In class, we looked at a recursive algorithm for multiplying two natural numbers. The algorithm was discovered by A. A. Karatsuba in 1960. Here is the time bound for that algorithm.

Let $T(n)$ be the worst-case time for multiplying two n-bit numbers. We derived the recurrence:
$T(n)$ is $O(1)$ for $n \leq 3$, and
$T(n) \leq T(\lfloor \frac{n}{2} \rfloor) + T(\lceil \frac{n}{2} \rceil) + T(\lceil \frac{n}{2} \rceil + 1) + an$ for $n > 3$.
(In the above, a is a constant.)

I claimed in class that $T(n)$ is $O(n^{\log_2 3})$. Here is a proof of that claim.

You might first try to prove that $T(n) \leq cn^{\log_2 3}$ (for some constant c). Unfortunately, if you try this, you will see that the induction hypothesis is not strong enough for the induction step to work.

So, to strengthen the induction hypothesis, we prove a stronger claim. (This is the same trick as described on page 85 of the textbook.)

Let $c = \max\{\frac{T(n)+2an}{(n-3)^{\log_2 3}} : n \in \{4, 5, 6, 7\}\}$.

Claim: for all $n \geq 4$, $T(n) \leq c(n-3)^{\log_2 3} - 2an$.

Base case ($n = 4, 5, 6, 7$): We chose c precisely so that the claim holds for these values of n.

Inductive Step: Let $n \geq 8$. Assume that $T(k) \leq c(k-3)^{\log_2 3} - 2ak$ for $4 \leq k < n$. We prove that $T(n) \leq c(n-3)^{\log_2 3} - 2an$.

Note that $4 \leq \lfloor \frac{n}{2} \rfloor \leq \lceil \frac{n}{2} \rceil < \lceil \frac{n}{2} \rceil + 1 \leq \frac{n+3}{2} < n$ since $n \geq 8$. Thus, the inductive hypothesis applies to $T(\lfloor \frac{n}{2} \rfloor), T(\lceil \frac{n}{2} \rceil)$ and $T(\lceil \frac{n}{2} \rceil + 1)$. So, we have

$$\begin{aligned}
T(n) &\leq T(\lfloor \tfrac{n}{2} \rfloor) + T(\lceil \tfrac{n}{2} \rceil) + T(\lceil \tfrac{n}{2} \rceil + 1) + an \\
&\leq c(\lfloor \tfrac{n}{2} \rfloor - 3)^{\log_2 3} - 2a\lfloor \tfrac{n}{2} \rfloor + c(\lceil \tfrac{n}{2} \rceil - 3)^{\log_2 3} - 2a\lceil \tfrac{n}{2} \rceil \\
&\qquad + c(\lceil \tfrac{n}{2} \rceil + 1 - 3)^{\log_2 3} - 2a(\lceil \tfrac{n}{2} \rceil + 1) + an \qquad \text{(by induction hypothesis)} \\
&\leq 3c(\tfrac{n+3}{2} - 3)^{\log_2 3} - 2a(\lfloor \tfrac{n}{2} \rfloor + \lceil \tfrac{n}{2} \rceil + \lceil \tfrac{n}{2} \rceil + 1) + an \\
&\leq 3c(\tfrac{n+3}{2} - 3)^{\log_2 3} - 2a(\tfrac{3n}{2}) + an \\
&= 3c(\tfrac{n-3}{2})^{\log_2 3} - 2an \\
&= c(n-3)^{\log_2 3} - 2an
\end{aligned}$$

This completes the proof of the claim.

It follows from the claim that $T(n) \leq cn^{\log_2 3}$ for $n \geq 4$, so $T(n)$ is $O(n^{\log_2 3})$.

Remark: How did I come up with this proof? First, I tried proving $T(n) \leq c(n-b)^{\log_2 3}$ for some constants b, c. When I did the induction step, I saw that choosing $b = 3$ handled the floors and ceilings and the $+1$ inside the arguments to T, but it didn't quite handle the $+an$. So then I made the claim even stronger: $T(n) \leq c(n-3)^{\log_2 3} - dn$ and I found that taking $d = 2a$ made the induction step work (for $n > 3$). Then I noticed that the claim was false for $n = 3$, so I started the induction at $n = 4$. Then I saw that the induction step could only apply the induction hypothesis if $n \geq 8$, so I handled $n = 4, 5, 6, 7$ separately in the base case by choosing the right c to make those cases work out.