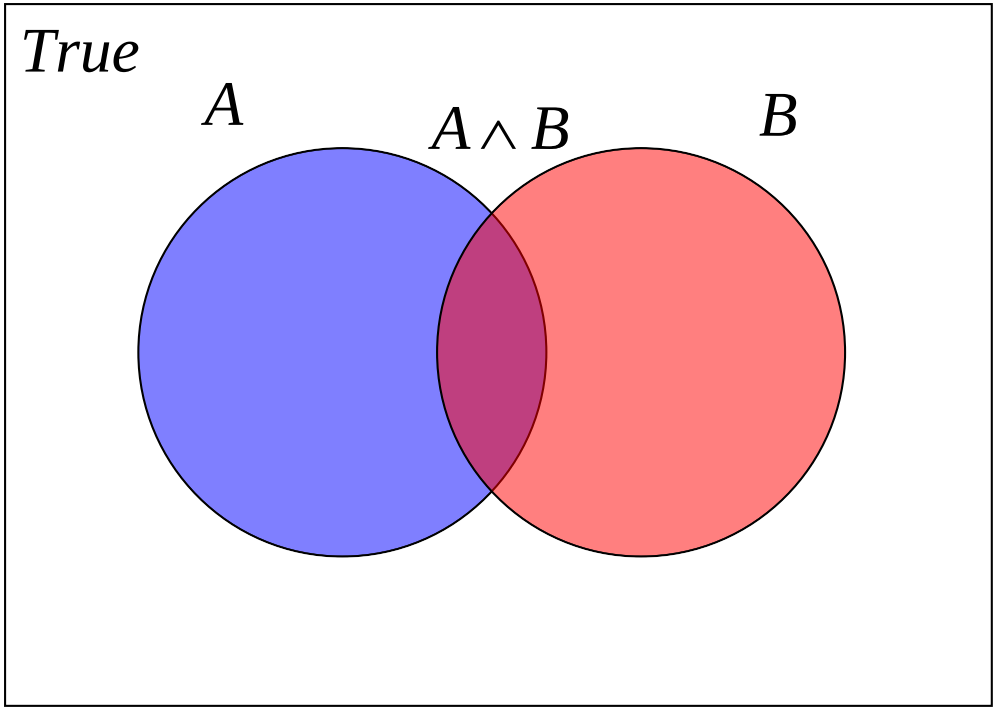
Axioms of Probability Theory

Pr*(A)* denotes probability that proposition *A* is true.

- $$0 \le \Pr(A) \le 1$$
- $$\Pr(True) = 1 \qquad \Pr(False) = 0$$
- $$\Pr(A \vee B) = \Pr(A) + \Pr(B) - \Pr(A \wedge B)$$

A Closer Look at Axiom 3

$$\Pr(A \vee B) = \Pr(A) + \Pr(B) - \Pr(A \wedge B)$$



Using the Axioms

$$\begin{aligned}
\Pr(A \vee \neg A) &= \Pr(A) + \Pr(\neg A) - \Pr(A \wedge \neg A) \\
\Pr(True) &= \Pr(A) + \Pr(\neg A) - \Pr(False) \\
1 &= \Pr(A) + \Pr(\neg A) - 0 \\
\Pr(\neg A) &= 1 - \Pr(A)
\end{aligned}$$

Discrete Random Variables

- X denotes a random variable.
- X can take on a countable number of values in $\{x_1, x_2, \ldots, x_n\}$.
- $P(X=x_i)$, or $P(x_i)$, is the probability that the random variable X takes on value x_i.
- $P(\cdot)$ is called probability mass function.

Discrete Random Variables

- fair coin

$$P(X=\text{heads}) = P(X=\text{tails}) = 1/2$$

- fair dice

$$P(X=1) = P(X=2) = P(X=3) = P(X=4) = P(X=5) = P(X=6) = 1/6$$

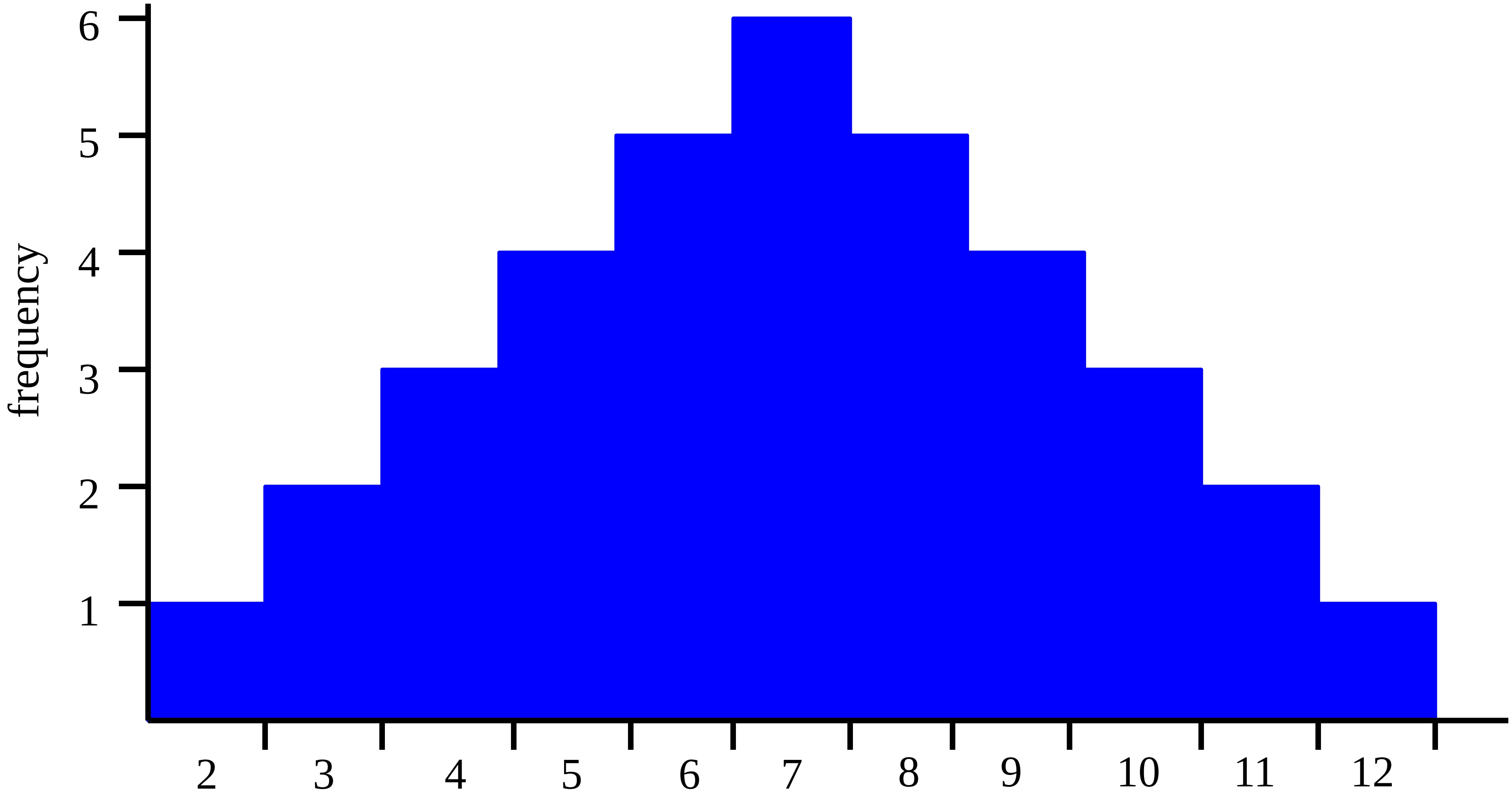
Discrete Random Variables

- sum of two fair dice

P(X=2)	(1,1)	1/36
P(X=3)	(1,2), (2,3)	2/36
P(X=4)	(1,3), (2,2), (3,1)	3/36
P(X=5)	(1,4), (2,3), (3,2), (4,1)	4/36
P(X=6)	(1,5), (2,4), (3,3), (4,2), (5,1)	5/36
P(X=7)	(1,6), (2,5), (3,4), (4,3), (5,2), (6, 1)	6/36
P(X=8)	(2, 6), (3, 5), (4,4), (5,3), (6, 2)	5/36
P(X=9)	(3, 6), (4, 5), (5, 4), (6, 3)	4/36
P(X=10)	(4, 6), (5, 5), (6, 4)	3/36
P(X=11)	(5, 6), (6, 5)	2/36
P(X=12)	(6, 6)	1/36

