Homework Assignment #7
Due: March 16, 2018 at 2:30 p.m.

1. There are n people working in a factory on a particular day. The people work different shifts. Worker i starts working at time s_i and finishes working at time f_i. The factory supervisor wants to repeatedly make an announcement so that every worker hears the announcement at least once. However, to avoid unnecessary disruptions, the supervisor wants to make the minimum possible number of announcements. (We assume that the announcement is very short, so that if it is made at time t, any worker i who has $s_i \leq t \leq f_i$ will hear it: if $t = f_i$, the worker will hear the announcement while walking out of the factory.) The supervisor comes up with the following idea to figure out what times to make the announcements.

She will make a list of all the workers, sorted by their finish times. She will make the announcement when the shift of the first worker in L ends. Then, she will remove all workers who heard that announcement from L. She will repeat this (announce and remove some workers from L) until L is empty.

To do the removals efficiently, she uses, an array of pointers to nodes in L as an auxiliary data structure. This array is sorted by *starting* times. Her greedy algorithm is as follows.

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1  sort the workers so that f_1 ≤ f_2 ≤ f_3 ≤ ··· ≤ f_n
2  create a doubly-linked list L of workers in this sorted order
3  let A[1..n] be an array of pointers to the nodes in the linked list
4  sort A by starting times (so that, for all j, start time of worker A[j] is less than or equal
                 to start time of worker A[j + 1]) // this step does not affect the order of L
5  j ← 1
6  loop
7      exit when L is empty
8      t ← finish time of the first worker in L
9      make the announcement at time t
10     loop
11         // remove from L all workers who heard the announcement at time t
12         exit when j = n + 1 or start time of worker A[j] > t
13         remove A[j] from L
14         j ← j + 1
15     end loop
16 end loop
```

Let T_i be the set of announcement times chosen in the first i iterations of the outer loop.

The outer loop has the following loop invariant: L contains exactly those workers who were not working at any of the times in T_i (i.e., L contains the workers who have not heard the announcement yet). You may use this fact without proving it.

A set of announcement times is *legal* if every worker hears at least one of the announcements. A legal set is *optimal* if there is no legal set that is smaller.

Prove that, for all i, there is a legal optimal set T^* that extends T_i. More formally, this means that $T_i \subseteq T^*$ **and** no element of $T^* - T_i$ is less than an element of T_i.