Homework Assignment #2
Due: January 26, 2018 at 2:30 p.m.

1. Ronald Drumpf is the president of a large country. He likes to brag about how well the stock market has been doing during his term in office. However, while bragging, he often makes wildly inaccurate claims, which gets him in trouble with the pesky newspaper reporters, who like to report on actual "facts". He has decided to turn over a new leaf and only make true statements from now on. So, he would like some accurate data to use in his speeches and tweets.

He told his minions to enter data from the stock market into an array A. $A[i]$ stores the value of his country's main stock market index (MSMI) at the end of the ith day of Drumpf's reign. Drumpf himself then sat down to write a programme to compute the length of the longest stretch of consecutive days when the MSMI was strictly higher at the end of the day than it was at the end of the previous day. However, Drumpf quickly became bored with this task and decided to go golfing instead. Here's what he wrote:

```
1  BRAG(A[1..n])
2      % Precondition: n is a positive integer and A[i] ∈ ℕ for 1 ≤ i ≤ n
3      m ← 0
4      t ← 0
5      for i ← 2..n
6          % invariant: ______________________________
7          if ______________________________ then
8              t ← t + 1
9              if t > m then
10                 ______________________________
11             end if
12         else
13             t ← 0
14         end if
15     end for
16     output "I made the stock market grow for m consecutive days!!"
17     % Postcondition:
18     % m = max{ℓ ∈ ℕ : ∃s ∈ ℕ such that 1 ≤ s ≤ n − ℓ and A[s] < A[s+1] < ··· < A[s+ℓ]}
19     % Alternative (equivalent) postcondition in words:
20     % m is the length of the longest contiguous subarray of A[1..n] where each entry in
21     % the subarray is greater than the previous entry of A
22 end BRAG
```

(a) Fill in the blanks on line 7 and 10 of Drumpf's code so that it will satisfy the postcondition when it terminates. Each blank should take $O(1)$ steps to perform and should not modify the values stored in A.

(b) State a loop invariant for line 6 that is sufficiently strong so that you could prove it is an invariant by induction, and so that you could use it to prove the postconditions.

You can state your invariant using mathematical notation or in plain but precise English.

(You don't have to hand in the proof that your statement is an invariant, but your invariant should be strong enough that you could do it. In fact, it's probably a good idea for you to do the proof, even though you don't have to hand it in, so that you can check that your invariant really is strong enough.)

(c) Prove that your loop invariant in part (b) implies the postcondition when the loop terminates. You may use either version of the postcondition.

2. Consider the following game. Alviine gets to roll a 10-sided die n times. The die's faces are labelled 1 to 10. It is a fair die, meaning that all of the 10 possible outcomes are equally likely. After any roll, Alviine can say "pay". When she says "pay", she is given a prize of k dollars, where k is the value on the upper face of the die at that time. However, she can only say "pay" once during the game.

Suppose Alviine makes decisions so as to maximize the average prize she will get. We would like to determine what this average will be.

(For those who have taken a course on probability, "average" just means the expected value paid to Alviine. For those who haven't, "average" just means the average payout over all 10^{12} possible sequences of 12 die rolls.)

(a) If $n = 1$, what would the average prize be?

(b) If $n = 2$, what should Alviine's strategy be? I.e., which outcomes of the first die roll should cause her to say "pay" and which outcomes should cause her to remain silent (assuming she wants to maximize her average prize)? What would the average payout be if Alviine follows this optimal strategy?

(c) Write an algorithm PRIZE(n) to compute the average prize if the game lasts n rounds (where n is any positive integer). Give a brief explanation of why your algorithm is correct. (You do not have to give a formal proof.)

(d) What is the output of PRIZE(16)? Your answer should be accurate to 5 places after the decimal.