Loop Invariants and Binary Search

Learning Outcomes

- From this lecture, you should be able to:
  - Use the loop invariant method to think about iterative algorithms.
  - Prove that the loop invariant is established.
  - Prove that the loop invariant is maintained in the ‘typical’ case.
  - Prove that the loop invariant is maintained at all boundary conditions.
  - Prove that progress is made in the ‘typical’ case
  - Prove that progress is guaranteed even near termination, so that the exit condition is always reached.
  - Prove that the loop invariant, when combined with the exit condition, produces the post-condition.
  - Trade off efficiency for clear, correct code.

Outline

- Iterative Algorithms, Assertions and Proofs of Correctness
- Binary Search: A Case Study

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- **Iterative Algorithms, Assertions and Proofs of Correctness**
- Binary Search: A Case Study

Assertions

- An **assertion** is a statement about the state of the data at a specified point in your algorithm.
- An assertion is not a task for the algorithm to perform.
- You may think of it as a comment that is added for the benefit of the reader.

Loop Invariants

- Binary search can be implemented as an **iterative algorithm** (it could also be done recursively).
- **Loop Invariant:** An **assertion** about the current state useful for designing, analyzing and proving the correctness of iterative algorithms.

Other Examples of Assertions

- Preconditions: Any assumptions that must be true about the input instance.
- Postconditions: The statement of what must be true when the algorithm/program returns.
- Exit condition: The statement of what must be true to exit a loop.

Iterative Algorithms

Take one step at a time

towards the final destination

```
loop
    take step
end loop
```

Establishing Loop Invariant

From the Pre-Conditions on the input instance we must establish the loop invariant.

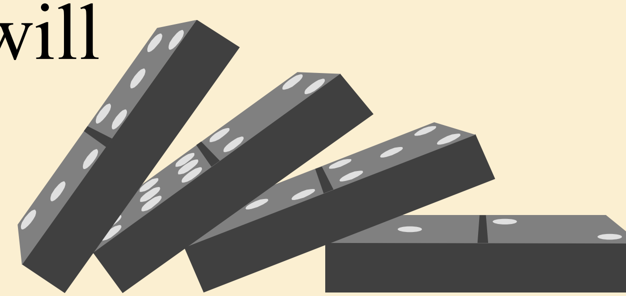
Maintain Loop Invariant

- Suppose that
  - We start in a safe location (pre-condition)
  - If we are in a safe location, we always step to another safe location (loop invariant)
- Can we be assured that the computation will always be in a safe location?

- By what principle?

Maintain Loop Invariant

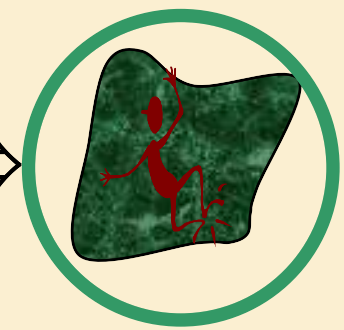
- By Induction the computation will always be in a safe location.



$$\Rightarrow S(0)$$

$$\Rightarrow \forall i, S(i) \Rightarrow S(i+1)$$

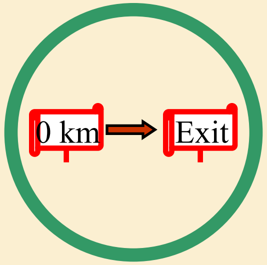
$$\left.\begin{matrix} \\ \\ \end{matrix}\right\} \Rightarrow \forall i, S(i) \Rightarrow$$



Ending The Algorithm

- Define Exit Condition

- Termination: With sufficient progress, the exit condition will be met.



- When we exit, we know
  - exit condition is true
  - loop invariant is true

  from these we must establish the post conditions.

Definition of Correctness

<PreCond> & <code> ➔<PostCond>

If the input meets the preconditions,

then the output must meet the postconditions.

If the input does not meet the preconditions, then nothing is required.

Outline

- Iterative Algorithms, Assertions and Proofs of Correctness
- **Binary Search: A Case Study**

Define Problem: Binary Search

- PreConditions
  - Key 25
  - Sorted List

3	5	6	13	18	21	21	25	36	43	49	51	53	60	72	74	83	88	91	95

- PostConditions
  - Find key in list (if there).

3	5	6	13	18	21	21	25	36	43	49	51	53	60	72	74	83	88	91	95

Define Loop Invariant

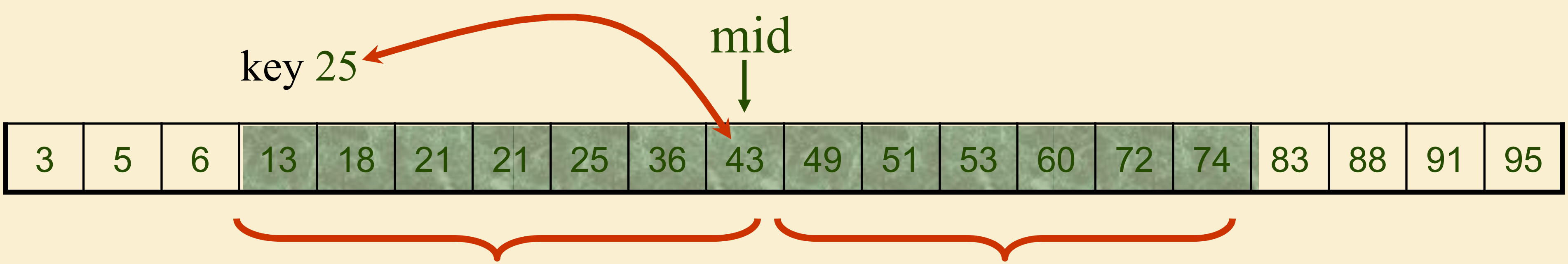
- Maintain a sublist.
- If the key is contained in the original list, then the key is contained in the sublist.

key 25

3	5	6	13	18	21	21	25	36	43	49	51	53	60	72	74	83	88	91	95

Define Step

- Cut sublist in half.
- Determine which half the key would be in.
- Keep that half.

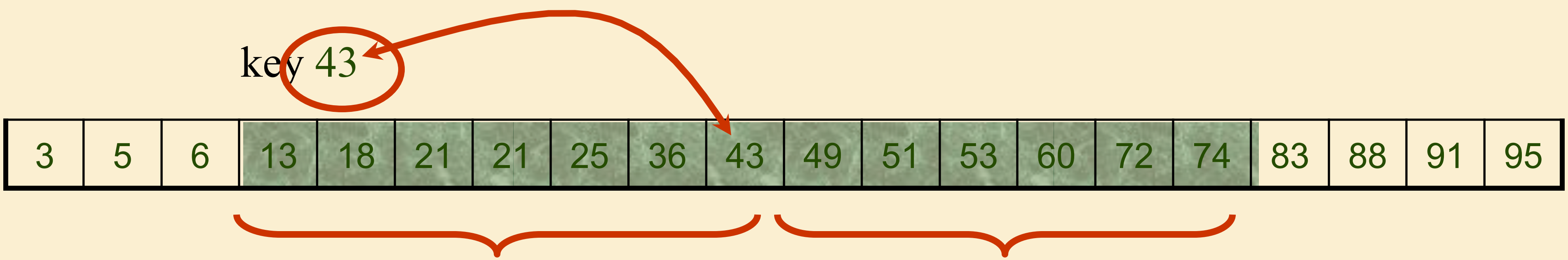


If key $\leq$ mid,
then key is in
left half.

If key > mid,
then key is in
right half.

Define Step

- It is faster not to check if the middle element is the key.
- Simply continue.