Discrete Random Variables

- plotting the frequency of each possible value yields the histogram

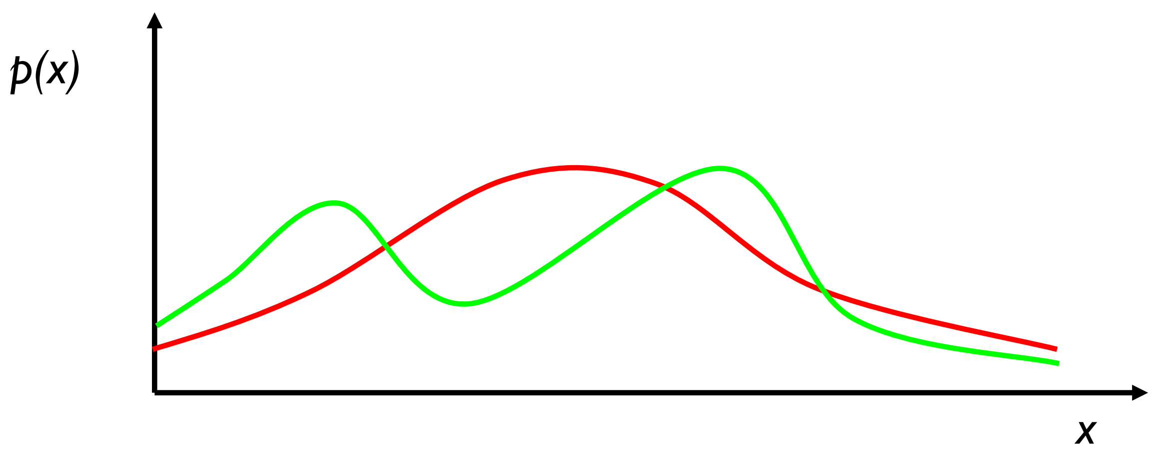


Continuous Random Variables

- X takes on values in the continuum.
- $p(X=x)$, or $p(x)$, is a probability density function.

$$\Pr(x \in (a,b)) = \int_a^b p(x)dx$$

- E.g.



Continuous Random Variables

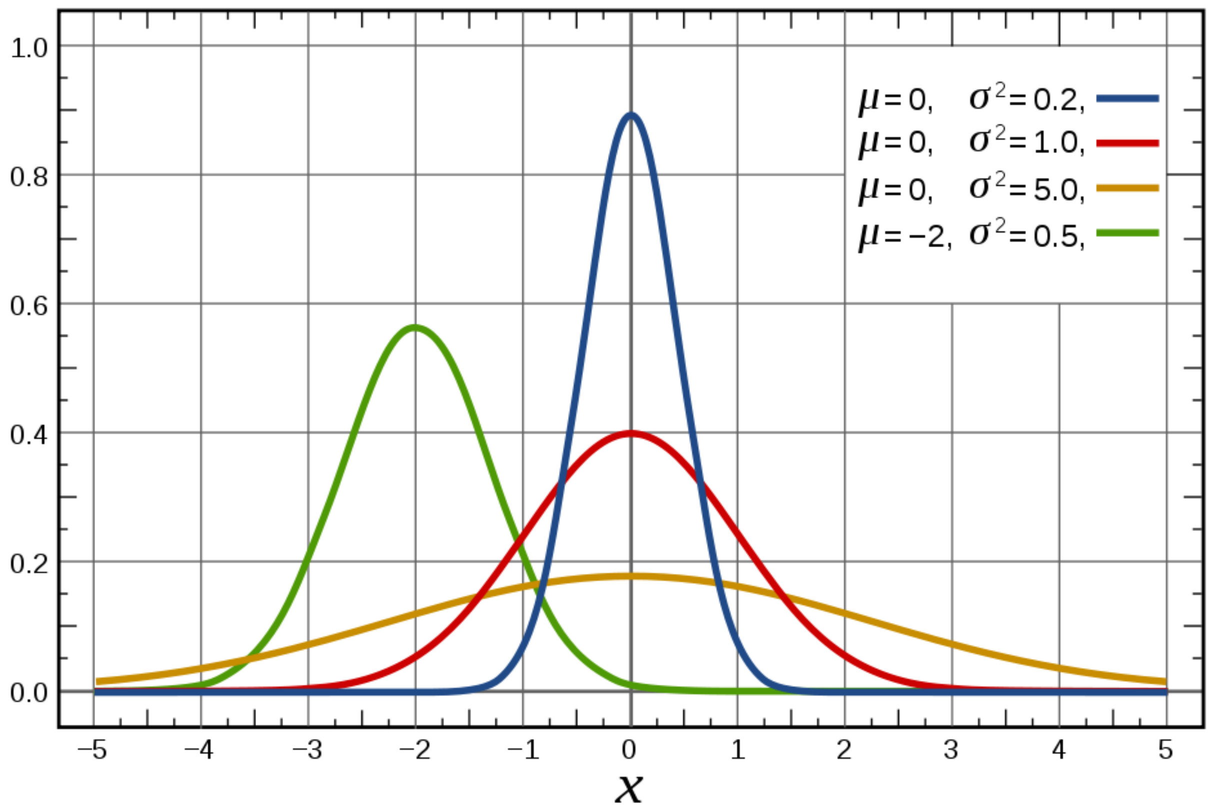
- unlike probabilities and probability mass functions, a probability density function can take on values greater than 1
  - e.g., uniform distribution over the range [0, 0.1]
- however, it is the case that

