EECS4221/5324: Lab 1 Written Questions

Posted: Sun Jan 14, 2018
Due: in class Wed Jan 25, 2018

1. Find the 4×4 homogeneous transformation matrix (showing the numeric values for all 16 elements) T_1^0 where:

   (a) $\{1\}$ has the same orientation as $\{0\}$ and the origin of $\{1\}$ is translated relative to the origin of $\{0\}$ by $d_1^0 = \begin{bmatrix} 5 & -5 & 10 \end{bmatrix}^T$.

$$T_1^0 = \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1}$$

   (b) The origin of $\{1\}$ is coincident with the origin of $\{0\}$, and $\hat{x}_1^0 = -\hat{z}_0^0$, $\hat{y}_1^0 = -\hat{x}_0^0$, and $\hat{z}_1^0 = \hat{y}_0^0$.

$$T_1^0 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2}$$

   (c) The origin of $\{1\}$ is translated relative to the origin of $\{0\}$ by $d_1^0 = \begin{bmatrix} 0 & 0 & -10 \end{bmatrix}^T$, and the orientation of $\{1\}$ relative to $\{0\}$ is the same as in part (b).

$$T_1^0 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & -10 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3}$$

   (d) The origin of $\{0\}$ is translated relative to the origin of $\{1\}$ by $d_0^1 = \begin{bmatrix} 0 & 0 & -10 \end{bmatrix}^T$, and the orientation of $\{1\}$ relative to $\{0\}$ is

   $\hat{x}_1^0 = \begin{bmatrix} 0.9971 & -0.0292 & -0.0705 \end{bmatrix}^T$,
   $\hat{y}_1^0 = \begin{bmatrix} -0.0292 & 0.7083 & -0.7053 \end{bmatrix}^T$, and
   $\hat{z}_1^0 = \begin{bmatrix} 0.0705 & 0.7053 & 0.7053 \end{bmatrix}^T$.

   Show how you derived the solution for part (d).

   You are given R_1^0 and d_0^1, which allows you to compute T_0^1:

$$T_0^1 = \begin{bmatrix} & (R_1^0)^T & & d_0^1 \\ & & & \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Finding $(T_0^1)^{-1}$ (perhaps using Matlab) yields T_1^0:

$$T_1^0 = \begin{bmatrix} 0.9971 & -0.0292 & 0.0705 & 0.0705 \\ -0.0292 & 0.7083 & 0.7053 & 7.053 \\ -0.0705 & -0.7053 & 0.7053 & 7.053 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

2. Find the missing elements of the following rotation matrices. Show your work, or explain your reasoning. It may be the case that there is no unique solution, in which case you should find all possible solutions. *Hint: Consider using the cross product.*

(a) $\begin{bmatrix} \cdot & 1 & 0 \\ \cdot & 0 & 0 \\ \cdot & 0 & -1 \end{bmatrix}$

The columns of the rotation matrix are x_1^0, y_1^0, and z_1^0. x_1^0 is equal to the cross product of y_1^0 and z_1^0:

$$x_1^0 = y_1^0 \times z_1^0 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{5}$$

(b) $\begin{bmatrix} \cdot & \sqrt{3}/2 & 0 \\ \cdot & 0 & 1 \\ \sqrt{3}/2 & \cdot & 0 \end{bmatrix}$

The magnitude of each column of the rotation matrix is equal to 1. Therefore, the missing element r_{32} of the second column can be found as:

$$(\sqrt{3}/2)^2 + 0^2 + r_{32}^2 = 1^2 \Rightarrow r_{32} = \pm 1/2$$

The first column can then be found using the cross product; if $r_{32} = 1/2$ then:

$$x_1^0 = y_1^0 \times z_1^0 = \begin{bmatrix} -1/2 \\ 0 \\ \sqrt{3}/2 \end{bmatrix}$$

If $r_{32} = -1/2$ then:

$$x_1^0 = y_1^0 \times z_1^0 = \begin{bmatrix} 1/2 \\ 0 \\ \sqrt{3}/2 \end{bmatrix}$$

(c) $\begin{bmatrix} 0 & 0 & 1 \\ \cdot & \cdot & 0 \\ \cdot & \cdot & 0 \end{bmatrix}$