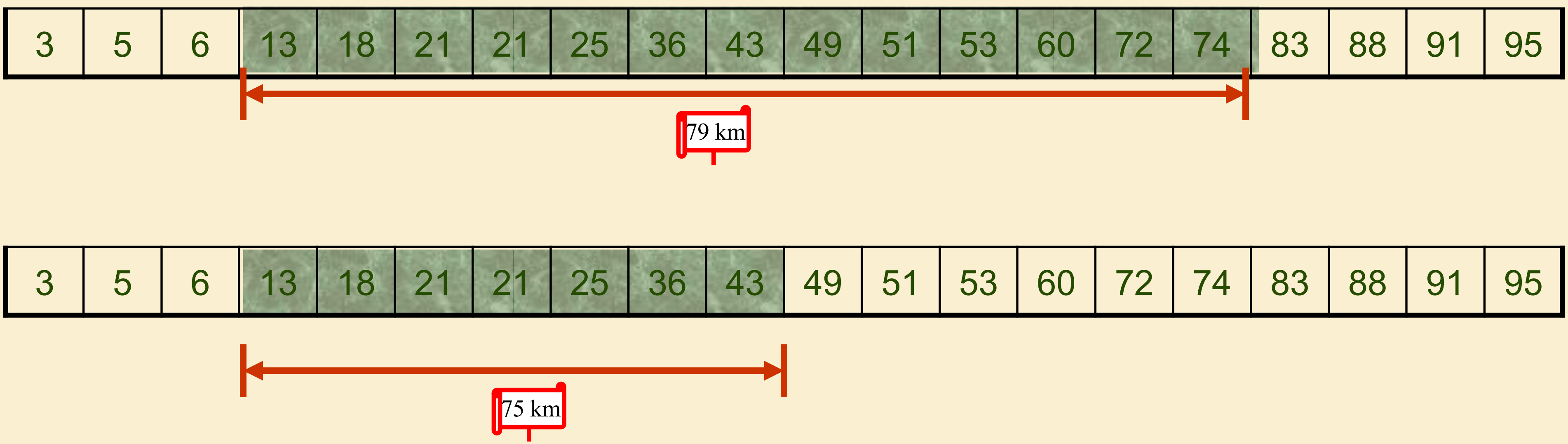


If key ≤ mid, then key is in left half.

If key > mid, then key is in right half.

Make Progress

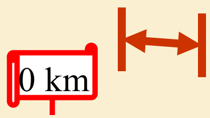
- The size of the list becomes smaller.



Exit Condition

key 25

3	5	6	13	18	21	21	25	36	43	49	51	53	60	72	74	83	88	91	95



- If the key is contained in the original list,

  then the key is contained in the sublist.
- Sublist contains one element.

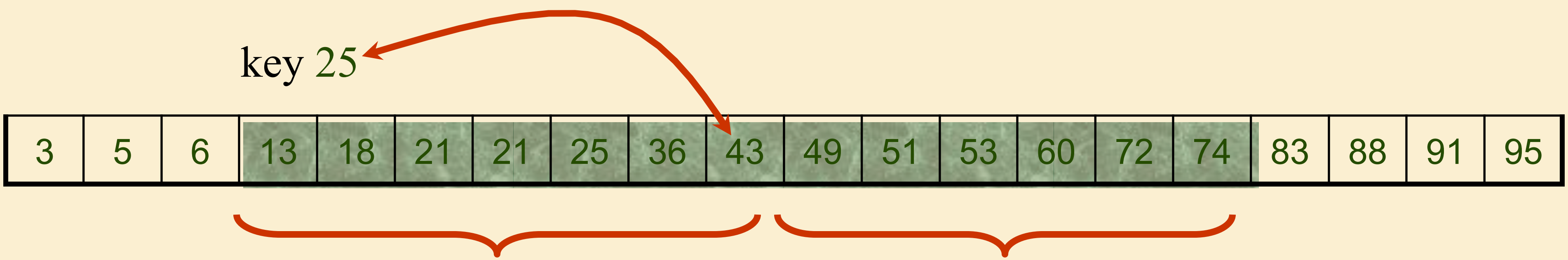
- If element = key, return associated entry.
- Otherwise return false.

Running Time

The sublist is of size n, $^n/_2$, $^n/_4$, $^n/_8$,…,1

Each step O(1) time.

Total = O(log n)



If key ≤ mid, then key is in left half.

If key > mid, then key is in right half.

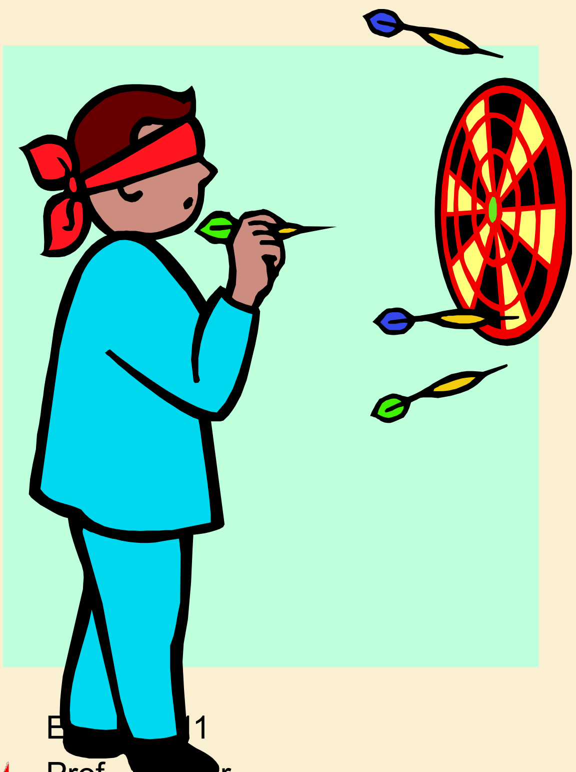
Running Time

- Binary search can interact poorly with the memory hierarchy (i.e. caching), because of its random-access nature.
- It is common to abandon binary searching for linear searching as soon as the size of the remaining span falls below a small value such as 8 or 16 or even more in recent computers.

```
BinarySearch(A[1..n], key)
<precondition>: A[1..n] is sorted in non-decreasing order
<postcondition>: If key is in A[1..n], algorithm returns its location
p = 1, q = n
while q > p
    < loop-invariant>: If key is in A[1..n], then key is in A[p..q]
    mid = ⌊(p + q)/2⌋
    if key ≤ A[mid]
        q = mid
    else
        p = mid + 1
    end
end
if key = A[p]
    return(p)
else
    return("Key not in list")
end
```

Simple, right?

- Although the concept is simple, binary search is notoriously easy to get wrong.
- Why is this?



Boundary Conditions

- The basic idea behind binary search is easy to grasp.
- It is then easy to write pseudocode that works for a 'typical' case.
- Unfortunately, it is equally easy to write pseudocode that fails on the *boundary conditions*.

Boundary Conditions

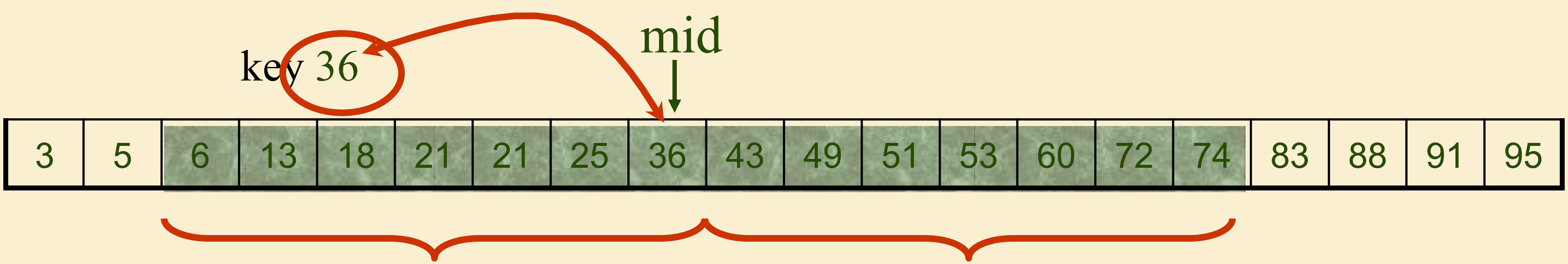
$$\text{if } key \le A[mid] \quad q = mid \quad \text{else} \quad p = mid + 1 \quad \text{end}$$

or

~~$$\text{if } key < A[mid] \quad q = mid \quad \text{else} \quad p = mid + 1 \quad \text{end}$$~~

What condition will break the loop invariant?

Boundary Conditions



Code: key ≥ A[mid] → select right half

Bug!!