$$\int_{-\infty}^{\infty} p(x)dx = 1$$

Continuous Random Variables

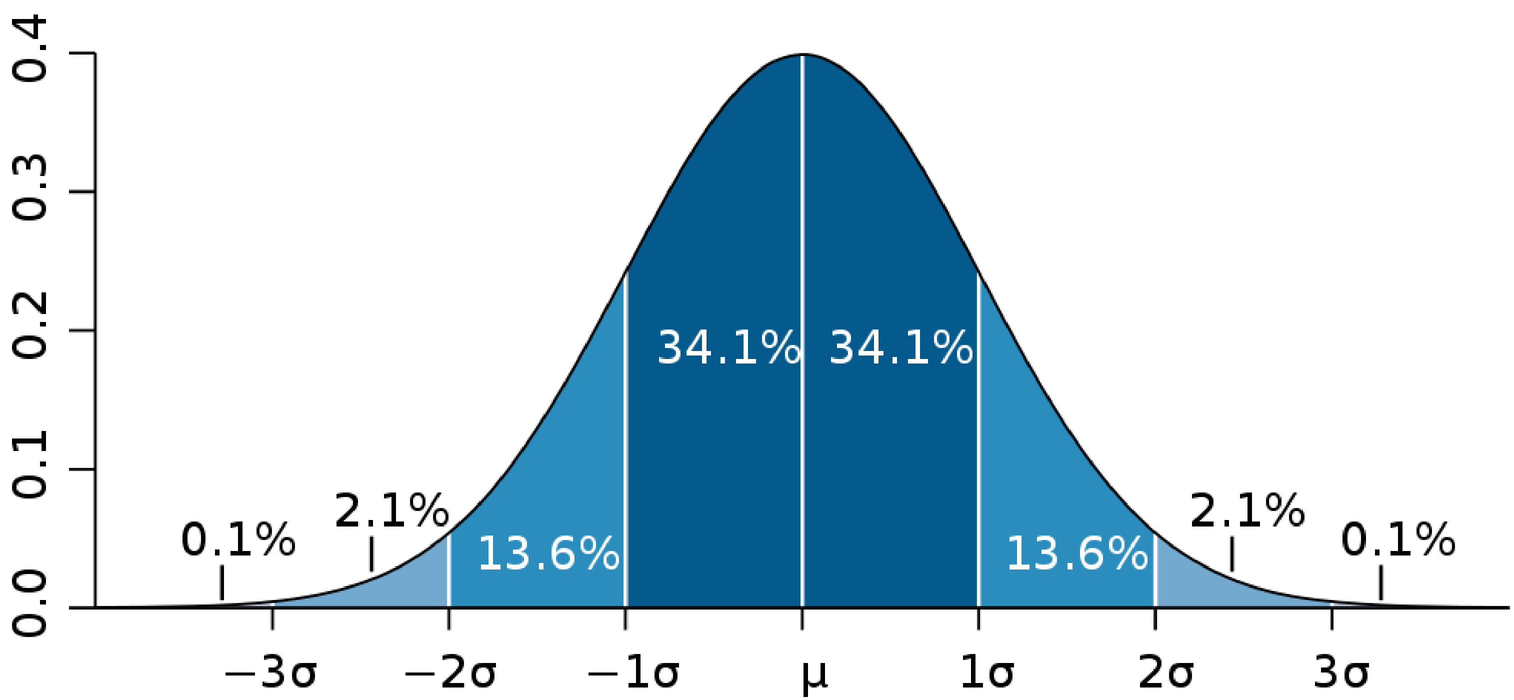
- normal or Gaussian distribution in 1D

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$



Continuous Random Variables

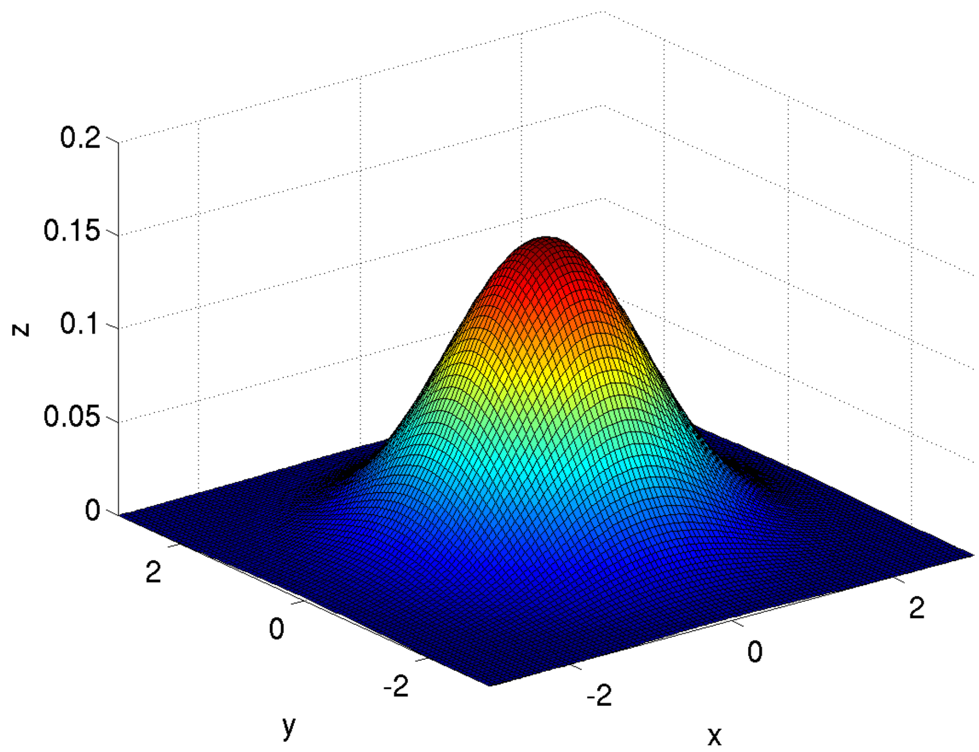
- $1D$ normal, or Gaussian, distribution
  - μ mean
  - σ standard deviation
  - $\Sigma = \sigma^2$ variance



Continuous Random Variables

- $2D$ normal, or Gaussian, distribution
  - μ mean
  - Σ covariance matrix

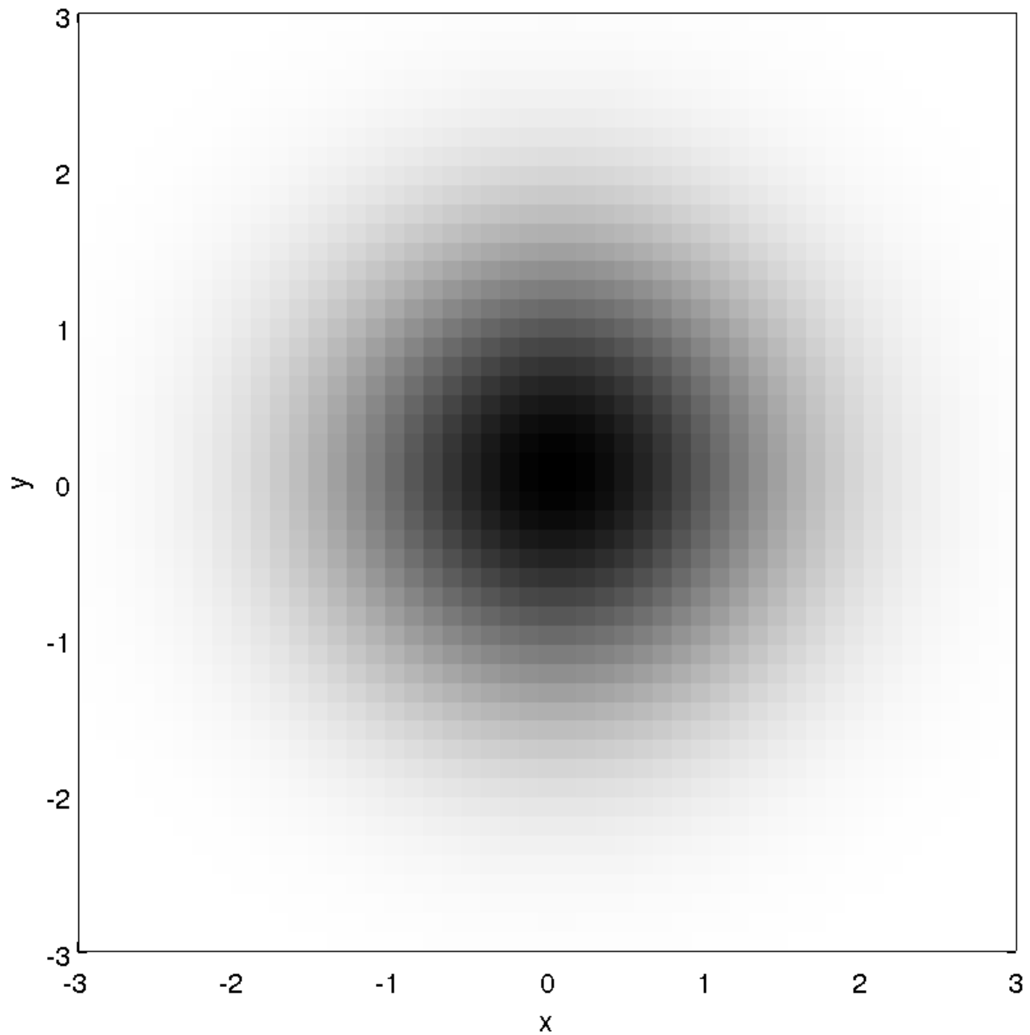
$$p(x) = \frac{1}{\sqrt{\det(2\pi\Sigma)}} e^{-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)}$$



