Next…

1. Covered basics of a simple design technique (Divide-and-conquer) – Ch. 2 of the text.
2. Next, more sorting algorithms.

Sorting

Switch from design paradigms to applications.
Sorting and order statistics (Ch 6 – 9).

First:

Heapsort

–Heap *data structure* and priority queue *ADT*

Quicksort

–a popular algorithm, very fast on average

Why Sorting?

"When in doubt, sort" – one of the principles of algorithm design. Sorting used as a subroutine in many of the algorithms:

- Searching in databases: we can do binary search on sorted data
- A large number of computer graphics and computational geometry problems
- Closest pair, element uniqueness

- A large number of sorting algorithms are developed representing different algorithm design techniques.
- A lower bound for sorting $\Omega(n \log n)$ is used to prove lower bounds of other problems.

Sorting algorithms so far

- Insertion sort, selection sort
  - Worst-case running time $\Theta(n^2)$; in-place
- Merge sort
  - Worst-case running time $\Theta(n \log n)$, but requires additional memory $\Theta(n)$; (WHY?)

Selection sort

```
Selection-Sort(A[1..n]):
   For i → n downto 2
A:     Find the largest element among A[1..i]
B:     Exchange it with A[i]
```

- A takes $\Theta(n)$ and B takes $\Theta(1)$: $\Theta(n^2)$ in total
- Idea for improvement: use a *data structure*, to do both A and B in *O(lg n)* time, balancing the work, achieving a better trade-off, and a total running time *O(n log n)*.

Heap sort

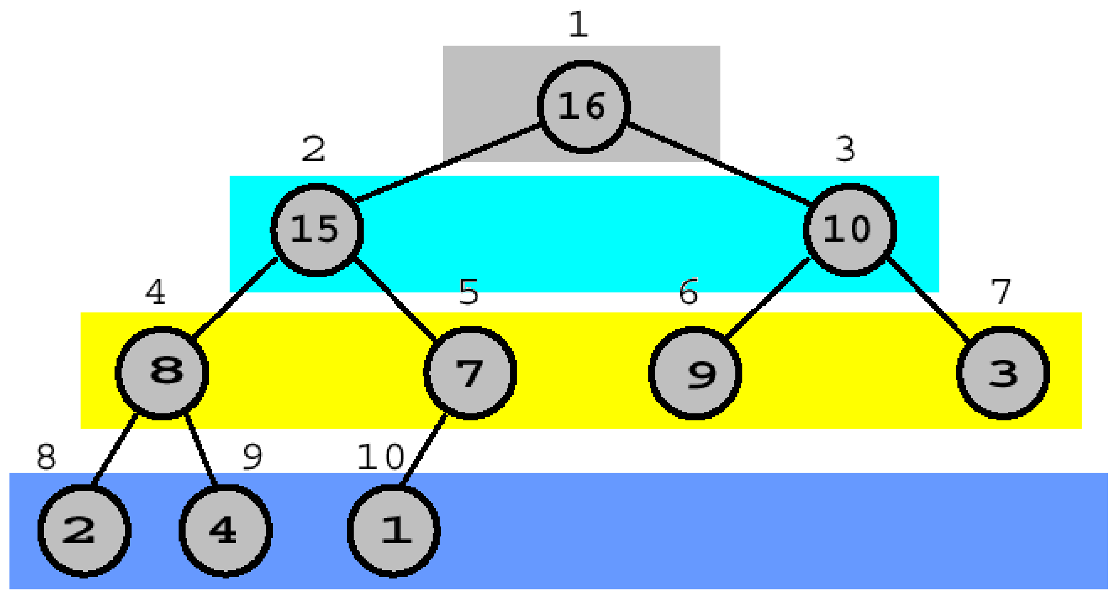
- Binary heap data structure *A*
  - array
  - Can be viewed as a nearly complete binary tree
    - All levels, except the lowest one are completely filled
  - The key in root is greater or equal than all its children, and the left and right subtrees are again binary heaps
- Two attributes
  - length[*A*]
  - heap-size[*A*]

Heap sort

Parent (*i*)
return $\lfloor i/2 \rfloor$

Left (*i*)
return $2i$

Right (*i*)
return $2i+1$



Heap property:

$A[\text{Parent}(i)] \geq A[i]$

1	2	3	4	5	6	7	8	9	10
16	15	10	8	7	9	3	2	4	1

Level: 3 2 1 0

Heap sort

- Notice the implicit tree links; children of node i are $2i$ and $2i+1$
- Why is this useful?
  - In a binary representation, a multiplication/division by two is left/right shift
  - Adding 1 can be done by adding the lowest bit

