Minimum and Maximum

Problem: Find the maximum and the minimum of n elements.

- Naïve algorithm 1: Find the minimum, then find the maximum -- 2(n-1) comparisons.
- Naïve algorithm 2: Find the minimum, then find the maximum of n-1 elements -- (n-1) + (n-2) = 2n -3 comparisons.

Minimum and Maximum – better algorithms

Problem: Find the maximum and the minimum of n elements.

Approach 1

- Sort n/2 pairs. Find min of losers, max of winners.

comparisons: n/2 + n/2 –1 + n/2-1 = 3n/2 –2.

Is this the best possible?

Approach 2

- Divide into n/2 pairs. Compare the first pair, set winner to current max, loser to current min.
- Sort next pair, compare winner to current max, loser to current min.

#comparisons: 1 + 3(n/2 –1) = 3n/2 –2.

Lower bounds for the MIN and MAX

Claim: Every comparison-based algorithm for finding both the minimum and the maximum of n elements requires at least (3n/2)-2 comparisons.

Idea: Use similar argument as for the minimum

Max = maximum and Min=minimum only if:

Every element other than min has won at least 1

Every element other than max has lost at least 1

A proof?

“Proof” from the web: For each comparison, x<y, score a point if this is first comparison that x loses or if y wins and 2 points if both occur. Before the algorithm can terminate n-2 must both win and lose (since they aren't min or max) and 2 elements must either win or lose. Thus, 2(n-2)+2 points are scored before termination.

Define A to be the set of elements that have not won or lost a comparison. All comparisons between elements in A must score 2 points. All other comparisons can score at most 1 point. Let X be A-A comparisons. Let Y be number of other comparisons. We want to minimize X+Y such that $2X+Y \geq 2n-2$ & $X \leq n/2$ (assume n is even). Given the constraints we want to make X as big as possible. So set X=n/2. Then $Y \geq 2n-2-2X \Rightarrow Y \geq 2n-2-n \Rightarrow Y \geq n-2 \Rightarrow X+Y \geq n/2 + n - 2$.

Is the previous proof correct?

Lower bounds for the MIN and MAX

Idea: Define 4 sets:

U: has not participated in a comparison

W: has won all comparisons

L: has lost all comparisons

N: has won and lost at least one comparison

Note: All these sets are disjoint.

1. Initially all elements in U.
2. Finally no elements in U, 1 each in W,L and n-2 in N.
3. Each element in N comes from U via W or L.

Lower bounds for the MIN and MAX - contd

Idea: Score a point when an element enters W or L or N for the first time.

Question: Can we ensure that only U-U comparisons result in two points being scored?

Answer: YES! The adversary argument!

The adversary constructs a worst-case input by revealing as little as possible about the inputs.

Lower bounds for the MIN and MAX - contd

Adversary strategy:

U-U: any

U-W: make element of W winner

U-L: make element of L loser

U-N: any

W-W: any (be consistent with before)

W-L/N: make element of W winner

L-L: any (be consistent with before)

L-N: make element of L loser

Lower bounds for the MIN and MAX – contd.

We need to score 2n–2 points. At most n/2 U-U comparisons can be made – gives n points.

To move n-2 elements to N, we need another n-2 comparisons.

Next: Linear sorting

Q: Can we beat the Ω(n log n) lower bound for sorting?

A: In general no, but in some special cases YES!

Ch 7: Sorting in linear time

Non-Comparison Sort – Bucket Sort

- Assumption: uniform distribution
  - Input numbers are uniformly distributed in [0,1).
  - Suppose input size is n.
- Idea:
  - Divide [0,1) into n equal-sized subintervals (buckets).
  - Distribute n numbers into buckets
  - Expect that each bucket contains few numbers.
  - Sort numbers in each bucket (insertion sort as default).
  - Then go through buckets in order, listing elements