Boundary Conditions

$$\begin{array}{l} \text{if } key \le A[mid] \\ \quad q = mid \\ \text{else} \\ \quad p = mid + 1 \\ \text{end} \end{array}$$

OK

$$\begin{array}{l} \text{if } key < A[mid] \\ \quad q = mid - 1 \\ \text{else} \\ \quad p = mid \\ \text{end} \end{array}$$

OK

$$\begin{array}{l} \text{if } key < A[mid] \\ \quad q = mid \\ \text{else} \\ \quad p = mid + 1 \\ \text{end} \end{array}$$

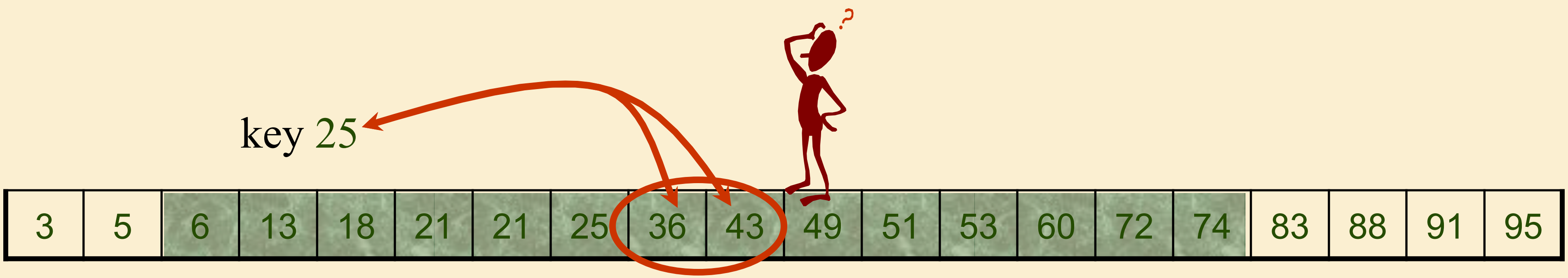
Not OK!!

Boundary Conditions

$$\text{mid} = \left\lfloor \frac{p+q}{2} \right\rfloor$$

or

$$\text{mid} = \left\lceil \frac{p+q}{2} \right\rceil$$



Shouldn't matter, right?

Select $\text{mid} = \left\lceil \frac{p+q}{2} \right\rceil$

Boundary Conditions

$\text{if } key \le A[mid]$

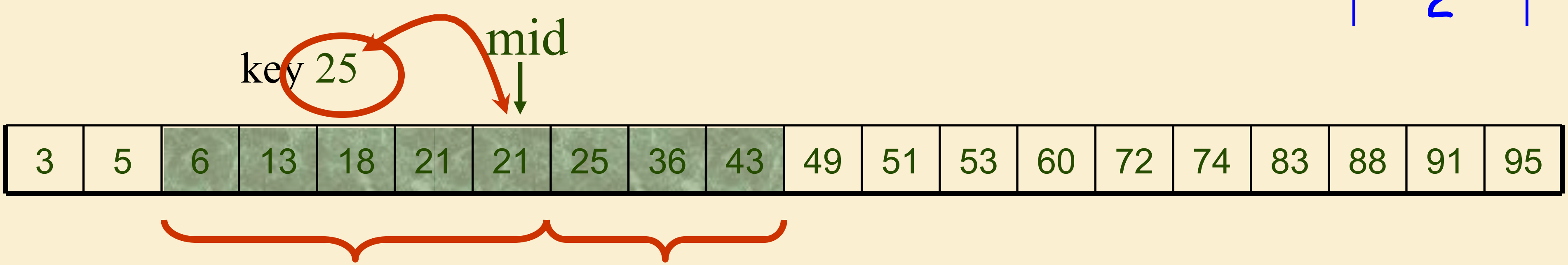
$\quad q = mid$

else

$\quad p = mid + 1$

end

Select $mid = \left\lceil \frac{p+q}{2} \right\rceil$



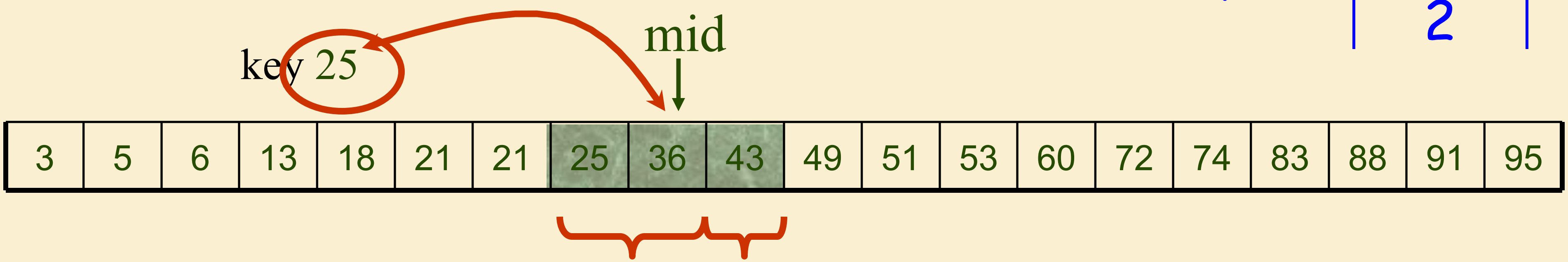
If key $\le$ mid, then key is in left half.

If key $>$ mid, then key is in right half.

Boundary Conditions

```
if key ≤ A[mid]
    q = mid
else
    p = mid + 1
end
```

Select $mid = \left\lceil \frac{p+q}{2} \right\rceil$



If key ≤ mid, then key is in left half.

If key > mid, then key is in right half.

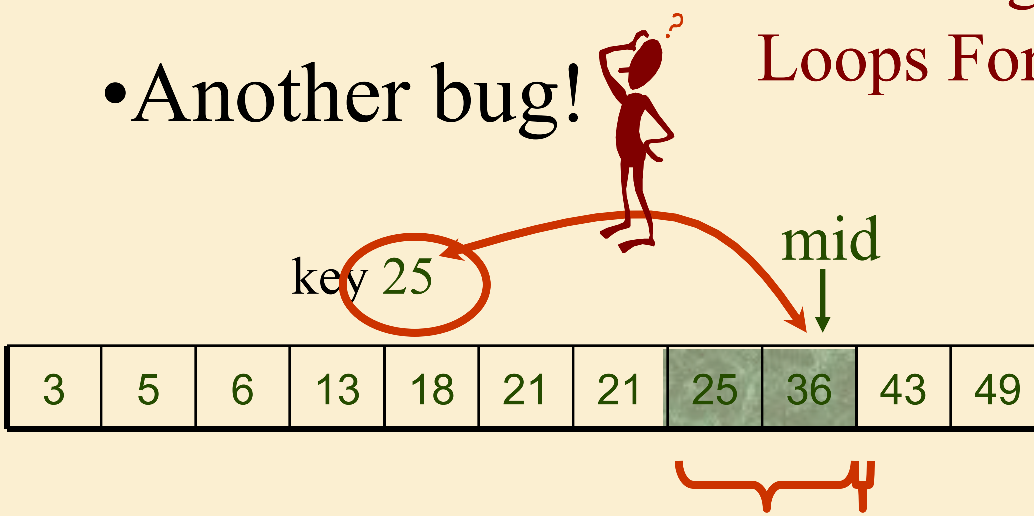
Boundary Conditions

if $key \le A[mid]$
    $q = mid$
else
    $p = mid + 1$
end

- Another bug!

No progress toward goal: Loops Forever!

Select $mid = \left\lceil \frac{p+q}{2} \right\rceil$



If $key \le mid$, then key is in left half.

If $key > mid$, then key is in right half.

Boundary Conditions

$$mid = \left\lfloor \frac{p+q}{2} \right\rfloor$$

if $key \le A[mid]$

    $q = mid$

else

    $p = mid + 1$

end

OK

$$mid = \left\lceil \frac{p+q}{2} \right\rceil$$

if $key < A[mid]$

    $q = mid - 1$

else

    $p = mid$

end

OK

$$mid = \left\lceil \frac{p+q}{2} \right\rceil$$

if $key \le A[mid]$

    $q = mid$

else

    $p = mid + 1$

end

Not OK!!

Getting it Right

- How many possible algorithms?
- How many **correct** algorithms?
- Probability of **guessing** correctly?

$$\text{mid} = \left\lfloor \frac{p+q}{2} \right\rfloor$$

or $\text{mid} = \left\lceil \frac{p+q}{2} \right\rceil$?

if $key \le A[mid]$

or if $key < A[mid]$?