Heapify

- i is index into the array A
- Binary trees rooted at Left(i) and Right(i) are heaps
- But, A[i] might be smaller than its children, thus violating the heap property
- The method Heapify makes A a heap once more by moving A[i] down the heap until the heap property is satisfied again

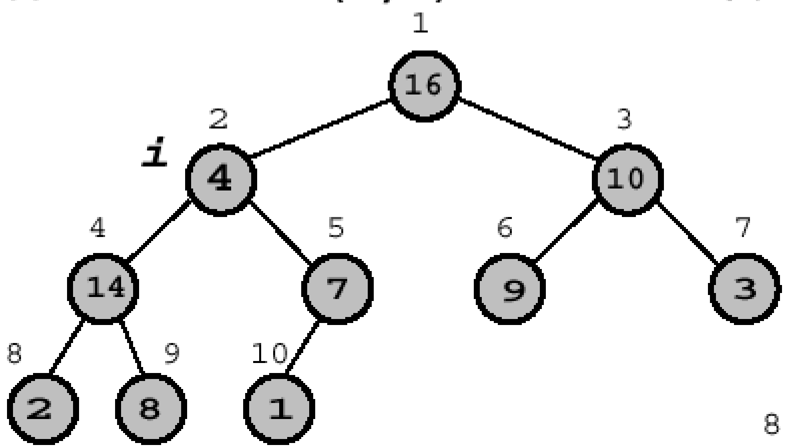
Heapify

n is total number of elements

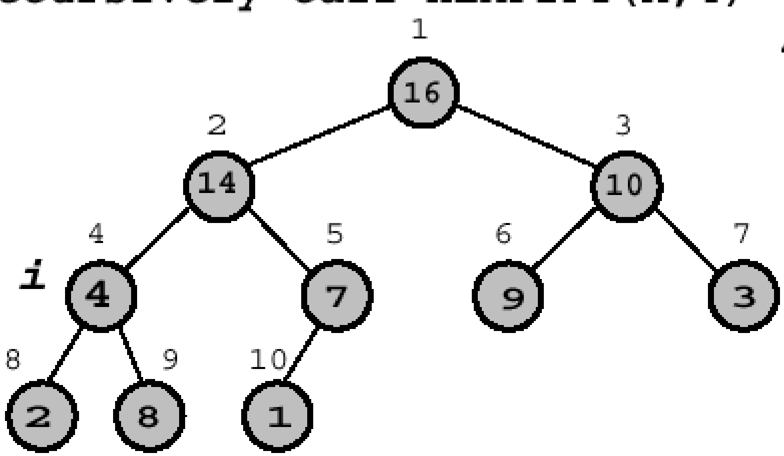
```
HEAPIFY(A, i)
  ▷ Left & Right subtrees of i are heaps.
  ▷ Makes subtree rooted at i a heap.
  l ← LEFT(i)              ▷ l = 2i
  r ← RIGHT(i)             ▷ r = 2i + 1
  if l ≤ n and A[l] > A[i]
      then largest ← l
      else largest ← i
  if r ≤ n and A[r] > A[largest]
      then largest ← r
  if largest ≠ i
      then exchange A[i] ↔ A[largest]
           HEAPIFY(A, largest)
```

Heapify Example

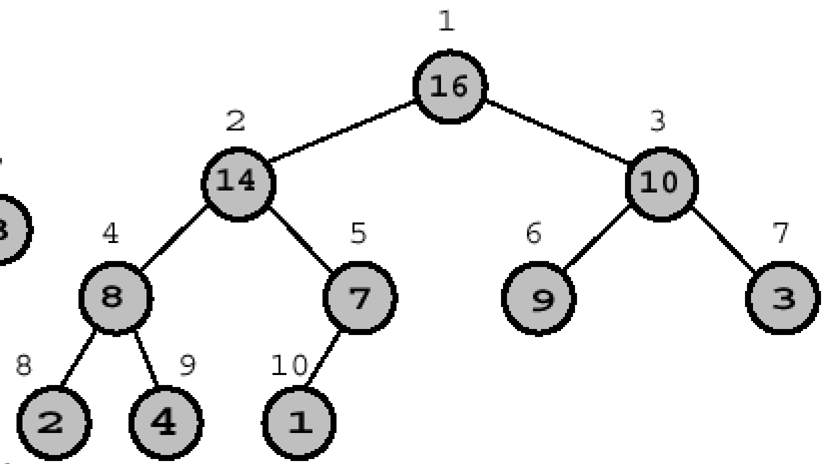
1. Call HEAPIFY(A,2)



