York University EECS 3101 August 7, 2018

Homework Assignment #8A
Due: August 14, 2018 at 7:00 p.m.

1. Jaanika is throwing a big party. She has invited n married couples to attend. We represent the people in these couples by $1_A, 1_B, 2_A, 2_B, 3_A, 3_B, \ldots, n_A, n_B$. The party room has m tables. Table i has room for c_i people to sit at it, where c_i is a positive integer. Jaanika would like to ensure that nobody sits at the same table as his or her spouse.

Jaanika comes up with the following greedy algorithm: Go through the couples, seating the members of each couple at the two tables that have the most room. She implements it using a priority queue Q that will store the set of non-full tables, where the priority of a table is the number of seats that are still free. It keeps track of the people sitting at each table using an array $T[1..m]$ of sets, where $T[j]$ is the set of people to be seated at table j.

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1  for j ← 1..m
2      p[j] ← c_j
3      T[j] ← {}
4      INSERT j into Q with priority p[j]
5  end for
6  for i ← 1..n
7      Invariant: the elements of Q are {j : |T[j]| < c_j} and for all j ∈ {1,..,m}, p[j] = c_j − |T[j]|
8      if Q contains fewer than 2 elements, output “impossible” and halt
9      x ← EXTRACT-MAX(Q)
10     y ← EXTRACT-MAX(Q)
11     T[x] ← T[x] ∪ {i_A}
12     T[y] ← T[y] ∪ {i_B}
13     p[x] ← p[x] − 1
14     p[y] ← p[y] − 1
15     if table x has positive priority, INSERT x into Q with priority p[x]
16     if table y has positive priority, INSERT y into Q with priority p[y]
17 end for
18 output the seating arrangement T[1..m]
```

The loop invariant on line 7 is very easy to prove. You can use it without proving it.

Let $T_i[1..m]$ be the value of the array T after i iterations of the algorithm. Your goal is to prove that *if* there is a legal seating arrangement, then there is a legal seating arrangement that extends T_i.

(a) State precisely what it means for a seating arrangement T^* to extend the partial arrangement T_i.

(b) Prove that if there is a legal seating arrangement for all n couples, then for $0 \leq i \leq n$, there is a legal seating arrangement that extends T_i.

(c) Prove that if the algorithm outputs a legal seating arrangement if one exists, and outputs “impossible” otherwise.