$q = mid$

else

$p = mid + 1$

end

or

$q = mid - 1$

else

$p = mid$

end

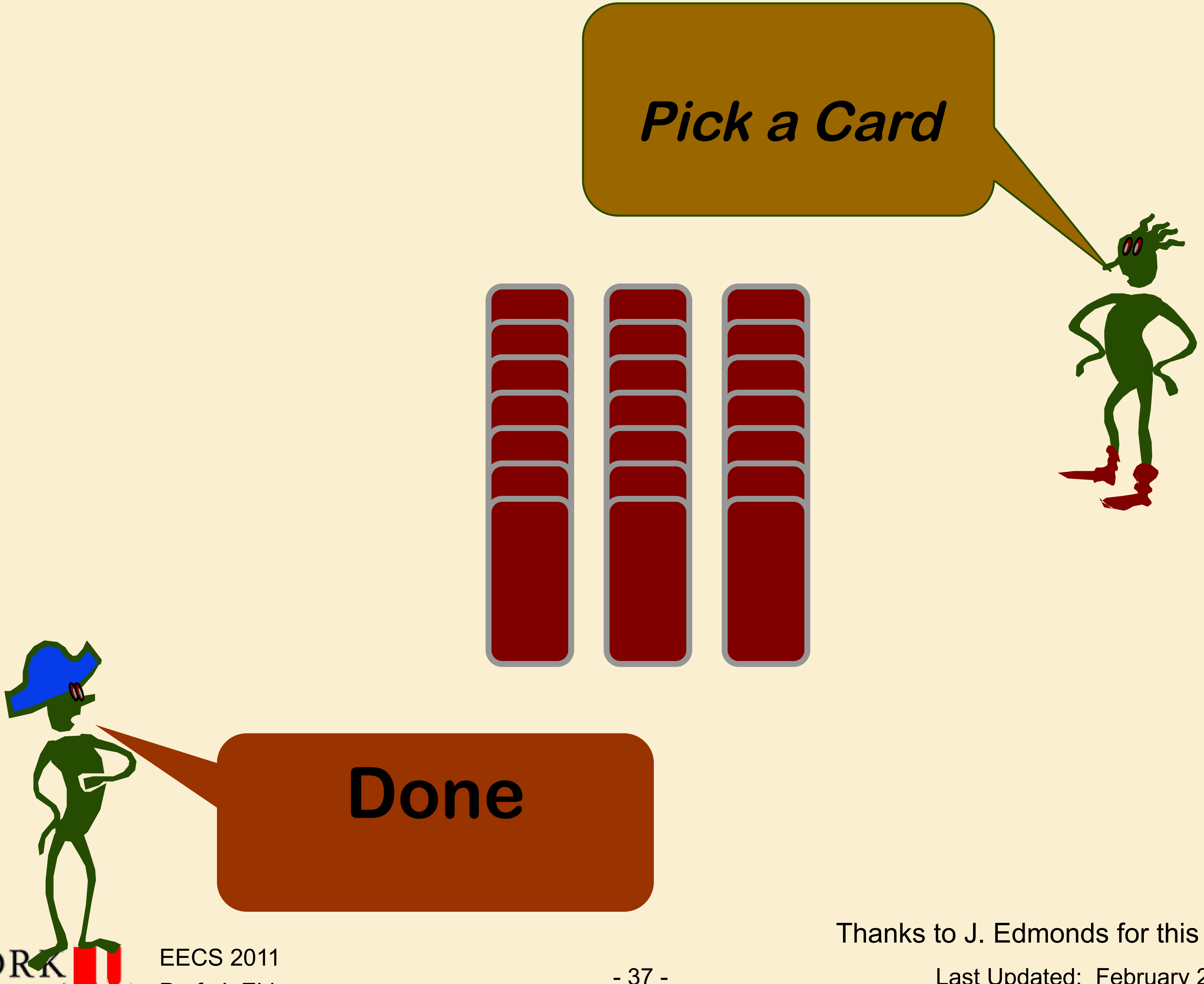
Alternative Algorithm: Less Efficient but More Clear

```
BinarySearch(A[1..n], key)
<precondition>:  A[1..n] is sorted in non-decreasing order
<postcondition>: If key is in A[1..n], algorithm returns its location
p = 1, q = n
while q ≥ p
    < loop-invariant>: If key is in A[1..n], then key is in A[p..q]
    mid = ⌊(p+q)/2⌋
    if key < A[mid]
        q = mid − 1
    else if key > A[mid]
        p = mid + 1
    else
        return(mid)
    end
end
return("Key not in list")
```

Still $\Theta(\log n)$, but with slightly larger constant.

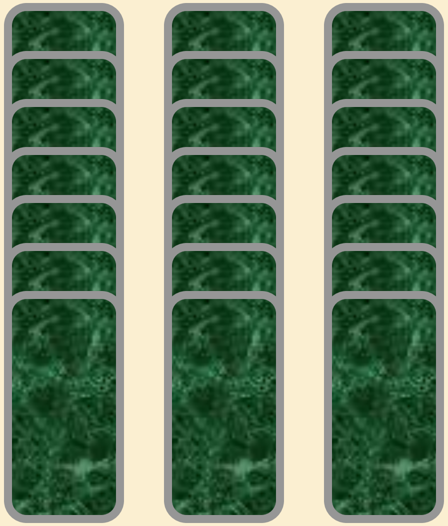
Card Trick

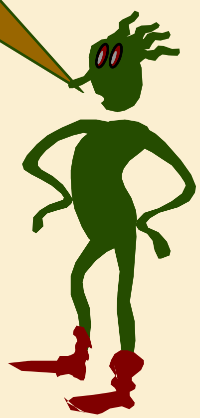
- A volunteer, please.