2. Exchange A[2] with A[4] and
recursively call HEAPIFY(A,4)



3. Exchange A[4] with A[9] and
recursively call HEAPIFY(A,9)



4. Node 9 has no children,
so we are done.

Heapify: Running time

- The running time of Heapify on a subtree of size n rooted at node i is
  - determining the relationship between elements: $\Theta(1)$
  - plus the time to run Heapify on a subtree rooted at one of the children of i, where 2n/3 is the worst-case size of this subtree.
  - Alternatively
    - Running time on a node of height h: O(h)

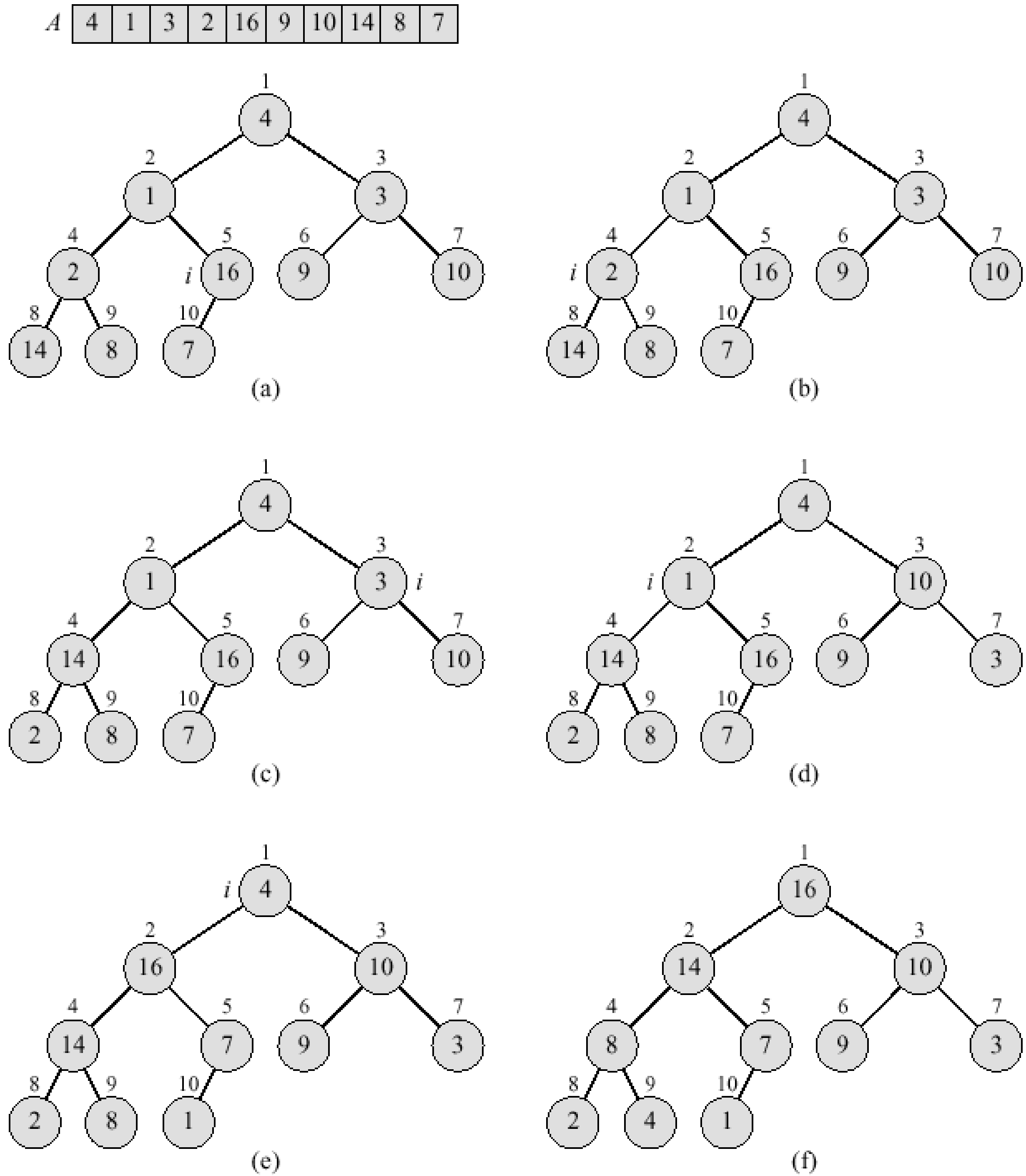
$$T(n) \le T(2n/3) + \Theta(1) \quad \Rightarrow \quad T(n) = O(\log n)$$

Building a Heap

- Convert an array A[1...n], where n = length[A], into a heap
- Notice that the elements in the subarray A[($\lfloor n/2 \rfloor$ + 1)...n] are already 1-element heaps to begin with!

```
BUILD-HEAP(A)
  for i ← ⌊n/2⌋ downto 1
        do HEAPIFY(A, i)
```