Continuous Random Variables

- in $2D$
  - isotropic

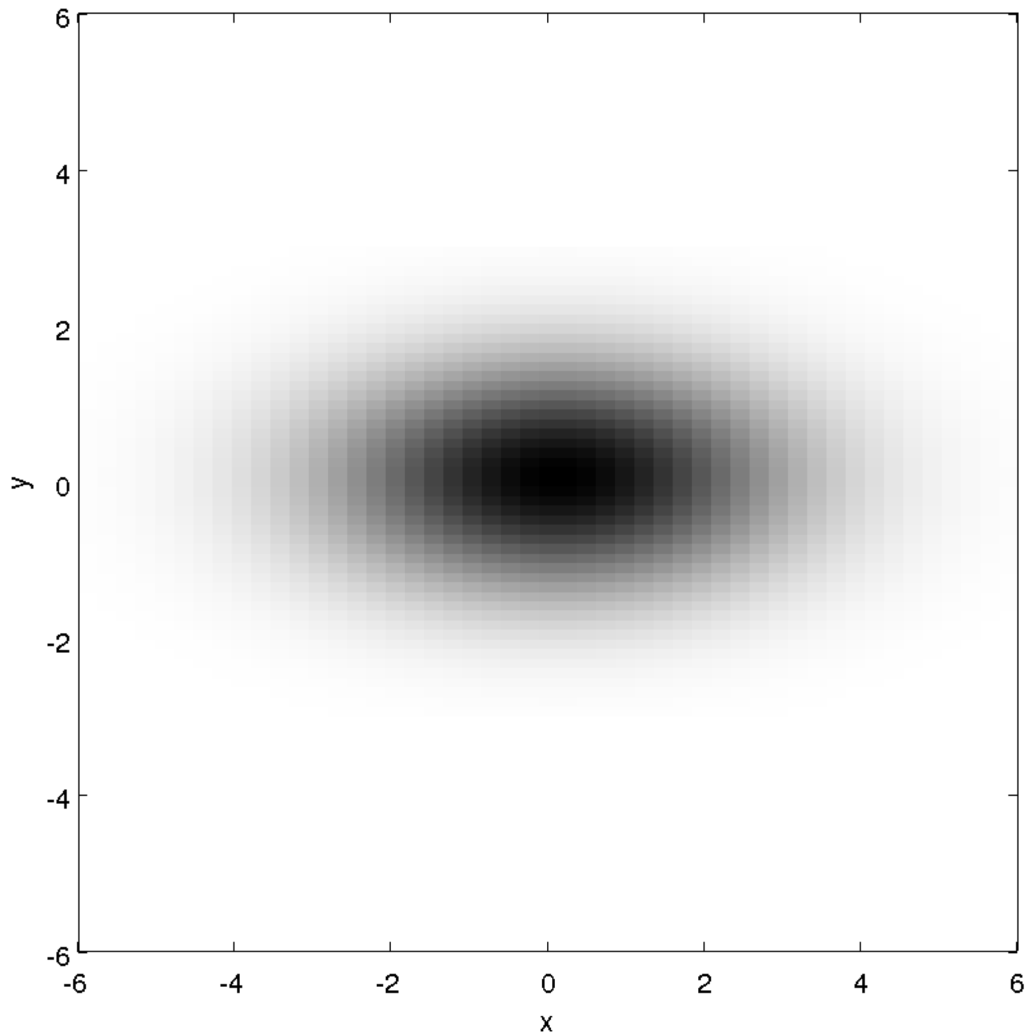
$$\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



Continuous Random Variables

- in $2D$
  - anisotropic

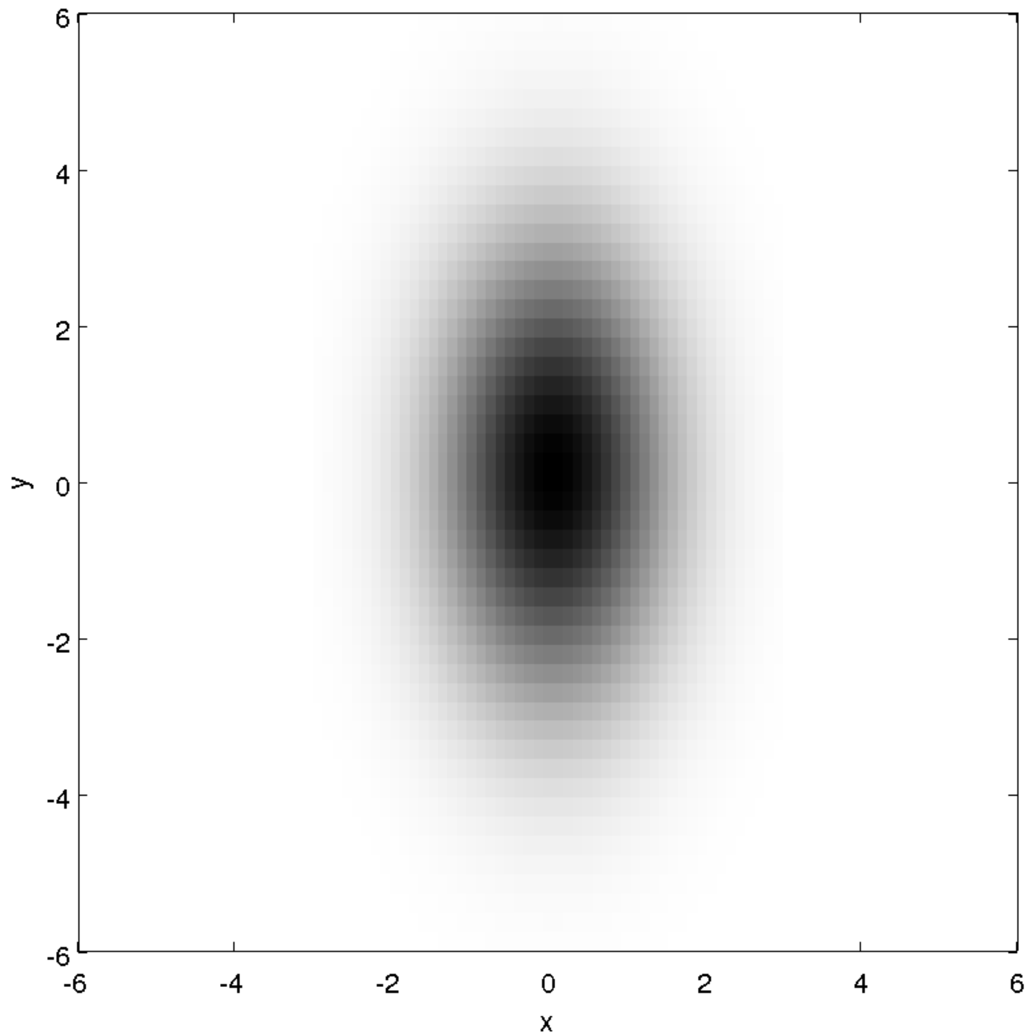
$$\Sigma = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$$