Can be shown to run in linear-time <u>on average</u>

Example of BUCKET-SORT

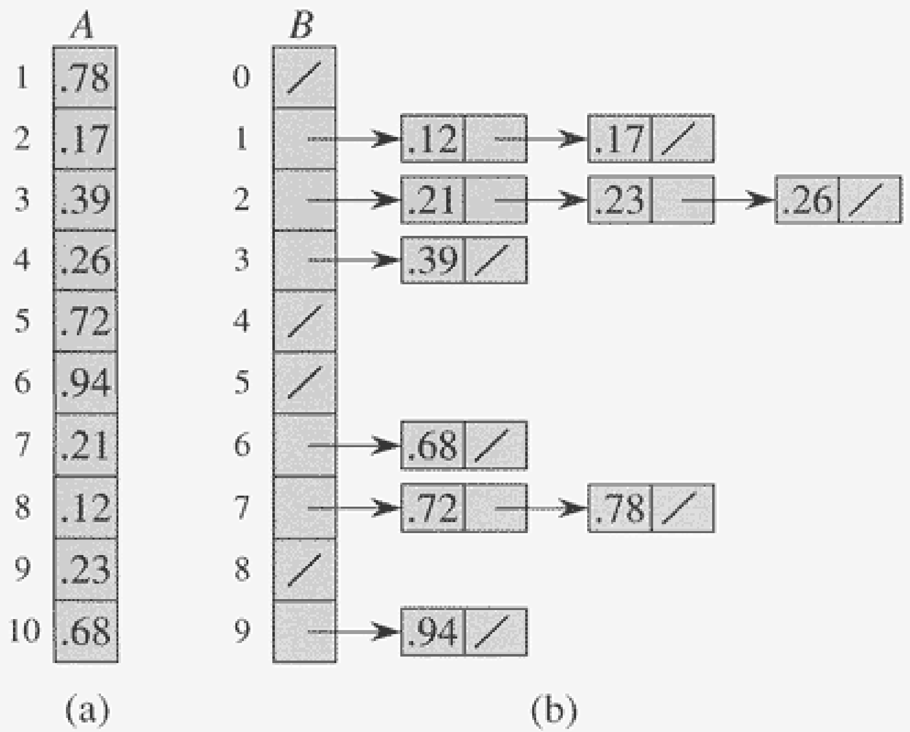


Figure 8.4 The operation of BUCKET-SORT. **(a)** The input array $A[1..10]$. **(b)** The array $B[0..9]$ of sorted lists (buckets) after line 5 of the algorithm. Bucket i holds values in the half-open interval $[i/10, (i+1)/10)$. The sorted output consists of a concatenation in order of the lists $B[0], B[1], \ldots, B[9]$.

Bucket Sort - generalizations

- What if input numbers are NOT uniformly distributed?
- What if the distribution is not known a priori?

Non-Comparison Sort – Counting Sort

- Assumption: n input numbers are integers in the range [0,k], k=O(n).
- Idea:
  - Determine the number of elements less than x, for each input x.
  - Place x directly in its position.

Counting Sort - pseudocode

Counting-Sort(A,B,*k*)

- **for** *i*←0 **to** k
- do C[*i*] ←0
- **for** *j* ←1 **to** length[A]
- **do** C[A[*j*]] ←C[A[*j*]]+1
- // C[*i*] contains number of elements equal to *i*.
- **for** *i* ←1 **to** *k*
- **do** C[*i*]=C[*i*]+C[*i*-1]
- // C[*i*] contains number of elements ≤ *i*.
- **for** *j* ←length[A] **downto** 1
- **do** B[C[A[*j*]]] ←A[*j*]
- C[A[*j*]] ←C[A[*j*]]-1