Building a heap



Building a Heap: Analysis

- Correctness: induction on i, all trees rooted at m > i are heaps
- Running time: less than n calls to Heapify = n O(lg n) = O(n lg n)
- Good enough for an O(n lg n) bound on Heapsort, but sometimes we build heaps for other reasons, would be nice to have a tight bound
  - Intuition: for most of the time Heapify works on smaller than n element heaps

Building a Heap: Analysis (2)

- Definitions
  - height of node: longest path from node to leaf
  - height of tree: height of root

```
BUILD-HEAP(A)
for i ← ⌊n/2⌋ downto 1
    do HEAPIFY(A, i)
```

  - time to Heapify = O(height of subtree rooted at i)
  - assume n = 2^k – 1 (a complete binary tree k = $\lfloor lg\ n \rfloor$)

$$
\begin{aligned}
T(n) &= O\left(\frac{n+1}{2} + \frac{n+1}{4}\cdot 2 + \frac{n+1}{8}\cdot 3 + \ldots + 1\cdot k\right) \\
&= O\left((n+1)\cdot \sum_{i=1}^{\lfloor \lg n \rfloor} \frac{i}{2^i}\right) \text{ since } \sum_{i=1}^{\lfloor \lg n \rfloor} \frac{i}{2^i} = \frac{1/2}{(1-1/2)^2} = 2 \\
&= O(n)
\end{aligned}
$$

Building a Heap: Analysis (3)

- How? By using the following "trick"

$$\sum_{i=0}^{\infty} x^i = \frac{1}{1-x} \text{ if } |x|<1 \quad \text{//differentiate}$$

$$\sum_{i=1}^{\infty} i \cdot x^{i-1} = \frac{1}{(1-x)^2} \quad \text{//multiply by } x$$

$$\sum_{i=1}^{\infty} i \cdot x^{i} = \frac{x}{(1-x)^2} \quad \text{//plug in } x = \frac{1}{2}$$

$$\sum_{i=1}^{\infty} \frac{i}{2^i} = \frac{1/2}{1/4} = 2$$

- Therefore Build-Heap time is O(n)