Continuous Random Variables

- in $2D$
  - anisotropic

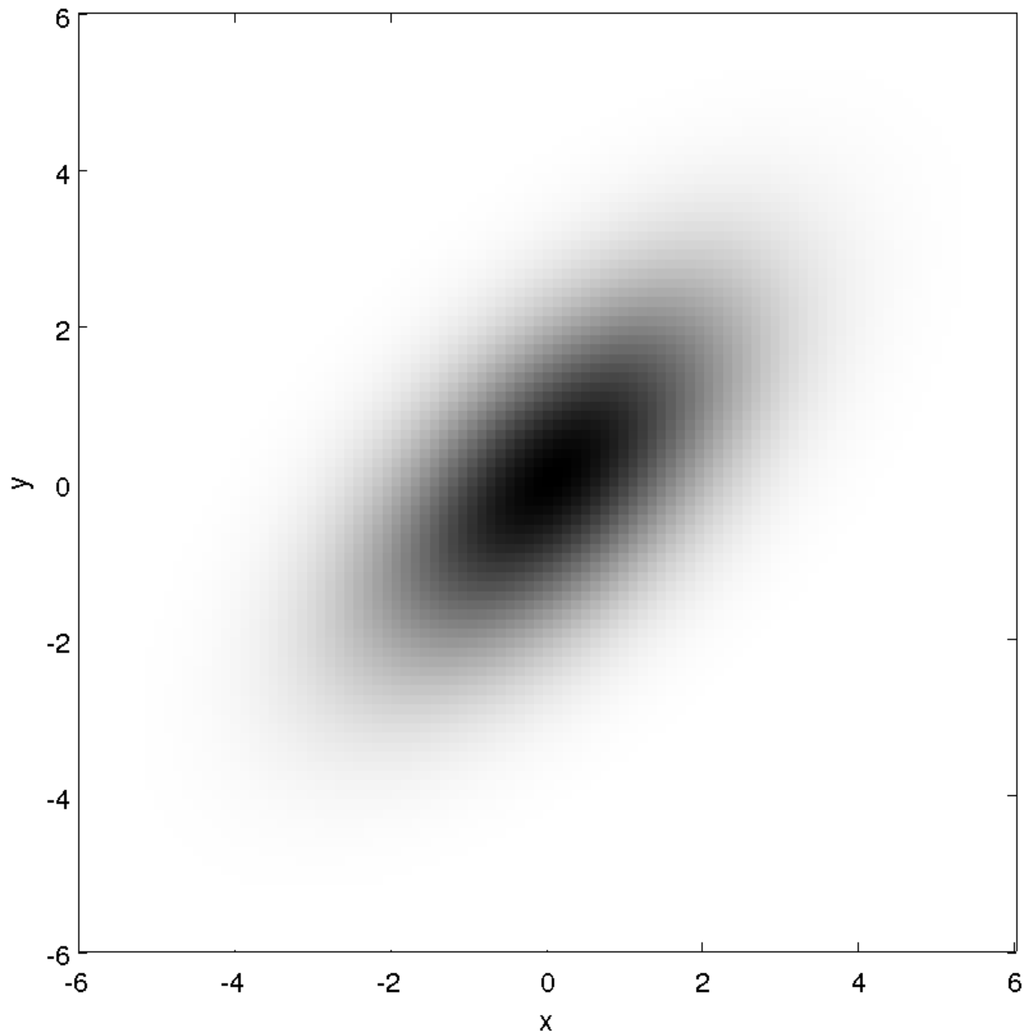
$$\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$



Continuous Random Variables

- in $2D$
  - anisotropic

$$\Sigma = \begin{bmatrix} 2.5 & 1.5 \\ 1.5 & 2.5 \end{bmatrix}$$



Covariance matrices

- the covariance matrix is always symmetric and positive semi-definite
- positive semi-definite:

$$x^T \Sigma x \geq 0 \text{ for all } x$$

- positive semi-definiteness guarantees that the eigenvalues of Σ are all greater than or equal to 0

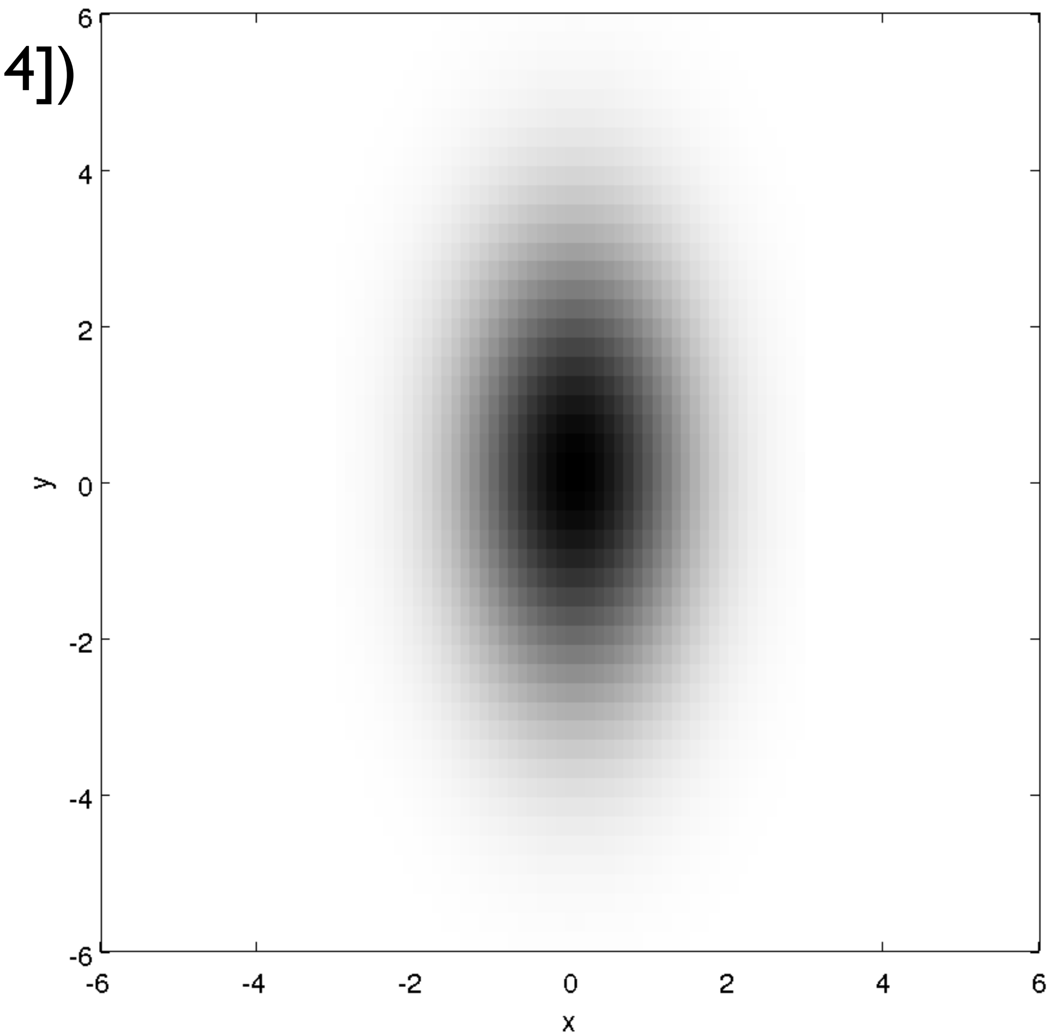
```
>> [v, d] = eig([1 0; 0 4])

v =

    1    0
    0    1


d =

    1    0
    0    4
```