Counting Sort - example

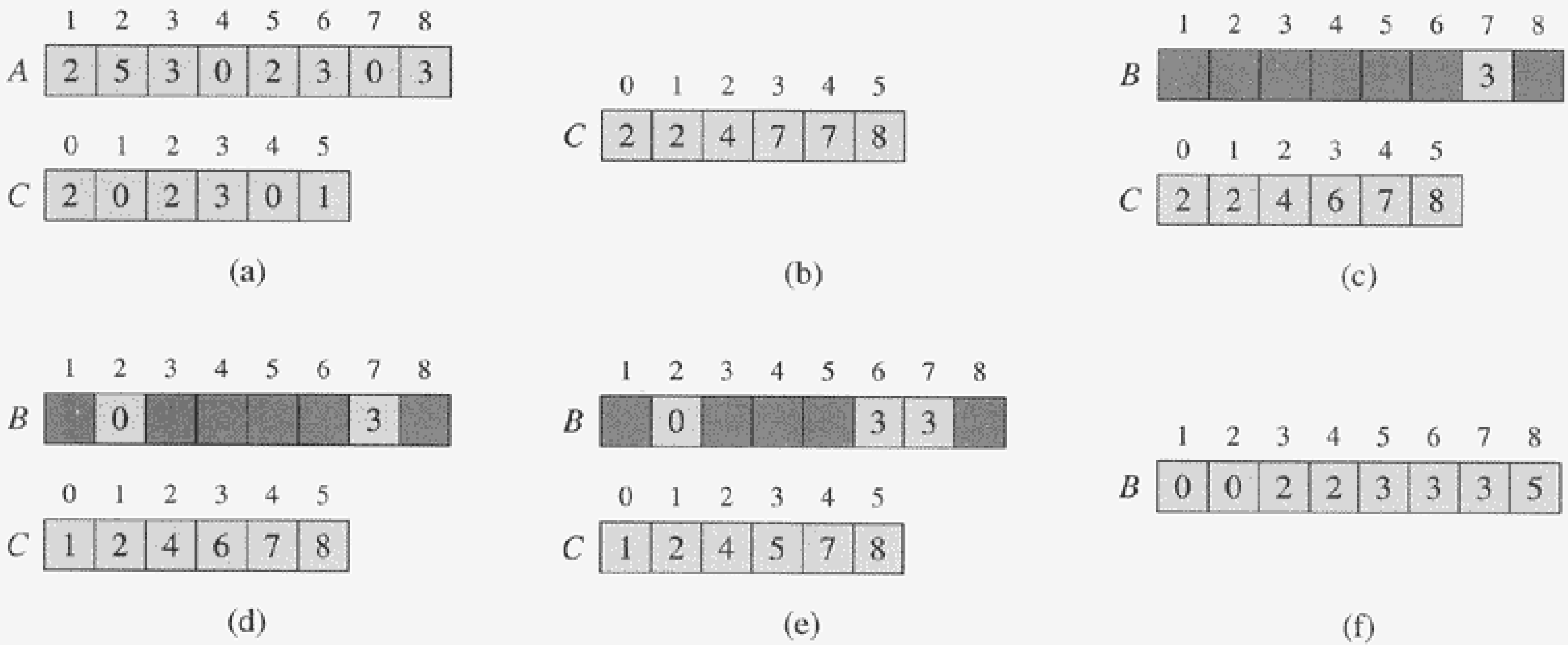


Figure 8.2 The operation of COUNTING-SORT on an input array $A[1..8]$, where each element of A is a nonnegative integer no larger than $k = 5$. **(a)** The array A and the auxiliary array C after line 4. **(b)** The array C after line 7. **(c)–(e)** The output array B and the auxiliary array C after one, two, and three iterations of the loop in lines 9–11, respectively. Only the lightly shaded elements of array B have been filled in. **(f)** The final sorted output array B.

Counting Sort - analysis

for $i \leftarrow 0$ **to** k	$\Theta(k)$
do $C[i] \leftarrow 0$	$\Theta(1)$
for $j \leftarrow 1$ **to** length[A]	$\Theta(n)$
do $C[A[j]] \leftarrow C[A[j]]+1$	$\Theta(1)$ $(\Theta(1)\ \Theta(n)=\Theta(n))$
// $C[i]$ contains number of elements equal to i. $\Theta(0)$	
for $i \leftarrow 1$ **to** k	$\Theta(k)$
do $C[i]=C[i]+C[i-1]$	$\Theta(1)$ $(\Theta(1)\ \Theta(n)=\Theta(n))$
// $C[i]$ contains number of elements $\leq i$.	$\Theta(0)$
for $j \leftarrow$ length[A] **downto** 1 $\Theta(n)$	
do $B[C[A[j]]] \leftarrow A[j]$	$\Theta(1)$ $(\Theta(1)\ \Theta(n)=\Theta(n))$
$C[A[j]] \leftarrow C[A[j]]-1$	$\Theta(1)$ $(\Theta(1)\ \Theta(n)=\Theta(n))$

Total cost is $\Theta(k+n)$, suppose $k=O(n)$, then total cost is $\Theta(n)$.

So, it beats the Ω(n log n) lower bound!

Stable sort

- Preserves order of elements with the same key.
- Counting sort is stable.

Crucial question: can counting sort be used to sort large integers efficiently?

Radix sort

Radix-Sort(A,d)

- **for** i$\leftarrow$1 **to** d
- **do** use a stable sort to sort A on digit i

Analysis:

Given n d-digit numbers where each digit takes on up to k values, Radix-Sort sorts these numbers correctly in $\Theta(d(n+k))$ time.

Radix sort - example

1019	2231	1019	1019	1019
3075	3075	2225	3075	2225
2225	2225	2231	2225	2231
2231	1019	3075	2231	3075

Sorted!

1019	1019
3075	2231
2231	2225
2225	3075

Not sorted!

Next: Medians and Order Statistics (Ch. 9)

Order statistics: The i^{th} order statistic of n elements $S=\{a_1, a_2,\ldots, a_n\}$: i^{th} smallest elements

- Minimum and maximum, Median
- finding the kth largest element in an unsorted array.

Already seen:

1. $k=1$: $\Theta(n)$ algorithm optimal.
2. Also, Heapify + Extract-max: $\Theta(n)$ algorithm.
   Same bounds hold for any constant k.
3. Sorting solves it for any k. $\Theta(n \log n)$ algorithm.

What about $k=n/2$? Can we do better than $\Theta(n \log n)$ algorithm?

Medians and Order Statistics

To select the i^{th} smallest element of $S=\{a_1, a_2,\ldots, a_n\}$

- Can we use PARTITION?
  - if we are very lucky, we will get it in the first try!
  - otherwise we should have a smaller set to recurse on.

- No guarantee of being lucky!

  How can we guarantee a significantly smaller set?

The algorithm is the most complicated divide-and-conquer algorithm in this course!