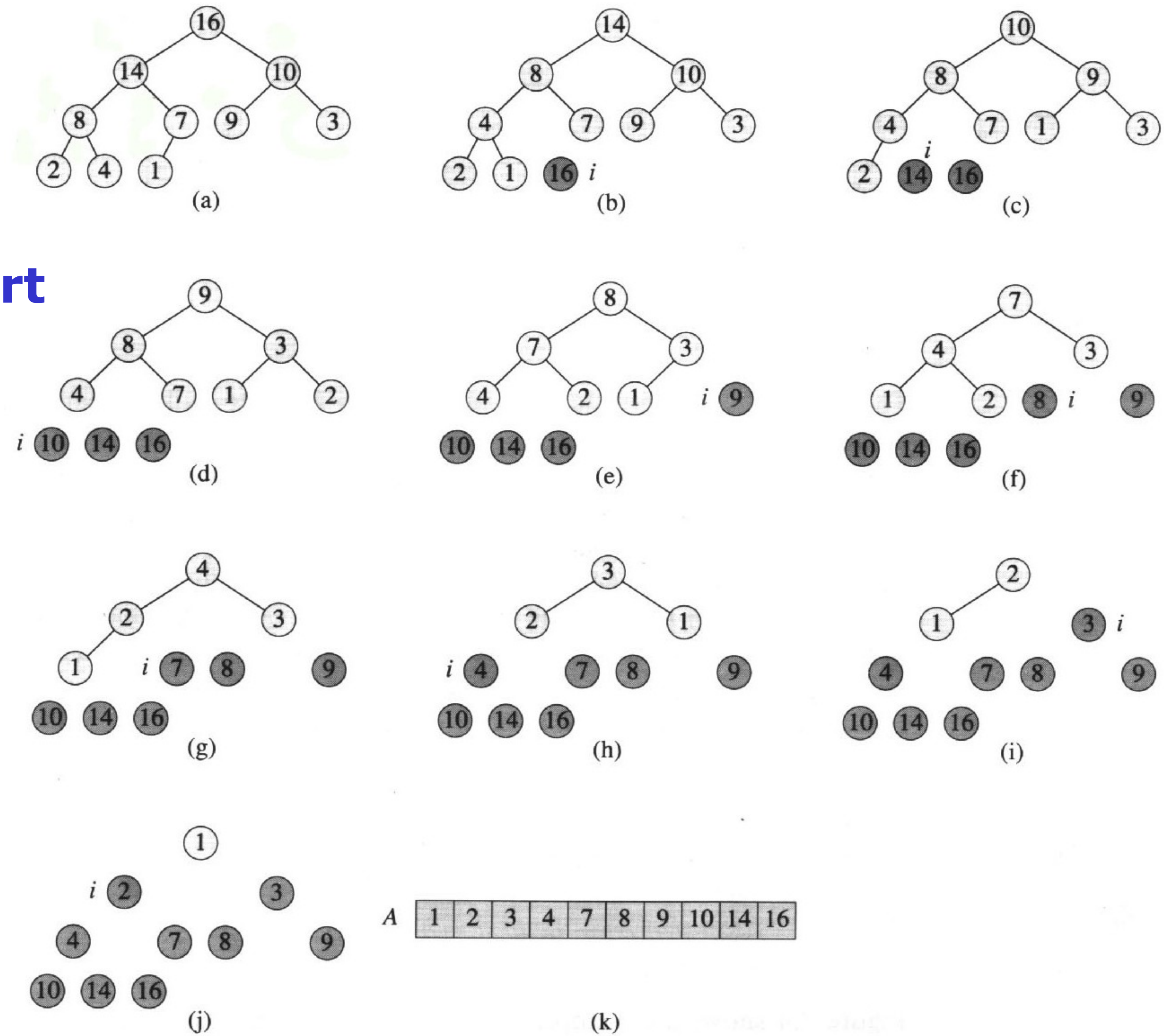
Heap sort

$\text{HEAPSORT}(A)$	**Analysis**
$\text{BUILD-HEAP}(A)$	?? $O(n)$
for $i \leftarrow n$ **downto** 2	n times
do exchange $A[1] \leftrightarrow A[i]$	$O(1)$
$n \leftarrow n - 1$	$O(1)$
$\text{HEAPIFY}(A, 1)$	$O(\lg n)$

The total running time of heap sort is O(n lg n)

+ Build-Heap(A) time, which is O(n)

Heap sort



Heap Sort: Summary

- Heap sort uses a heap data structure to improve selection sort and make the running time asymptotically optimal
- Running time is O(n log n) – like merge sort, but unlike selection, insertion, or bubble sorts
- Sorts in place – like insertion, selection or bubble sorts, but unlike merge sort

Priority Queues

- A priority queue is an ADT(abstract data type) for maintaining a set S of elements, each with an associated value called key
- A PQ supports the following operations
  - Insert(S,x) insert element x in set S $(S \leftarrow S \cup \{x\})$
  - Maximum(S) returns the element of S with the largest key
  - Extract-Max(S) returns and removes the element of S with the largest key

Priority Queues (2)

- Applications:
  - job scheduling shared computing resources (Unix)
  - Event simulation
  - As a building block for other algorithms
- A Heap can be used to implement a PQ

Priority Queues(3)

- Removal of max takes constant time on top of Heapify $\Theta(\lg n)$

```
HEAP-EXTRACT-MAX(A)
    ▷ Removes and returns largest element of A
    max ← A[1]
    A[1] ← A[n]
    n ← n − 1
    HEAPIFY(A, 1)            ▷ Remakes heap
    return max
```