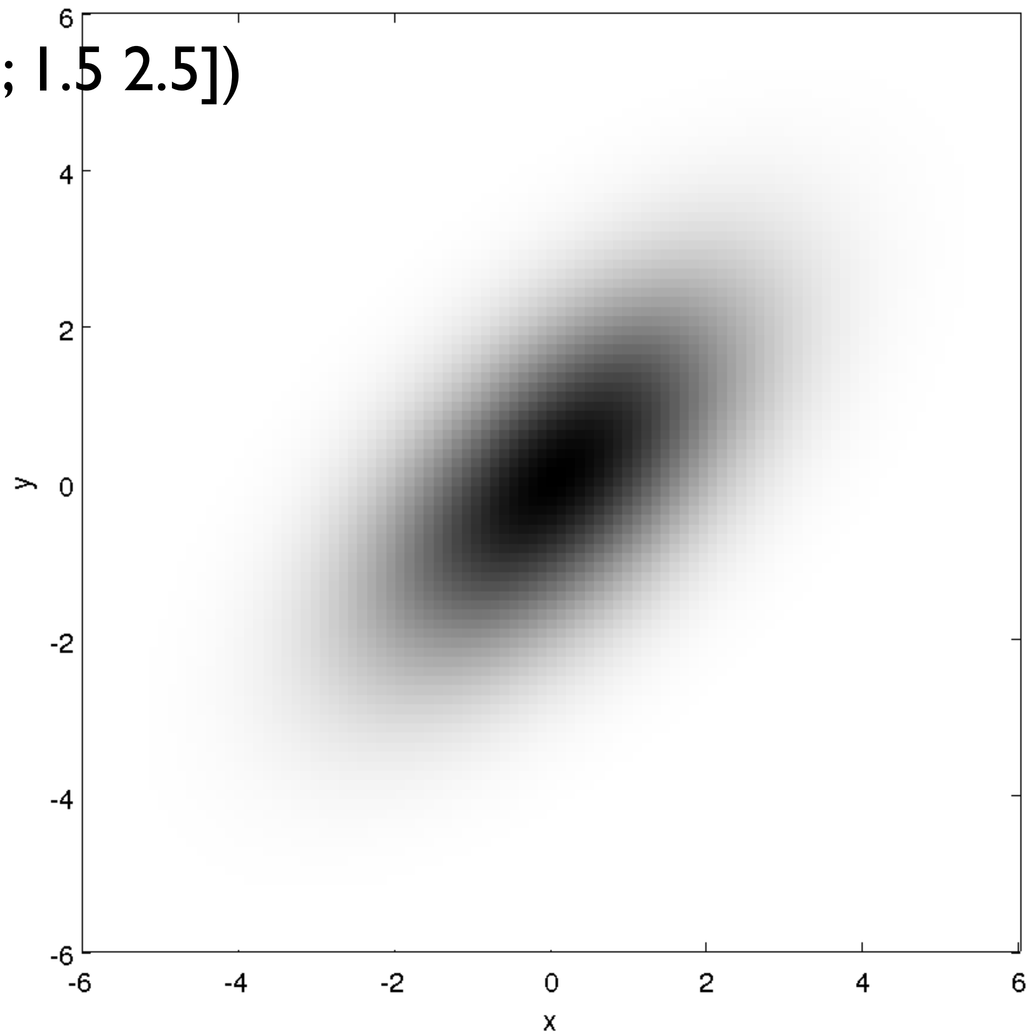
```
>> [v, d] = eig([2.5 1.5; 1.5 2.5])

v =

   -0.7071    0.7071
    0.7071    0.7071


d =

     1     0
     0     4
```



Joint Probability

- the joint probability distribution of two random variables

$$P(X{=}x \text{ and } Y{=}y) = P(x,y)$$

describes the probability of the event that X has the value x and Y has the value y

- If X and Y are independent then

$$P(x,y) = P(x)\, P(y)$$

Joint Probability

- the joint probability distribution of two random variables

$$P(X\!=\!x \text{ and } Y\!=\!y) = P(x,y)$$

  describes the probability of the event that X has the value x and Y has the value y

- example: two fair dice

$$P(X\!=\!\text{even and } Y\!=\!\text{even}) = 9/36$$

$$P(X\!=\!1 \text{ and } Y\!=\!\text{not } 1) = 5/36$$

Joint Probability

- example: insurance policy deductibles

y

		$0	**$100**	**$200**
x	**$100**	0.20	0.10	0.20
	$250	0.05	0.15	0.30

home

automobile

Joint Probability and Independence

- X and Y are said to be independent if

$$P(x,y) = P(x)\, P(y)$$

  for all possible values of x and y

- example: two fair dice

$$P(X\text{=even and }Y\text{=even}) = (1/2)\,(1/2)$$
$$P(X\text{=1 and }Y\text{=not 1}) = (1/6)\,(5/6)$$

- are X and Y independent in the insurance deductible example?

Marginal Probabilities

- the marginal probability distribution of X

$$P_X(x) = \sum_y P(x, y)$$

describes the probability of the event that X has the value x

- similarly, the marginal probability distribution of Y

$$P_Y(y) = \sum_x P(x, y)$$

describes the probability of the event that Y has the value y

Joint Probability

- example: insurance policy deductibles

y

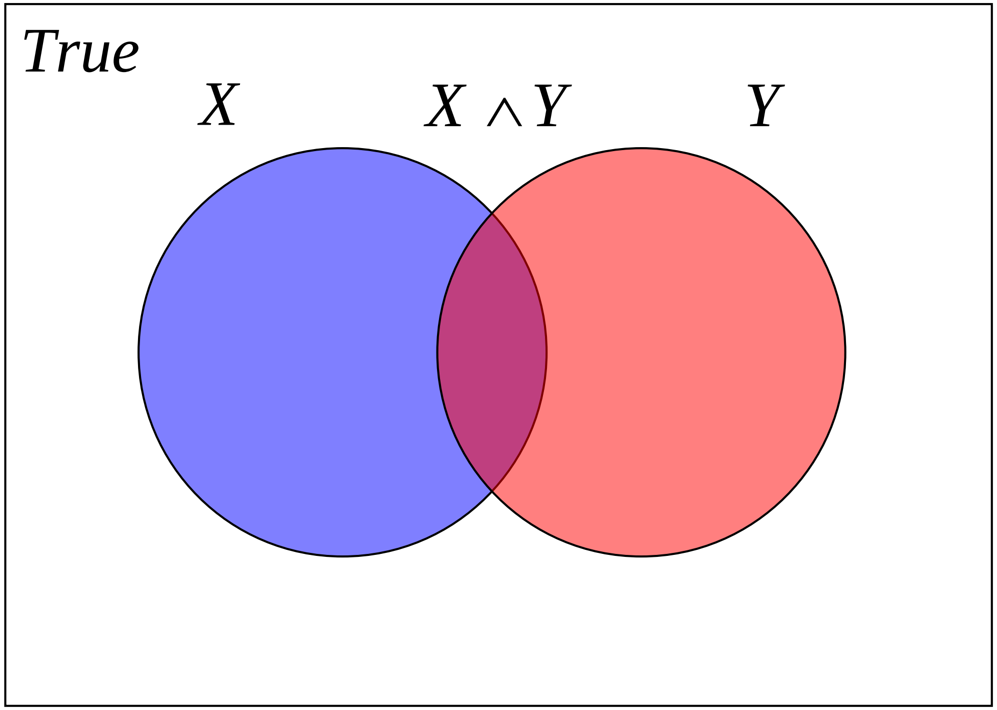
x		\$0	\$100	\$200
	\$100	0.20	0.10	0.20
	\$250	0.05	0.15	0.30

home

automobile

Conditional Probability

- the conditional probability $P(x \mid y) = P(X=x \mid Y=y)$ is the probability of $P(X=x)$ if $Y=y$ is known to be true
  - "conditional probability of x given y"