Order Statistics

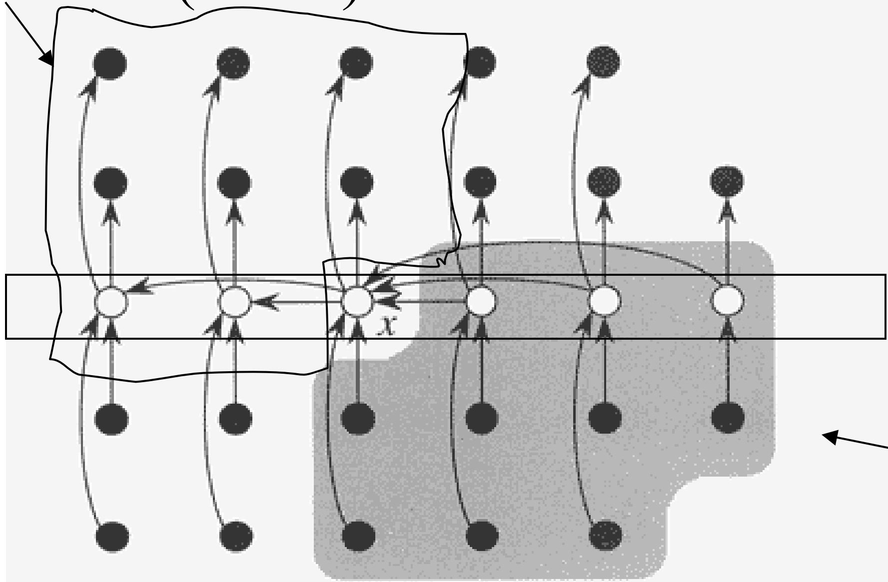
1. Divide n elements into $\lceil n/5 \rceil$ groups of 5 elements.
2. Find the median of each group.
3. Use SELECT recursively to find the median x of the above $\lceil n/5 \rceil$ medians.
4. Partition using x as pivot, and find position k of x.
5. If i=k return

   else recurse on the appropriate subarray.

What kind of split does this produce?

The Way to Select x

At least $(3n/10)$-6 elements $<x$



Divide elements into $\lceil n/5 \rceil$ groups of 5 elements each.
Find the median of each group
Find the median of the medians

At least $(3n/10)$-6 elements $>x$

Figure 9.1 Analysis of the algorithm SELECT. The n elements are represented by small circles, and each group occupies a column. The medians of the groups are whitened, and the median-of-medians x is labeled. (When finding the median of an even number of elements, we use the lower median.) Arrows are drawn from larger elements to smaller, from which it can be seen that 3 out of every full group of 5 elements to the right of x are greater than x, and 3 out of every group of 5 elements to the left of x are less than x. The elements greater than x are shown on a shaded background.

Analysis of SELECT

- Steps 1,2,4 take O(n),
- Step 3 takes $T(\lceil n/5 \rceil)$.
- Let us see step 5:

  - At least half of medians in step 2 are $\geq x$, thus at least $\lceil 1/2 \lceil n/5 \rceil \rceil -2$ groups contribute 3 elements which are $\geq x$.

  i.e, $3(\lceil 1/2 \lceil n/5 \rceil \rceil -2) \geq (3n/10)-6$.

  - Similarly, the number of elements $\leq x$ is also at least $(3n/10)-6$.
    - Thus, $|S_1|$ is at most $(7n/10)+6$, similarly for $|S_3|$.
    - Thus SELECT in step 5 is called recursively on at most $(7n/10)+6$ elements.
- Recurrence is:

$$T(n) = \begin{cases} O(1) & \text{if } n < 140 \\ T(\lceil n/5 \rceil) + T(7n/10+6) + O(n) & \text{if } n \geq 140 \end{cases}$$

Solve recurrence by substitution

- Suppose $T(n) \leq cn$, for some c.
- $T(n) \leq c\lceil n/5 \rceil + c(7n/10+6) + an$

  $\leq cn/5 + c + 7/10cn + 6c + an$

  $= 9/10cn + an + 7c$

  $= cn + (-cn/10 + an + 7c)$

  – Which is at most cn if $-cn/10+an+7c<0$.
  – i.e., $c \geq 10a(n/(n-70))$ when $n>70$.

  – So select $n=140$, and then $c \geq 20a$.

Note: n may not be 140, any integer >70 is OK.

Implication for Quicksort

- Worst case improves to O(n log n) BUT…

Test your understanding

1. Problem 9.3-7: Describe an O(n) algorithm that, given a set S of n distinct numbers and a positive integer k <= n, determines the k numbers in S that are closest to the median of S.

2. Problem 9.3-8: Let X[1..n], Y[1..n] be two sorted arrays. Give an O(lg n) algorithm to find the median of all 2n elements in arrays X,Y.