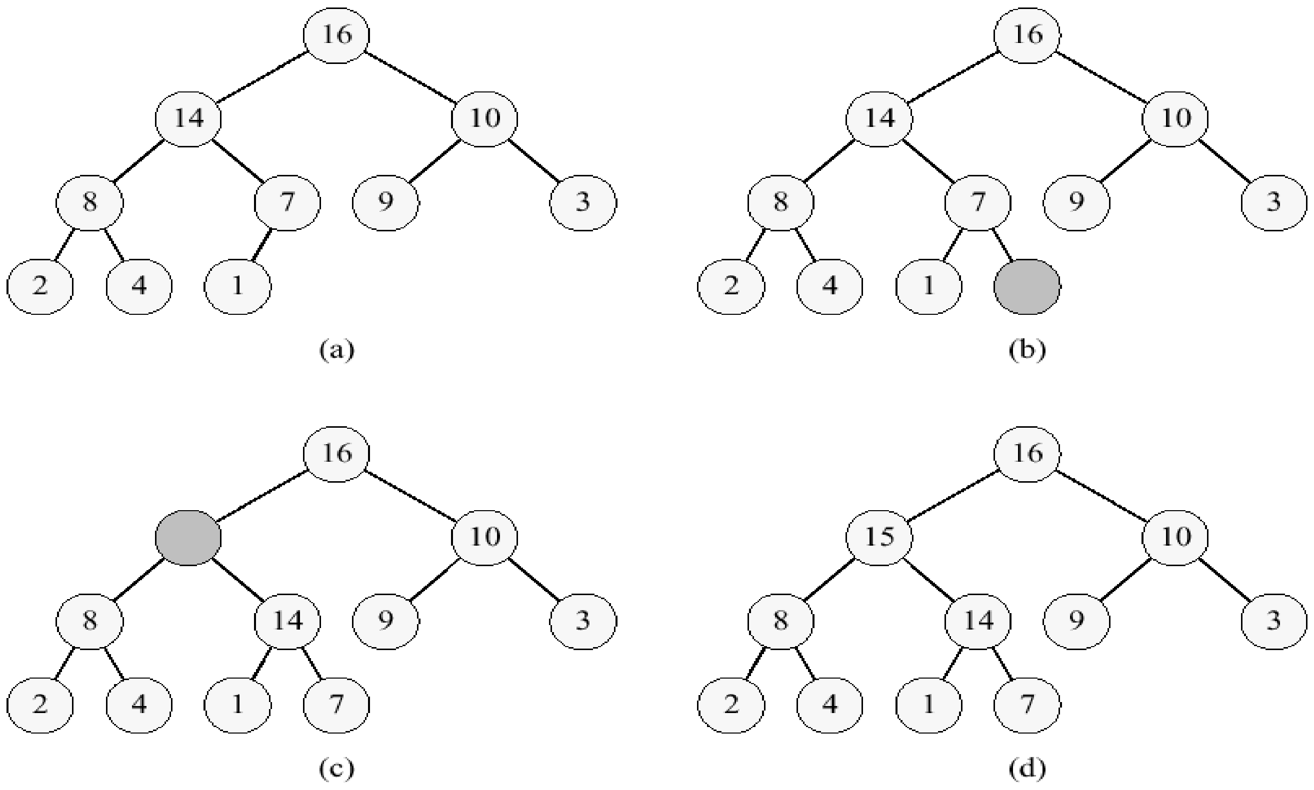
Priority Queues(4)

- Insertion of a new element
  - enlarge the PQ and propagate the new element from last place "up" the PQ
  - tree is of height lg n, running time: $\Theta(\lg n)$

```
HEAP-INSERT(A, key)
heap-size[A] ← heap-size[A] + 1
i ← heap-size[A]
while i > 1 and A[PARENT(i)] < key
        do A[i] ← A[PARENT(i)]
            i ← PARENT(i)
A[i] ← key
```

Priority Queues(5)

Insert a new element: 15



Quick Sort

- Characteristics
  - sorts "almost" in place, i.e., does not require an additional array, like insertion sort
  - Divide-and-conquer, like merge sort
  - very practical, average sort performance O(n log n) (with small constant factors), but worst case $O(n^2)$ [CAVEAT: this is true for the CLRS version]