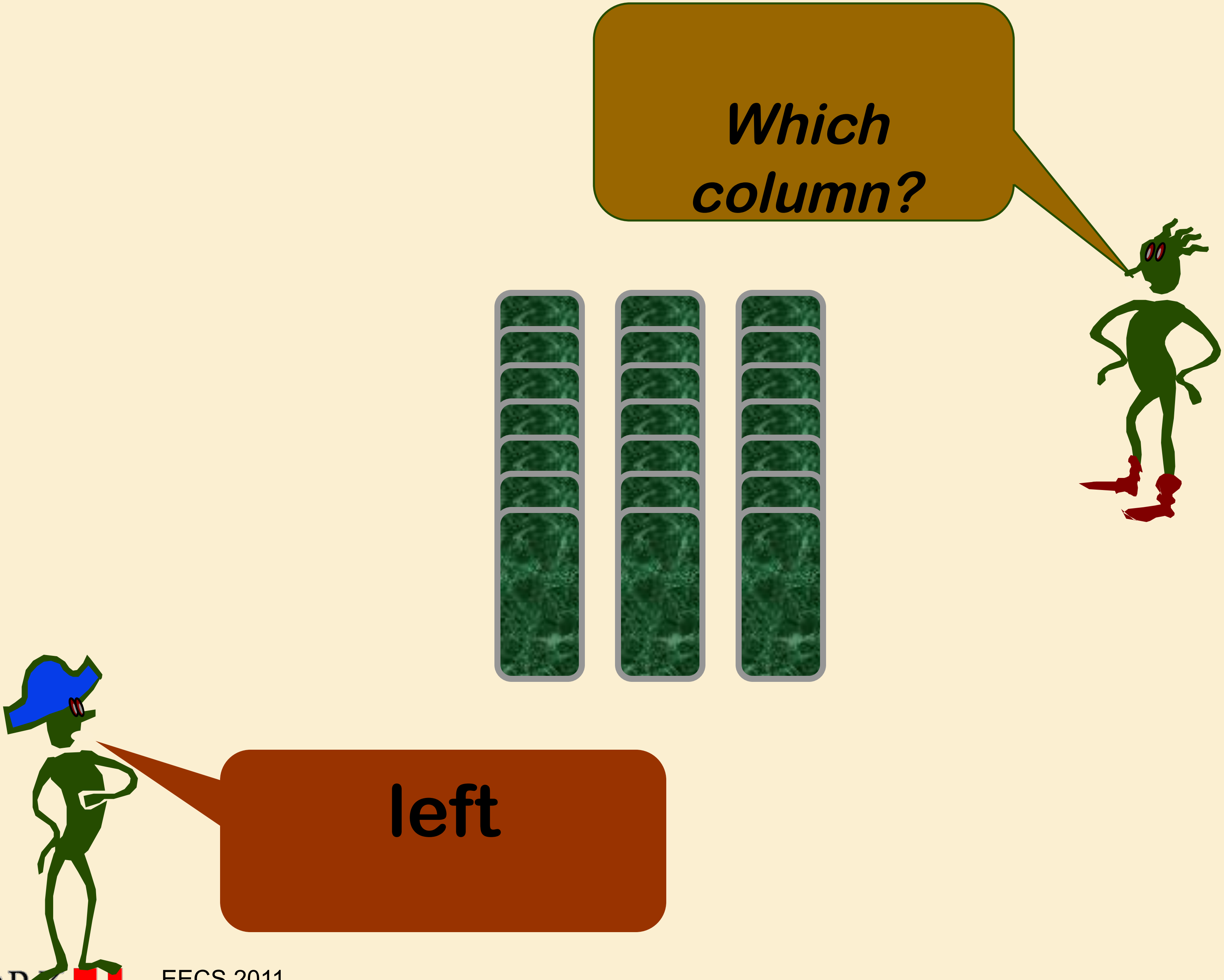


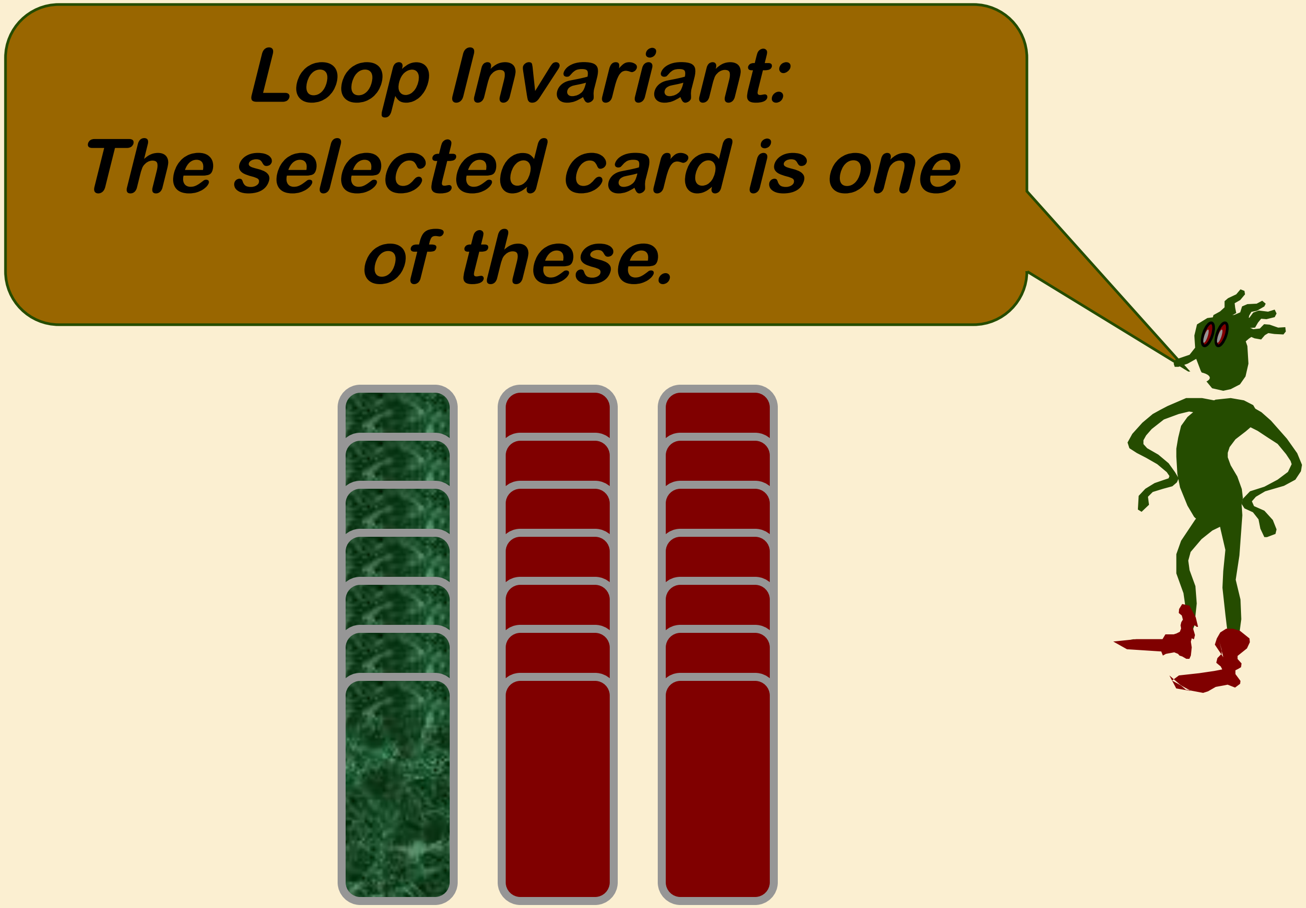
Thanks to J. Edmonds for this example.

Loop Invariant:
The selected card is one of these.

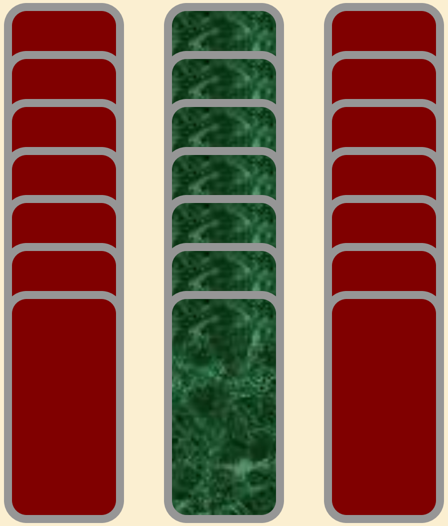


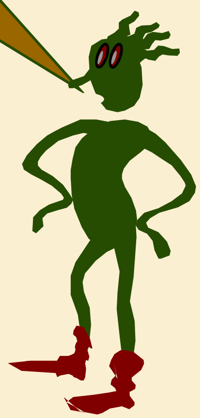


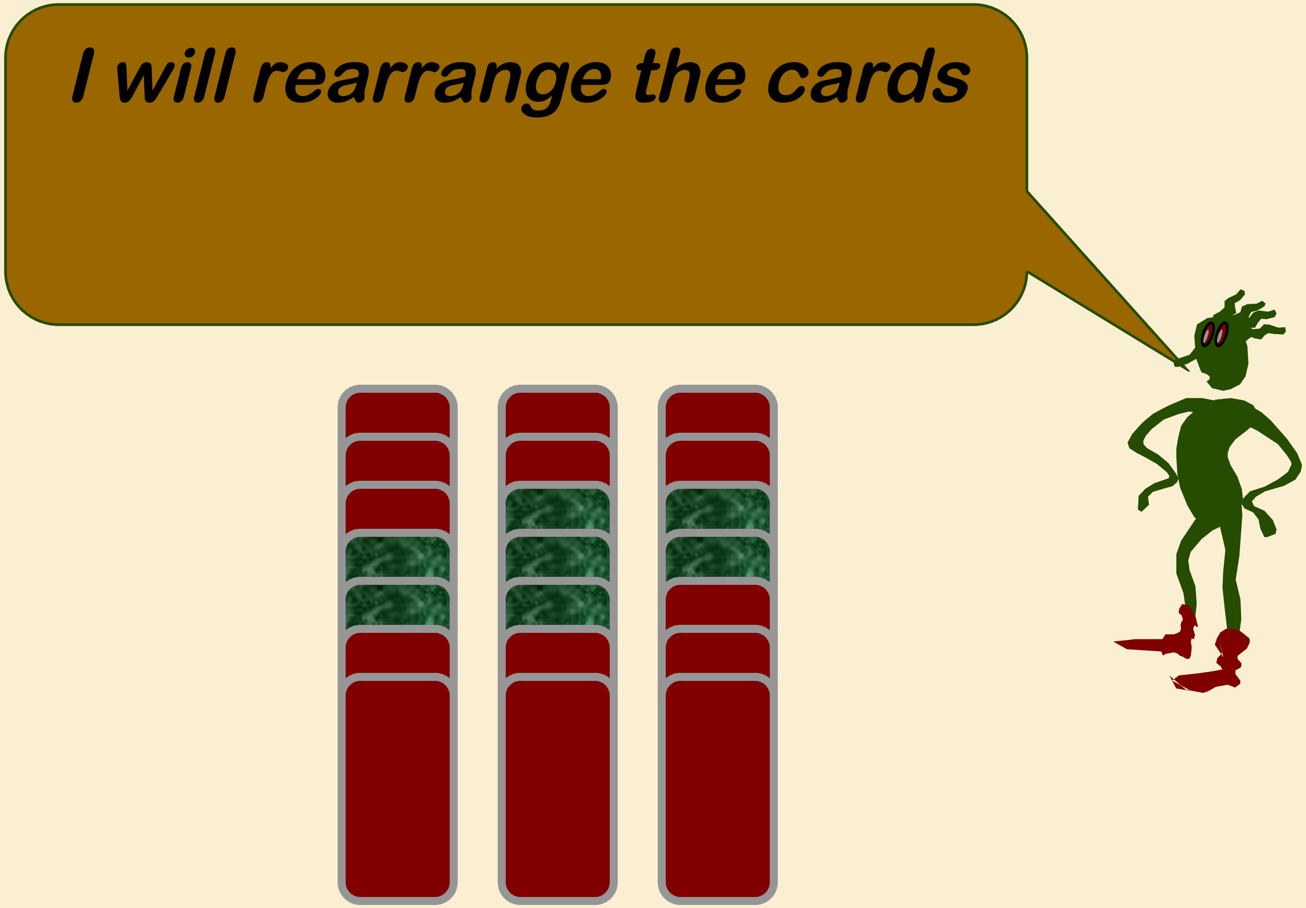

Which column?
left


Loop Invariant:
The selected card is one of these.