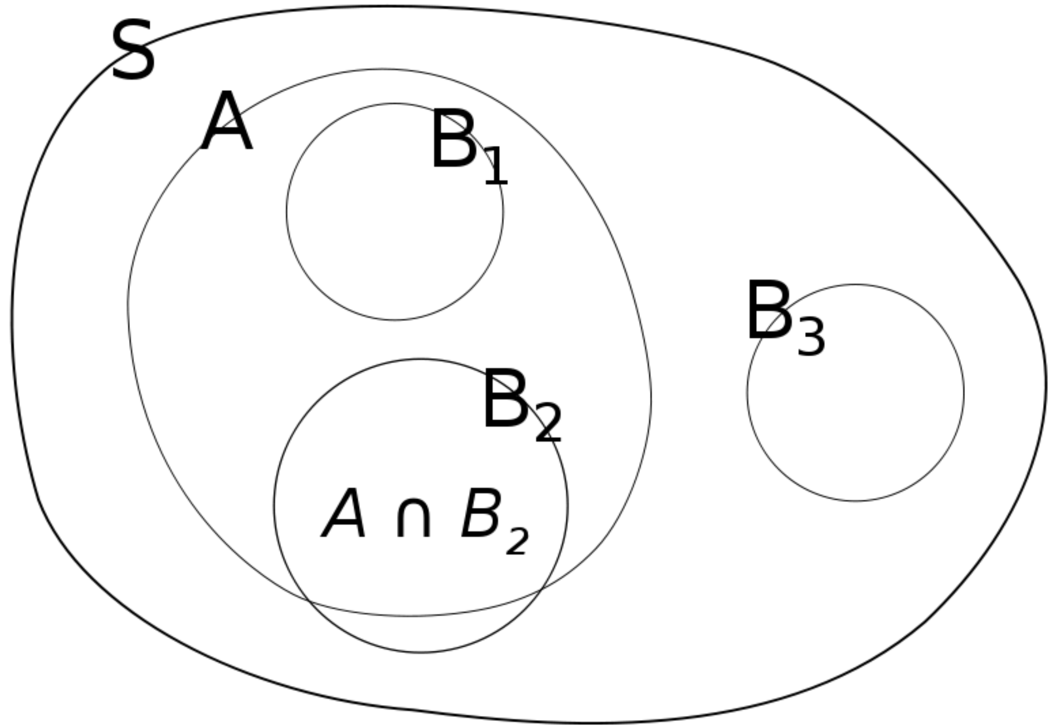
Conditional Probability

$P(A) \approx 0.3$

$P(A \mid B_3) = ?$

$P(A \mid B_1) = ?$

$P(A \mid B_2) = ?$



Conditional Probability

- “information changes probabilities”
- example:
  - roll a fair die; what is the probability that the number is a 3?
  - what is the probability that the number is a 3 if someone tells you that the number is odd? is even?

Conditional Probability

- "information changes probabilities"
- example:
  - pick a playing card from a standard deck; what is the probability that it is the ace of hearts?
  - what is the probability that it is the ace of hearts if someone tells you that it is an ace? that is a heart? that it is a king?

Conditional Probability

$$P(x \mid y) = \frac{P(x, y)}{P(y)}$$

- if X and Y are independent then

$$P(x, y) = P(x)P(y)$$

$$\therefore P(x \mid y) = \frac{P(x)P(y)}{P(y)} = P(x)$$

Bayes Formula

$$P(x,y) = P(x \mid y)P(y) = P(y \mid x)P(x)$$

$$\Rightarrow$$

$$\boxed{\underbrace{P(x \mid y)}_{\text{posterior}} = \frac{P(y \mid x)\ P(x)}{P(y)} = \frac{\text{likelihood} \cdot \text{prior}}{\text{evidence}}}$$

Bayes Rule with Background Knowledge

$$P(x \mid y, z) = \frac{P(y \mid x, z)\ P(x \mid z)}{P(y \mid z)}$$

Back to Kinematics

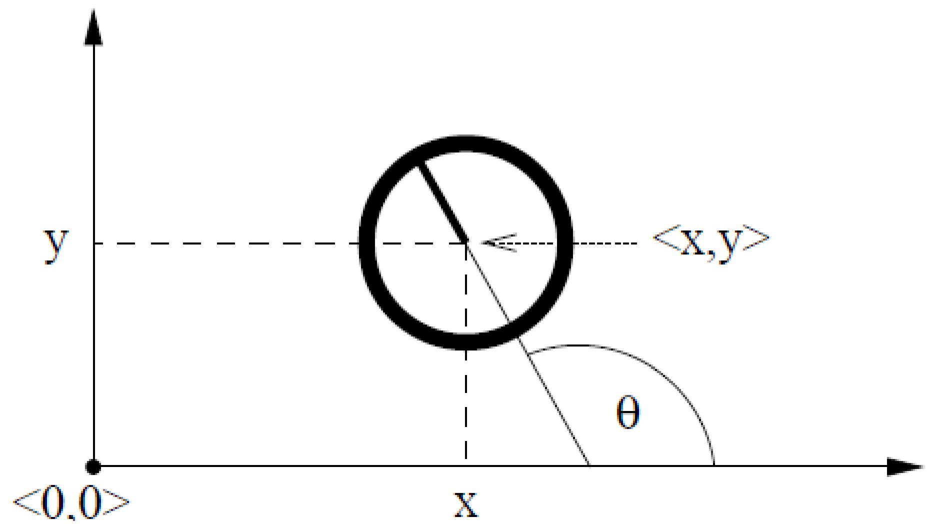


Figure 5.1 Robot pose, shown in a global coordinate system.

$$\text{pose vector or state}\quad x_t = \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \end{matrix}\right\} \text{location (in world frame)} \\ \left.\begin{matrix} \\ \end{matrix}\right\} \text{bearing or heading} \end{matrix}$$