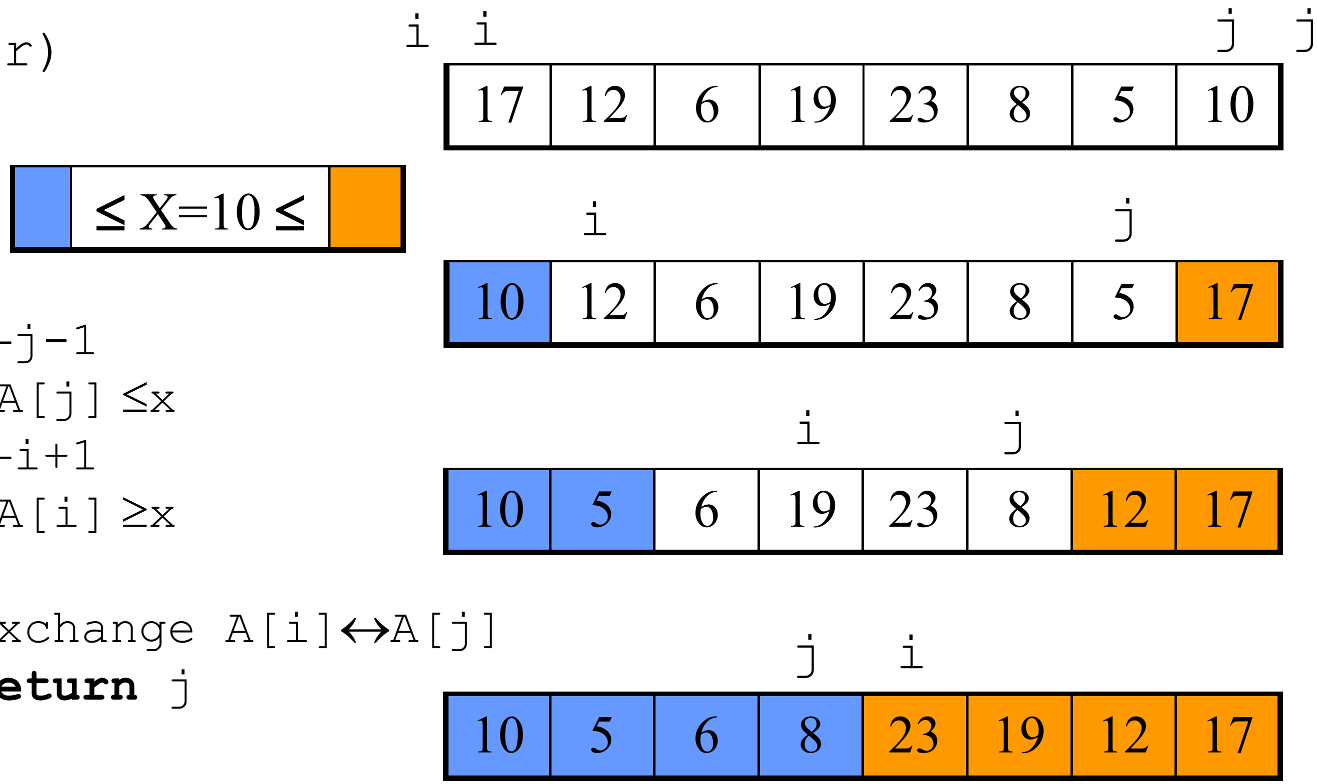
Quick Sort – the main idea

- To understand quick-sort, let's look at a high-level description of the algorithm
- A divide-and-conquer algorithm
  - Divide: partition array into 2 subarrays such that elements in the lower part <= elements in the higher part
  - Conquer: recursively sort the 2 subarrays
  - Combine: trivial since sorting is done in place

Partitioning

- Linear time partitioning procedure

```
Partition(A,p,r)
x←A[r]
i←p-1
j←r+1
while TRUE
    repeat j←j-1
        until A[j]≤x
    repeat i←i+1
        until A[i]≥x
    if i<j
        then exchange A[i]↔A[j]
        else return j
```



Quick Sort Algorithm

- Initial call Quicksort(A, 1, length[A])

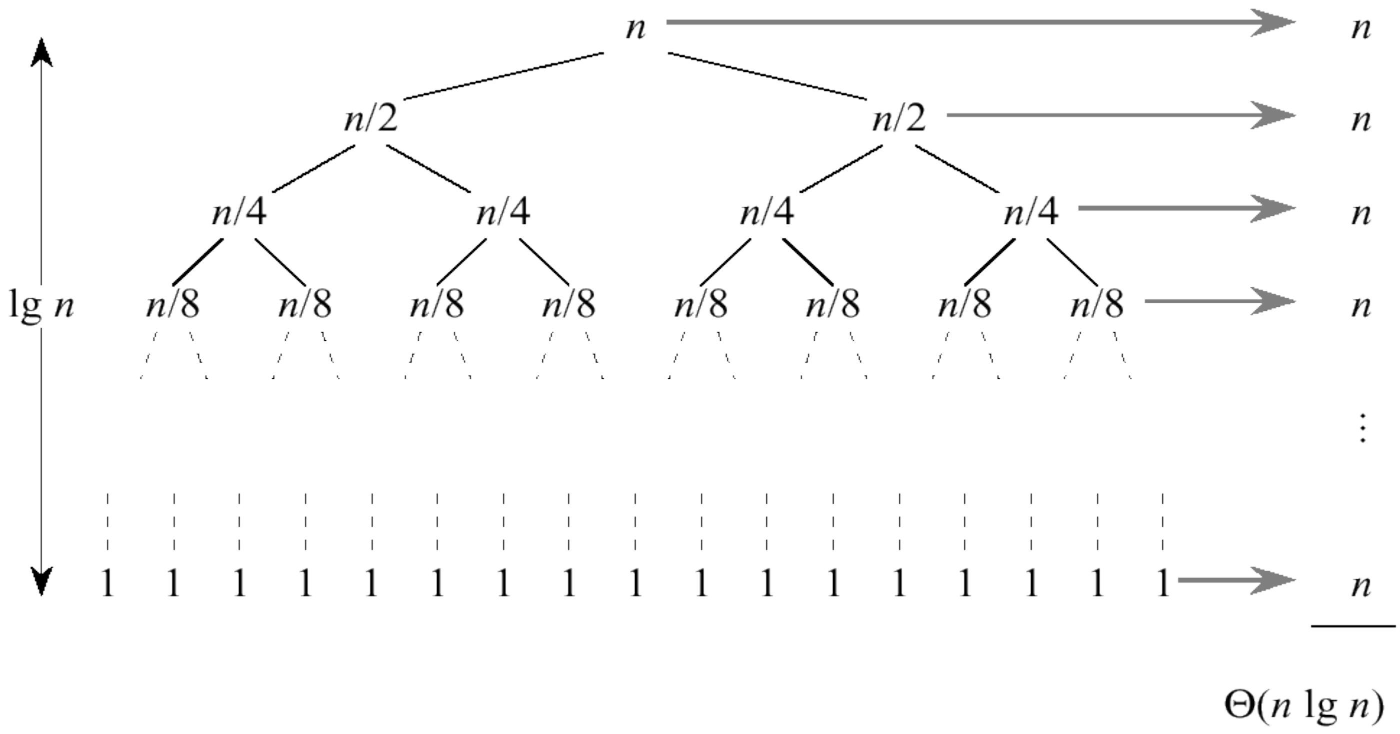
```
Quicksort(A,p,r)
if p<r
   then q←Partition(A,p,r)
        Quicksort(A,p,q)
        Quicksort(A,q+1,r)
```