Let the missing elements of the matrix be:

$$\begin{bmatrix} 0 & 0 & 1 \\ a & b & 0 \\ c & d & 0 \end{bmatrix}$$

Using the cross product we have:

$$x_1^0 \times y_1^0 = \begin{bmatrix} ad - bc \\ 0 \\ 0 \end{bmatrix} = z_1^0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Thus:

$$ad - bc = 1$$

The squared magnitude of the first column of the rotation matrix is:

$$a^2 + c^2 = 1$$

From the previous two equations we can conclude that:

$$d = a \text{ and } b = -c$$

and the rotation matrix is:

$$\begin{bmatrix} 0 & 0 & 1 \\ a & -c & 0 \\ c & a & 0 \end{bmatrix}$$

Using the trigonometric identity $\cos^2 \alpha + \sin^2 \alpha = 1$ we can always express the rotation matrix as:

$$\begin{bmatrix} 0 & 0 & 1 \\ \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \end{bmatrix}$$

for some angle α; however, α is not equal to the angle of the rotation.

But could we not choose $a = \sin\gamma$ and $c = \cos\gamma$ for some value γ? Well, yes, but:

$$\begin{bmatrix} 0 & 0 & 1 \\ \sin\gamma & -\cos\gamma & 0 \\ \cos\gamma & \sin\gamma & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ \cos(90-\gamma) & -\sin(90-\gamma) & 0 \\ \sin(90-\gamma) & \cos(90-\gamma) & 0 \end{bmatrix}$$

so substituting $\alpha = 90 - \gamma$ yields

$$\begin{bmatrix} 0 & 0 & 1 \\ \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \end{bmatrix}$$

Using various trigonometric identities, you can verify that all 8 possible choices of a and c yield rotation matrices that can be expressed as

$$\begin{bmatrix} 0 & 0 & 1 \\ \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \end{bmatrix}$$

for some value α. The 8 choices of a and c are:

a	c
$\cos\gamma$	$\sin\gamma$
$\cos\gamma$	$-\sin\gamma$
$-\cos\gamma$	$\sin\gamma$
$-\cos\gamma$	$-\sin\gamma$
$\sin\gamma$	$\cos\gamma$
$\sin\gamma$	$-\cos\gamma$
$-\sin\gamma$	$\cos\gamma$
$-\sin\gamma$	$-\cos\gamma$

3. Consider the following 4×4 homogeneous transformation matrices:

$$\begin{aligned}
R_{x,a} &: \text{rotation about } x \text{ by an angle } a \\
R_{y,a} &: \text{rotation about } y \text{ by an angle } a \\
R_{z,a} &: \text{rotation about } z \text{ by an angle } a \\
D_{x,a} &: \text{translation along } x \text{ by a distance } a \\
D_{y,a} &: \text{translation along } y \text{ by a distance } a \\
D_{z,a} &: \text{translation along } z \text{ by a distance } a
\end{aligned}$$

Write the matrix product giving the overall transformation for the following sequences (do not perform the actual matrix multiplications):

(a) The following rotations all occur in the moving frame.

i. Rotate about the current z-axis by angle ϕ.
ii. Rotate about the current y-axis by angle θ.
iii. Rotate about the current z-axis by angle ψ.

Note: This yields the ZYZ-Euler angle rotation matrix.

$R_{z,\phi}R_{y,\theta}R_{z,\psi}$

(b) The following rotations all occur in a fixed (world) frame.

i. Rotate about the world x-axis by angle ψ.
ii. Rotate about the world y-axis by angle θ.
iii. Rotate about the world z-axis by angle ϕ.

Note: This yields the roll, pitch, yaw (RPY) rotation matrix.

$R_{z,\phi}R_{y,\theta}R_{x,\psi}$

(c) The following transformations all occur in the moving frame.

i. Rotate about the current z-axis by angle θ_i.
ii. Translate along the current z-axis by a distance d_i.
iii. Translate along the current x-axis by a distance a_i.
iv. Rotate about the current x-axis by angle α_i.

Note: This is the Denavit-Hartenberg transformation matrix.

$R_{z,\theta_i} D_{z,d_i} D_{x,a_i} R_{x,\alpha_i}$

(d) The following transformations occur in a fixed (world) frame and in a moving frame.

i. Rotate about the world x-axis by angle ϕ.
ii. Rotate about the world z-axis by angle θ.
iii. Rotate about the current x-axis by angle ψ.
iv. Rotate about the world z-axis by angle α.

$R_{z,\alpha}((R_{z,\theta} R_{x,\phi}) R_{x,\psi})$
Note: The parentheses are not necessary and are shown to indicate the order in which the rotations occur.