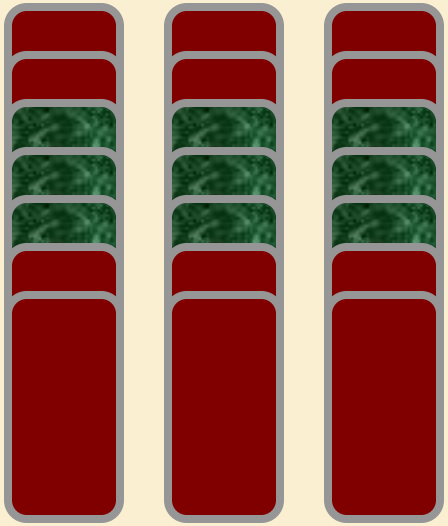
Selected column is placed in the middle

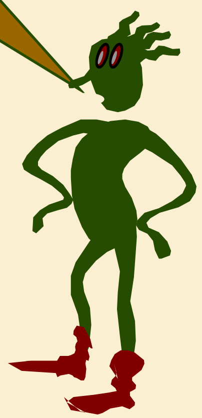


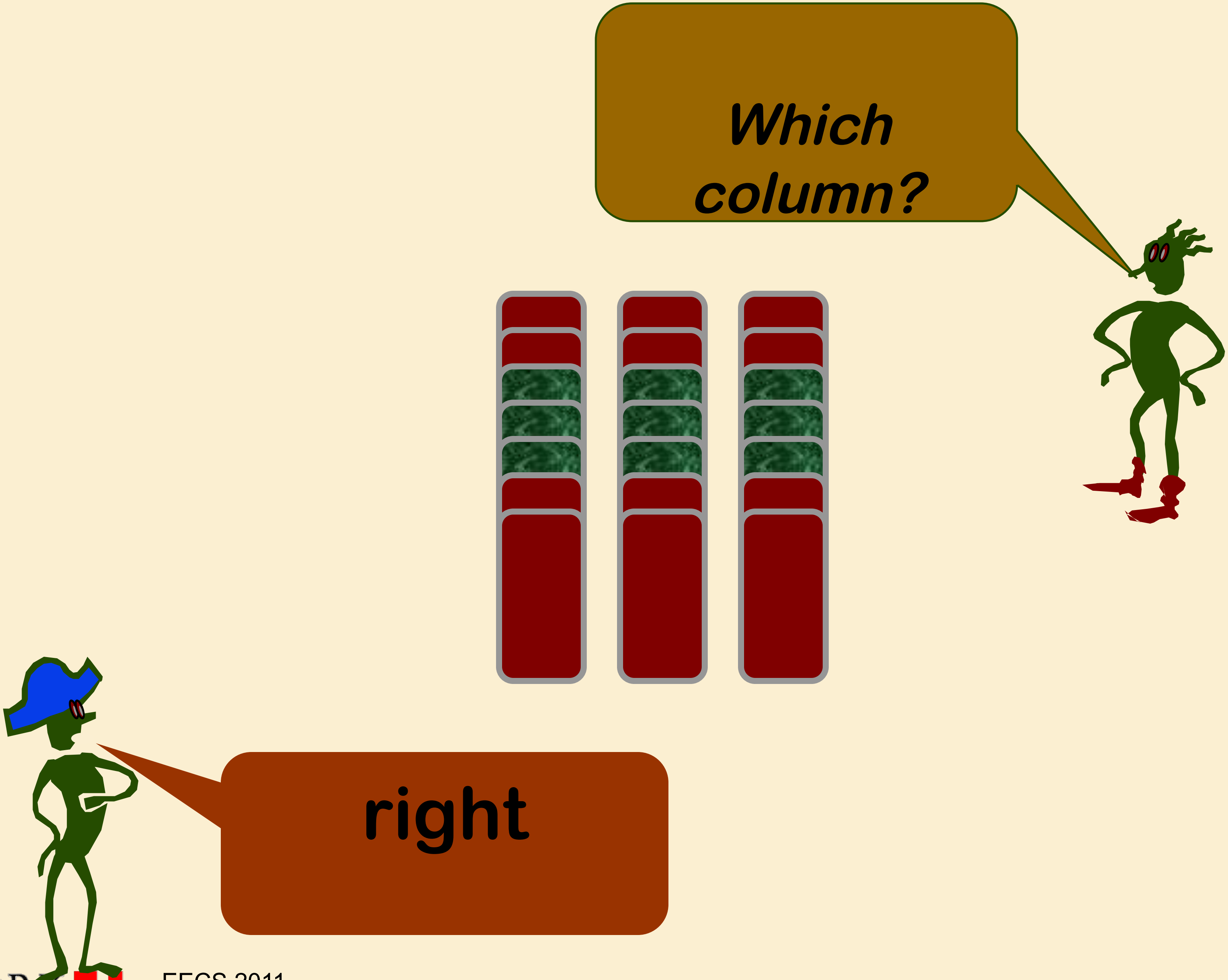



I will rearrange the cards

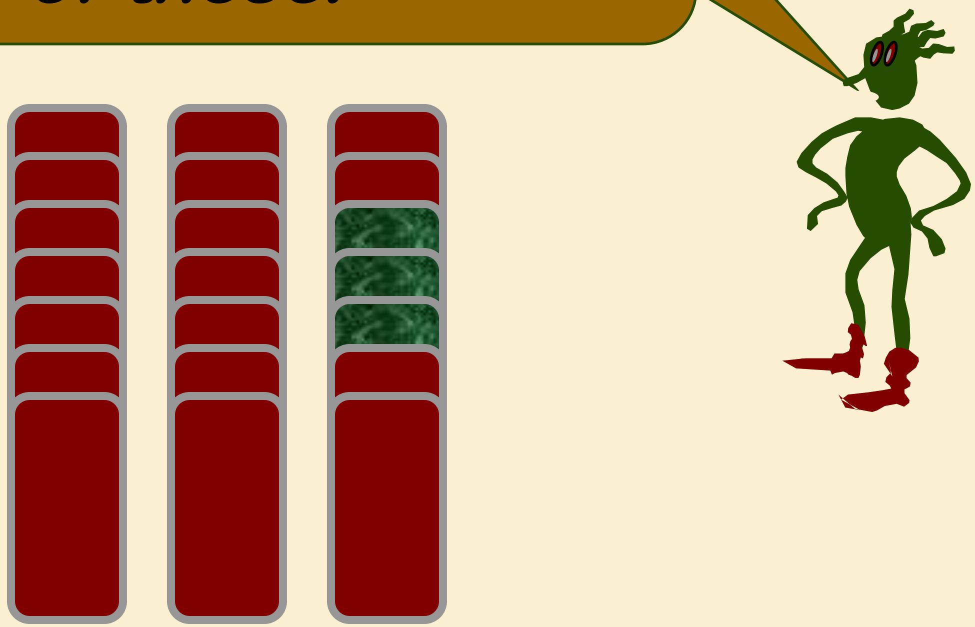
Relax Loop Invariant:
I will remember the same about each column.