Probabilistic Robotics

- we seek the conditional density

$$p(x_t \mid u_t, x_{t-1})$$

- what is the density of the state

$$x_t$$

given the motion command

$$u_t$$

performed at

$$x_{t-1}$$

Probabilistic Robotics

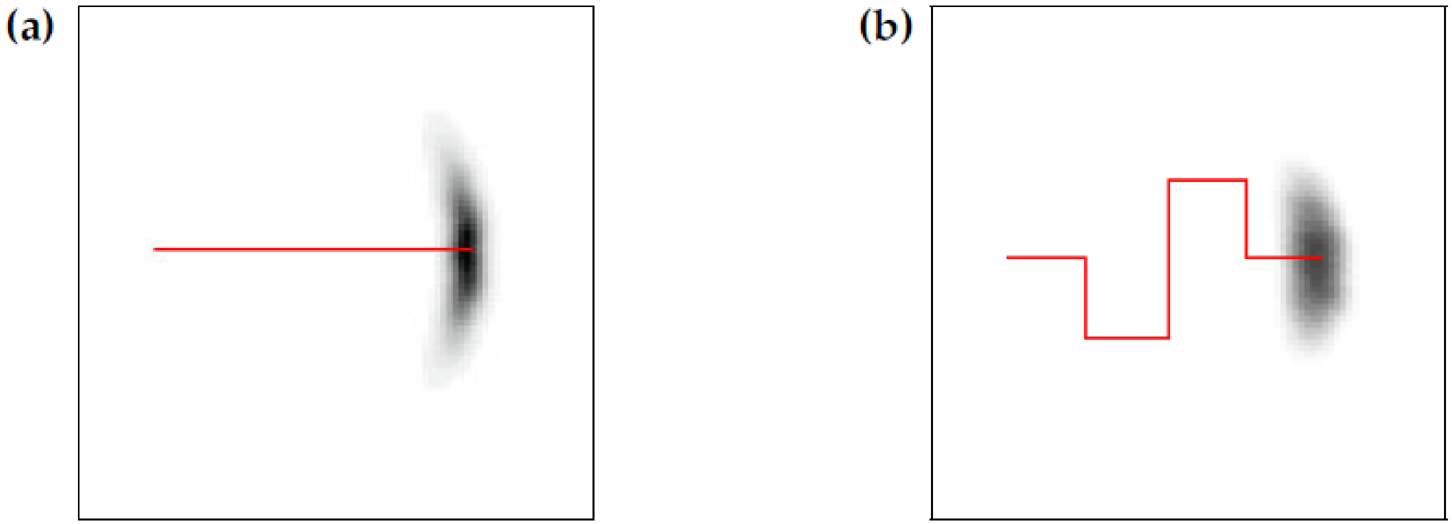


Figure 5.2 The motion model: Posterior distributions of the robot's pose upon executing the motion command illustrated by the solid line. The darker a location, the more likely it is. This plot has been projected into 2-D. The original density is three-dimensional, taking the robot's heading direction θ into account.

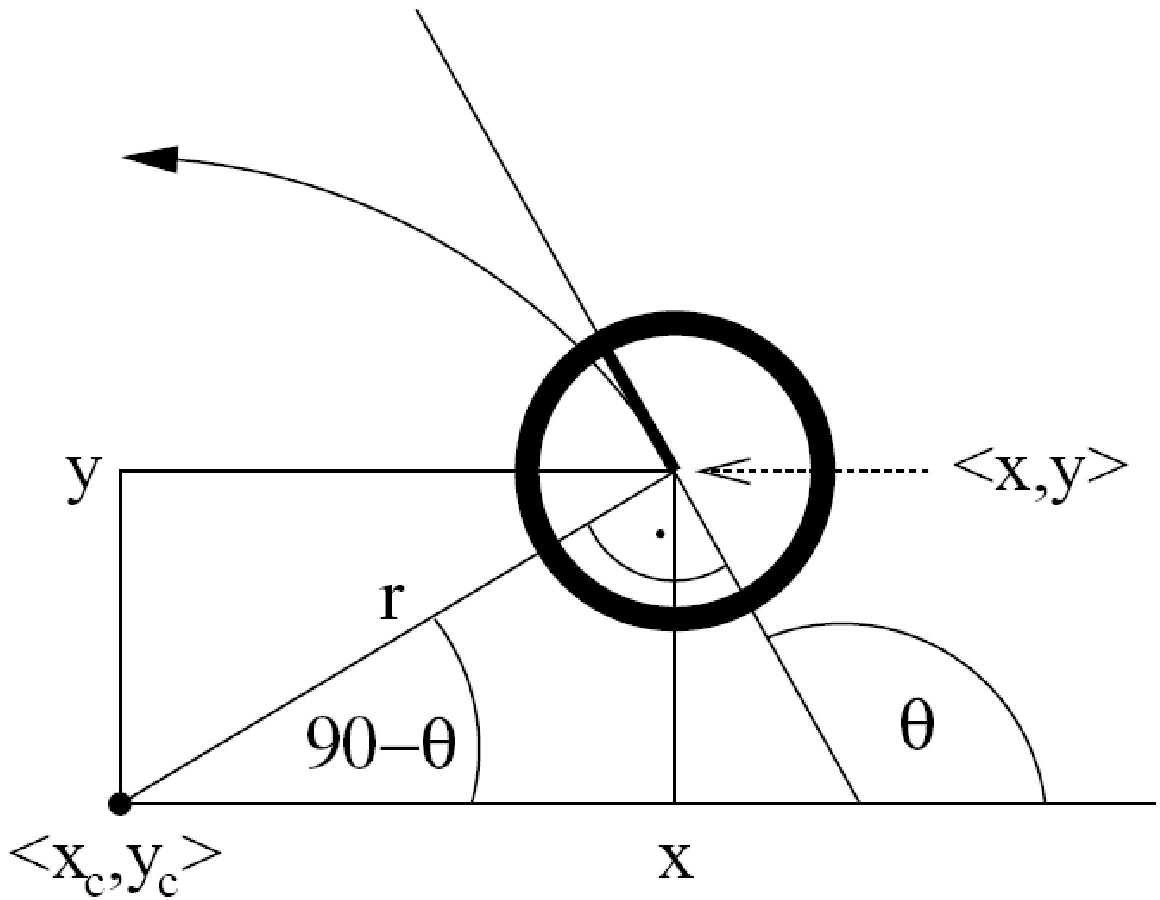
Velocity Motion Model

- assumes the robot can be controlled through two velocities
  - translational velocity v
  - rotational velocity ω
- our motion command, or control vector, is

$$u_t = \begin{pmatrix} v_t \\ \omega_t \end{pmatrix}$$

- positive values correspond to forward translation and counterclockwise rotation

Velocity Motion Model



Velocity Motion Model

- center of circle

$$\begin{pmatrix} x^* \\ y^* \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -\lambda \sin\theta \\ \lambda \cos\theta \end{pmatrix} = \begin{pmatrix} \frac{x+x'}{2} + \mu(y-y') \\ \frac{y+y'}{2} + \mu(x'-x) \end{pmatrix}$$

where

$$\mu = \frac{1}{2} \frac{(x-x')\cos\theta + (y-y')\sin\theta}{(y-y')\cos\theta - (x-x')\sin\theta}$$

Velocity Motion Model

Algorithm motion_model_velocity(x_t, u_t, x_{t-1})**:**

$$\mu = \frac{1}{2} \frac{(x - x')\cos\theta + (y - y')\sin\theta}{(y - y')\cos\theta - (x - x')\sin\theta}$$

$$x^* = \frac{x + x'}{2} + \mu(y - y')$$

$$y^* = \frac{y + y'}{2} + \mu(x' - x)$$

$$r^* = \sqrt{(x - x^*)^2 + (y - y^*)^2}$$

$$\Delta\theta = \mathrm{atan2}(y' - y^*, x' - x^*) - \mathrm{atan2}(y - y^*, x - x^*)$$

$$\hat{v} = \frac{\Delta\theta}{\Delta t}\, r^*$$

$$\hat{\omega} = \frac{\Delta\theta}{\Delta t}$$

$$\hat{\gamma} = \tfrac{\theta' - \theta}{\Delta t} - \hat{\omega}$$

return $\mathbf{prob}(v - \hat{v}, \alpha_1|v| + \alpha_2|\omega|) \;\cdot\; \mathbf{prob}(\omega - \hat{\omega}, \alpha_3|v| + \alpha_4|\omega|)$
$\cdot\, \mathbf{prob}(\hat{\gamma}, \alpha_5|v| + \alpha_6|\omega|)$