4. Prove or disprove the following statement:

$$D_k R_{k,\theta} = R_{k,\theta} D_k$$

where $R_{k,\theta}$ is the homogeneous form of the rotation matrix shown on Slide 67 of the first set of lecture slides, and D_k is the translation matrix:

$$D_k = \begin{bmatrix} 1 & 0 & 0 & k_x \\ 0 & 1 & 0 & k_y \\ 0 & 0 & 1 & k_y \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Note that $k_x^2 + k_y^2 + k_z^2 = 1$.

Partition the matrices into blocks like so:

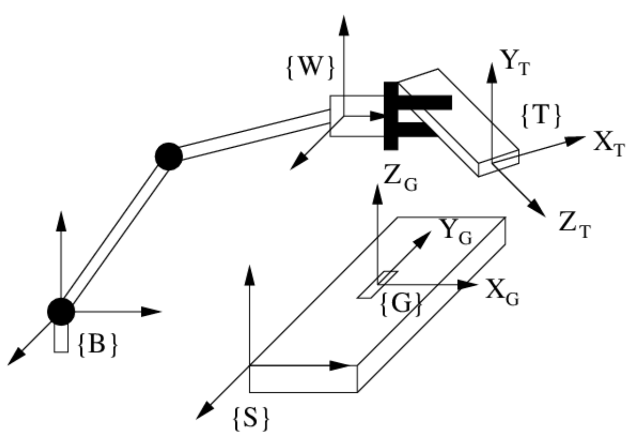
$$D_k R_{k,\theta} = \begin{bmatrix} I_{3\times3} & \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} & 1 \end{bmatrix} \begin{bmatrix} R & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} & 1 \end{bmatrix} = \begin{bmatrix} R & \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} & 1 \end{bmatrix}$$

$$R_{k,\theta} D_k = \begin{bmatrix} R & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} & 1 \end{bmatrix} \begin{bmatrix} I_{3\times3} & \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} & 1 \end{bmatrix} = \begin{bmatrix} R & R \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} & 1 \end{bmatrix}$$

Now we only need to prove or disprove $R\begin{bmatrix}k_x\\k_y\\k_z\end{bmatrix} = \begin{bmatrix}k_x\\k_y\\k_z\end{bmatrix}$

$$\begin{aligned}
R\begin{bmatrix}k_x\\k_y\\k_z\end{bmatrix} &= \begin{bmatrix} k_x^2 v_\theta + c_\theta & k_x k_y v_\theta - k_z s_\theta & k_x k_z v_\theta + k_y s_\theta \\ k_x k_y v_\theta + k_z s_\theta & k_y^2 v_\theta + c_\theta & k_y k_z v_\theta - k_x s_\theta \\ k_x k_z v_\theta - k_y s_\theta & k_y k_z v_\theta + k_x s_\theta & k_z^2 v_\theta + c_\theta \end{bmatrix} \begin{bmatrix}k_x\\k_y\\k_z\end{bmatrix} \\
&= \begin{bmatrix} k_x^2 k_x v_\theta + k_x c_\theta + k_y^2 k_x v_\theta - k_y k_z s_\theta + k_z^2 k_x v_\theta + k_y k_z s_\theta \\ k_x^2 k_y v_\theta + k_x k_z s_\theta + k_y^2 k_y v_\theta + k_y c_\theta + k_z^2 k_y v_\theta - k_x k_y s_\theta \\ k_x^2 k_z v_\theta - k_x k_y s_\theta + k_y^2 k_z v_\theta + k_x k_y s_\theta + k_z^2 k_z v_\theta + k_z c_\theta \end{bmatrix} \\
&= \begin{bmatrix} (k_x^2 + k_y^2 + k_z^2) k_x v_\theta + k_x c_\theta \\ (k_x^2 + k_y^2 + k_z^2) k_y v_\theta + k_y c_\theta \\ (k_x^2 + k_y^2 + k_z^2) k_z v_\theta + k_z c_\theta \end{bmatrix} \\
&= \begin{bmatrix} k_x(1 - c_\theta) + k_x c_\theta \\ k_y(1 - c_\theta) + k_y c_\theta \\ k_z(1 - c_\theta) + k_z c_\theta \end{bmatrix} \\
&= \begin{bmatrix}k_x\\k_y\\k_z\end{bmatrix}
\end{aligned}$$