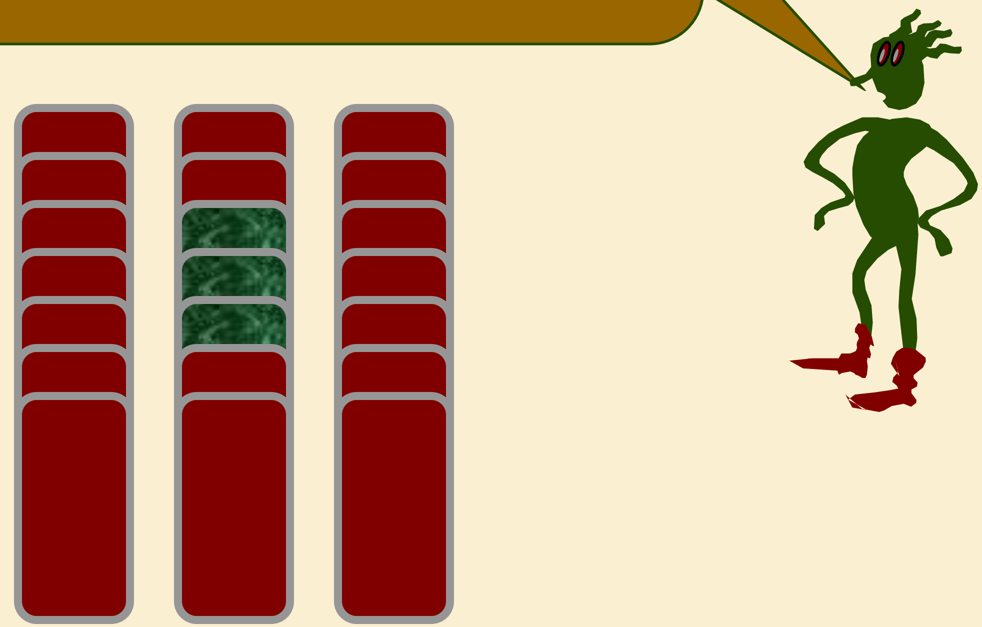


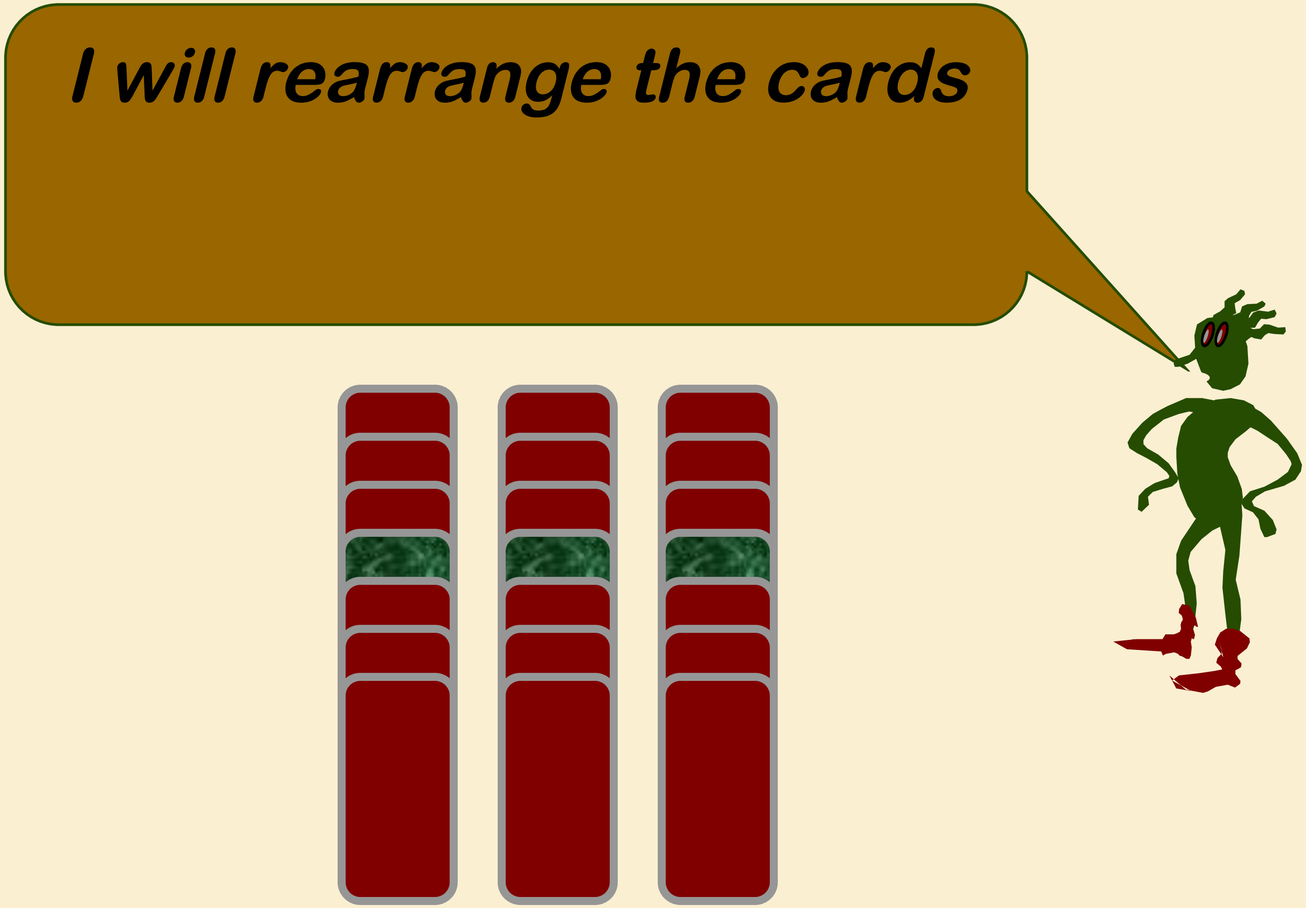

Which column?
right

Loop Invariant:
The selected card is one of these.