5. Consider the figure shown below.



(a) Suppose that we want to command the robot to align the part that it is holding (having coordinate frame $\{T\}$) with a slot (having coordinate frame $\{G\}$. Assume that the following transformations are known: T_W^B, T_T^W, T_S^B, and T_G^S. Find an expression for T_G^T (the pose of $\{G\}$ expressed in $\{T\}$) in terms of the known transformations.

$$T_G^T = (T_T^W)^{-1}(T_W^B)^{-1}T_S^B T_G^S$$

(b) If you used your answer to part (a) to move the robot, the robot would try to move so that frame $\{T\}$ was aligned with frame $\{G\}$ which doesn't work because the direction of the slot is y_G whereas the corresponding direction on the part is x_T. Suppose that we want the part aligned with the slot so that the origin of $\{T\}$ is coincident with the origin of $\{G\}$, x_T is parallel to y_G, and y_T is parallel with x_G. What transformation should you postmultiply T_G^T to obtain the desired alignment?

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

(c) Suppose that the robot has picked up a part (having coordinate frame $\{T\}$) but it does not know the pose of the part relative to the wrist (having coordinate frame $\{W\}$). One way to find T_T^W (the pose of the part relative to wrist) is to have the robot align the part so that the part has a known pose relative to the robot. This is known as the tool calibration problem. In the above figure, the calibration jig having coordinate frame $\{S\}$ has an accurately machined slot having coordinate frame $\{G\}$. When the part is aligned with the calibration slot, the origin of $\{T\}$ is coincident with the origin of $\{G\}$, x_T is parallel to y_G, and y_T is parallel with x_G. Assume that the transformations T_S^B and T_G^S are known and constant, and assume that the robot can measure T_W^B when the arm is moved. Find an expression for the pose of $\{T\}$ expressed in $\{W\}$ in terms of the known and measured transformations. Note that $\{G\}$ and $\{T\}$ are not identical when the alignment is achieved; there is a transformation not shown on the diagram that you must account for.

$$T_T^W = (T_W^B)^{-1} T_S^B T_G^S \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

6. **Graduate students only**

A quaternion is another representation of rotation in 3D. The quaternion $Q = (q_w, q_x, q_y, q_z)$ can be thought of as being a scalar q_w and a vector $\vec{q} = \begin{bmatrix} q_x & q_y & q_z \end{bmatrix}^T$. Given two quaternions $A = (a_w, \vec{a})$ and $B = (b_w, \vec{b})$, the quaternion product $C = AB$ is defined as

$$c_w = a_w b_w - \vec{a} \cdot \vec{b}$$
$$\vec{c} = a_w \vec{b} + b_w \vec{a} + \vec{a} \times \vec{b}$$

where $\vec{a} \times \vec{b}$ is the cross product of $\vec{a}$ and $\vec{b}$.

(a) Show that $Q_I Q = Q Q_I = Q$ for every unit quaternion Q where $Q_I = (1, 0, 0, 0)$, i.e., Q_I is the identity quaternion.

(b) The conjugate Q^* of a quaternion $Q = (q_w, \vec{q})$ is given by $Q^* = (q_w, -\vec{q})$. Show that $Q^* Q = Q Q^* = (1, 0, 0, 0)$, i.e., Q^* is the inverse of Q.

(c) The quaternion $Q = (q_w, \vec{q})$ where $q_w = \cos\frac{\theta}{2}$ and $\vec{q} = \begin{bmatrix} k_x \sin\frac{\theta}{2} & k_y \sin\frac{\theta}{2} & k_z \sin\frac{\theta}{2} \end{bmatrix}^T$ represents the rotation of angle θ about the unit vector $\hat{k} = \begin{bmatrix} k_x & k_y & k_z \end{bmatrix}^T$. A vector $\vec{p} = \begin{bmatrix} p_x & p_y & p_z \end{bmatrix}^T$ can be rotated using the quaternion product QPQ^* where P is the quaternion $(0, \vec{p})$. Show that this is true for a rotation of angle θ about the z-axis.

7. **Graduate students only**

Consider a vector v that is rotated about a unit vector $\hat{k}$ (passing through the origin) by an angle θ to form a new vector v':

$$v' = R_{k,\theta} v$$

Derive Rodrigues' rotation formula,

$$v' = v \cos\theta + (\hat{k} \times v) \sin\theta + \hat{k}(\hat{k} \cdot v)(1 - \cos\theta)$$

Do not replicate the Wikipedia derivation; instead, use the rotation matrix for rotation about an axis $\hat{k}$ by an angle θ (Slide 67 of the first set of lecture slides).