Selected column is placed in the middle

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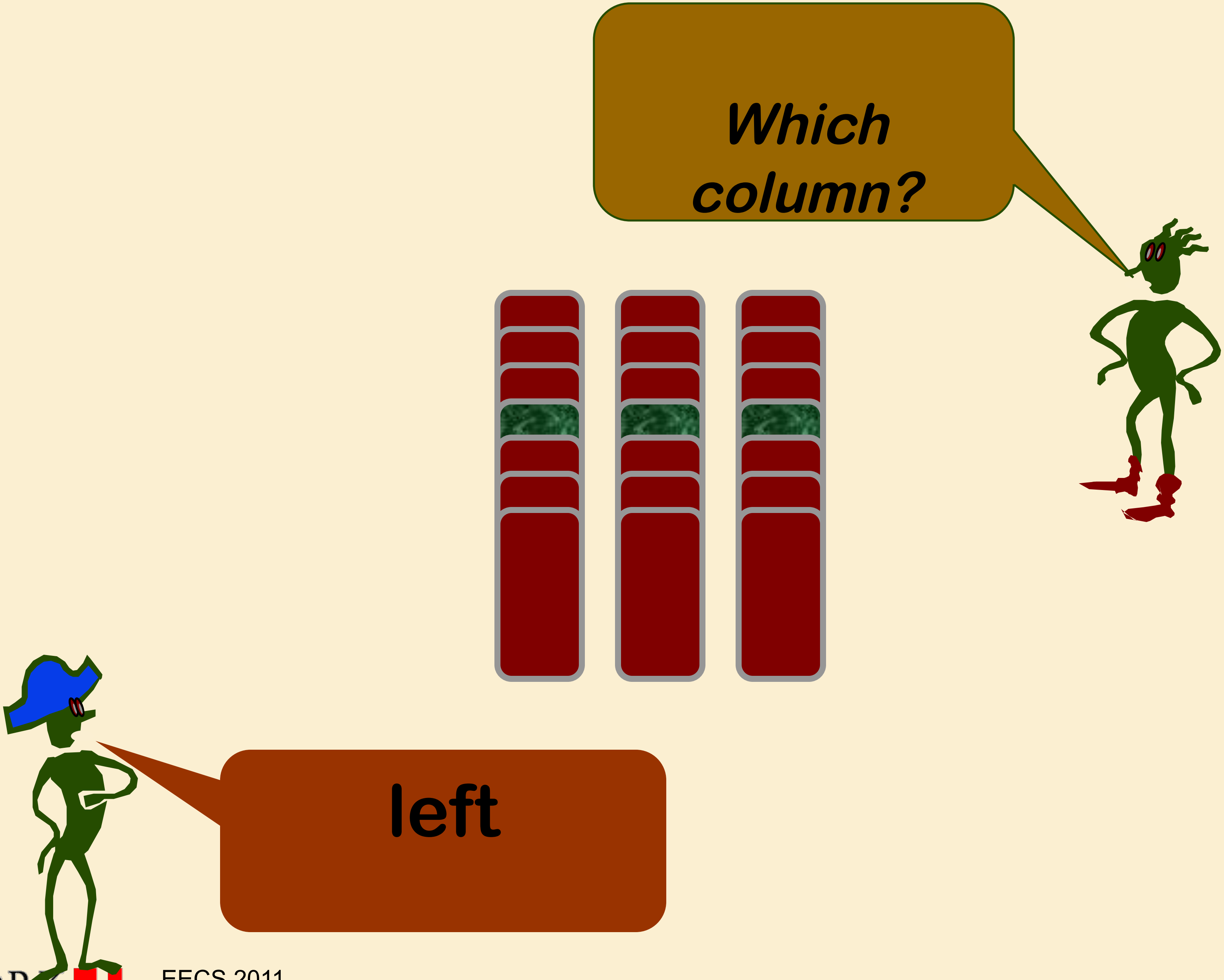
Which column?
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Loop Invariant:
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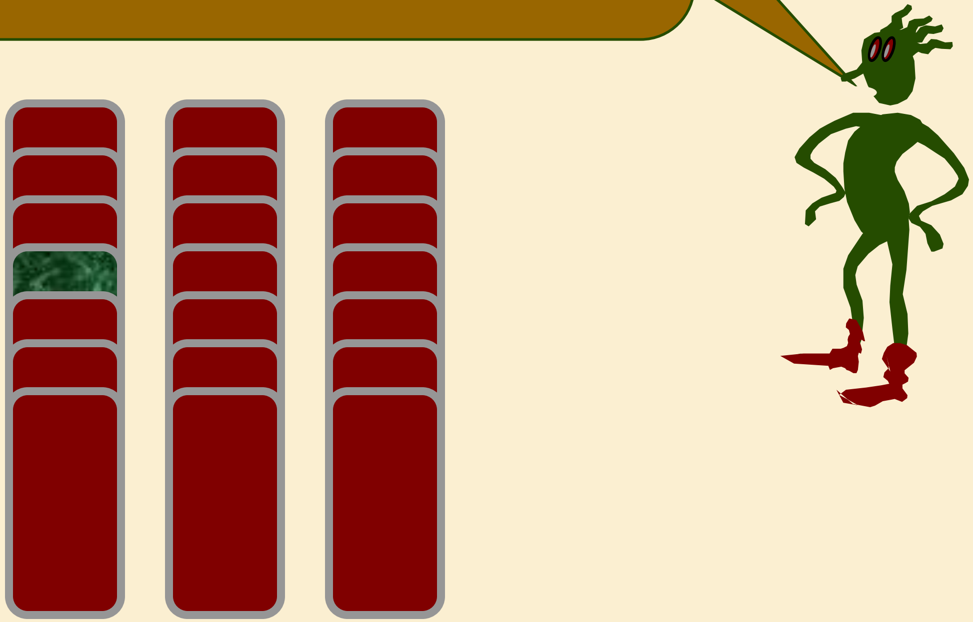
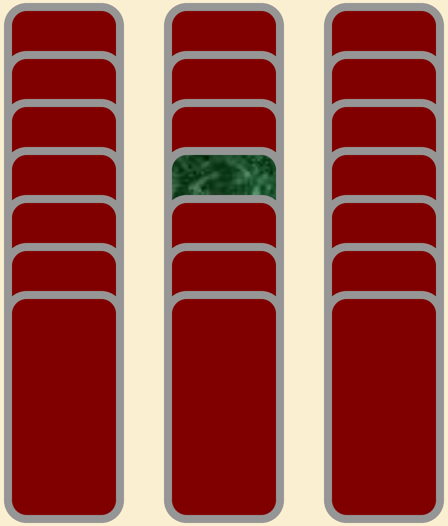
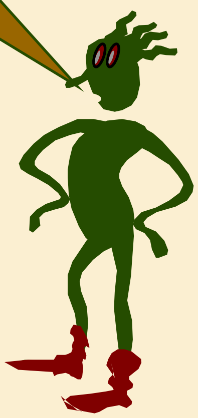
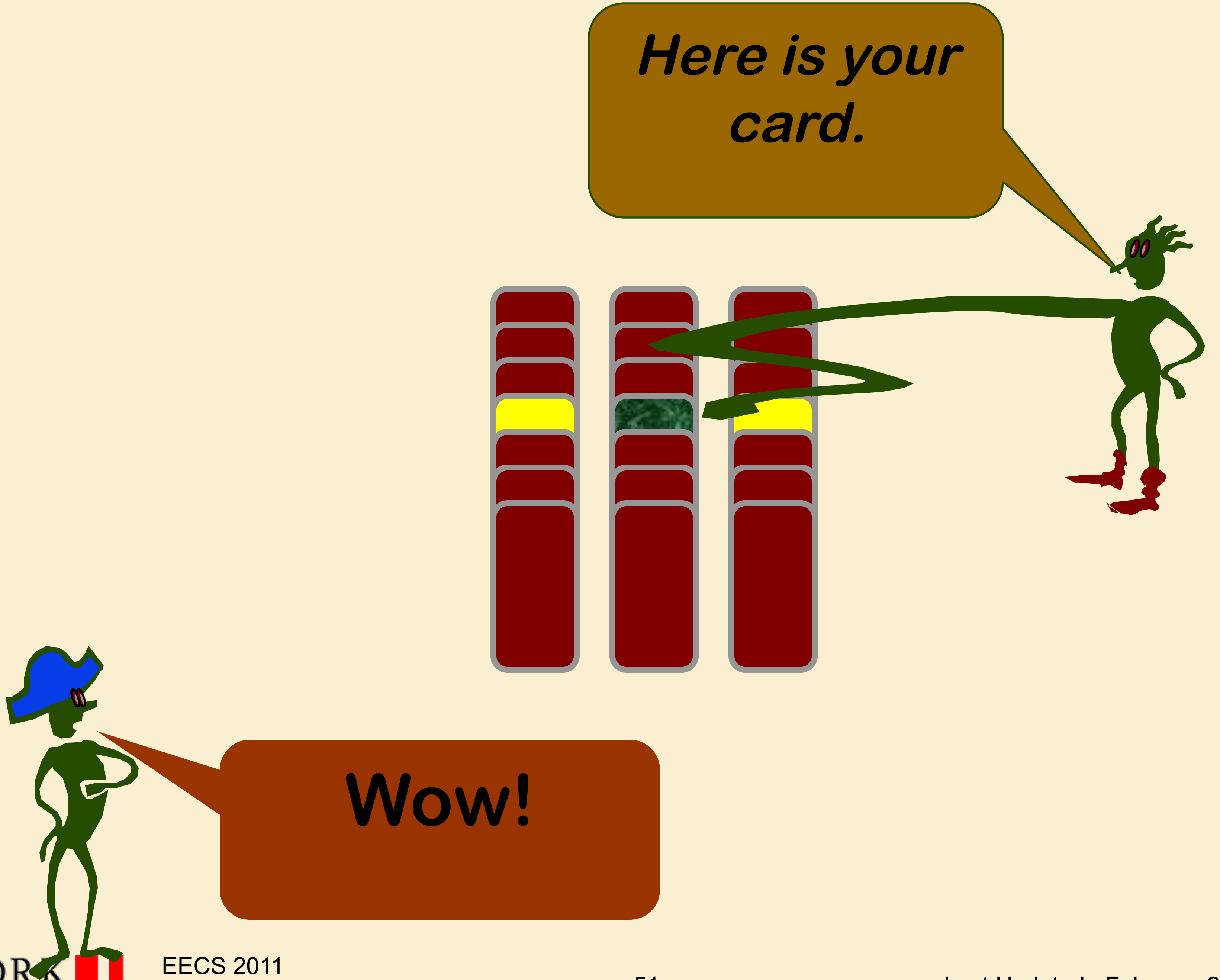


Selected column is placed in the middle






Here is your card.
Wow!

Ternary Search

- Loop Invariant: selected card in central subset of cards

  Size of subset = $\lceil n/3^i \rceil$
  Where
  n = total number of cards
  i = number of completed iterations

- How many iterations are required to guarantee success?

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