Analysis of Quicksort

- Assume that all input elements are distinct
- The running time depends on the distribution of splits

Best Case

- If we are lucky, Partition splits the array evenly $T(n) = 2T(n/2) + \Theta(n)$



Using the median as a pivot

- The recurrence in the previous slide works out, BUT……

Q: Can we find the median in linear-time?

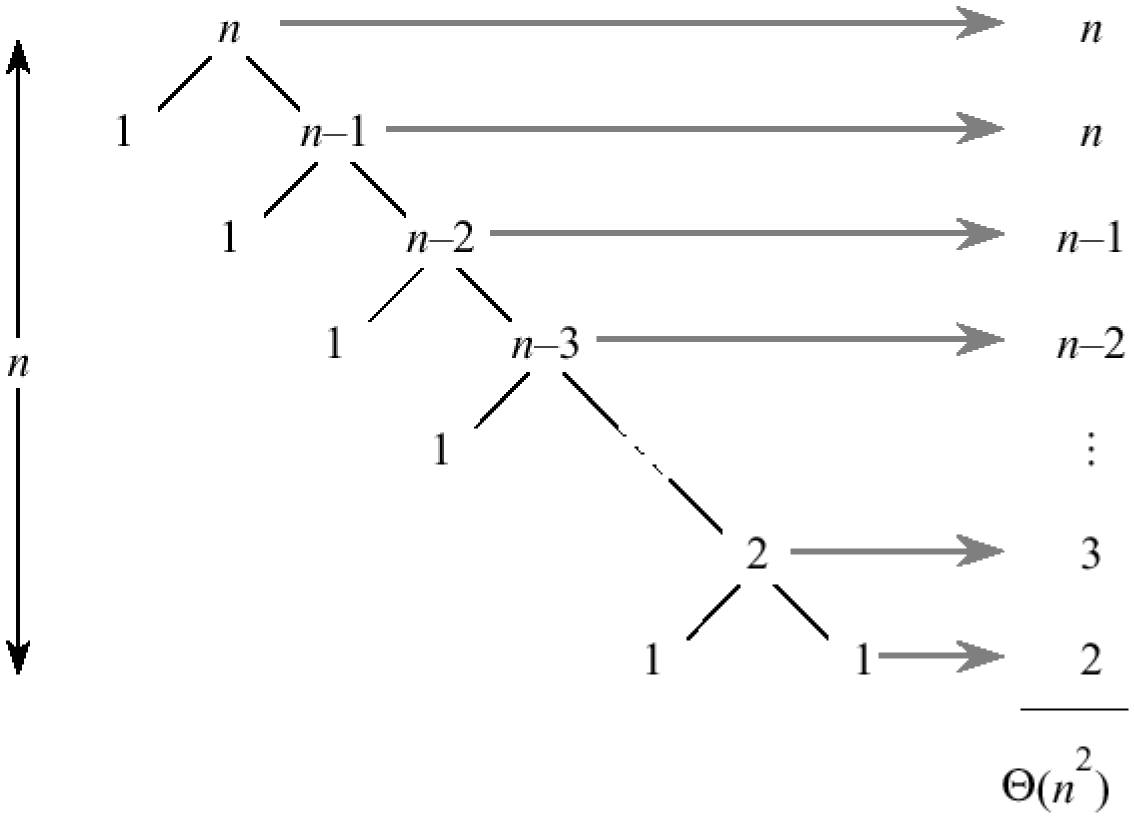
A: YES! But we need to wait until we get to Chapter 8…..

Worst Case

- What is the worst case?
- One side of the parition has only one element

$$
\begin{aligned}
T(n) = T(1) + T(n-1) + \Theta(n) \\
&= T(n-1) + \Theta(n) \\
&= \sum_{k=1}^{n} \Theta(k) \\
&= \Theta\left(\sum_{k=1}^{n} k\right) \\
&= \Theta(n^2)
\end{aligned}
$$

Worst Case (2)



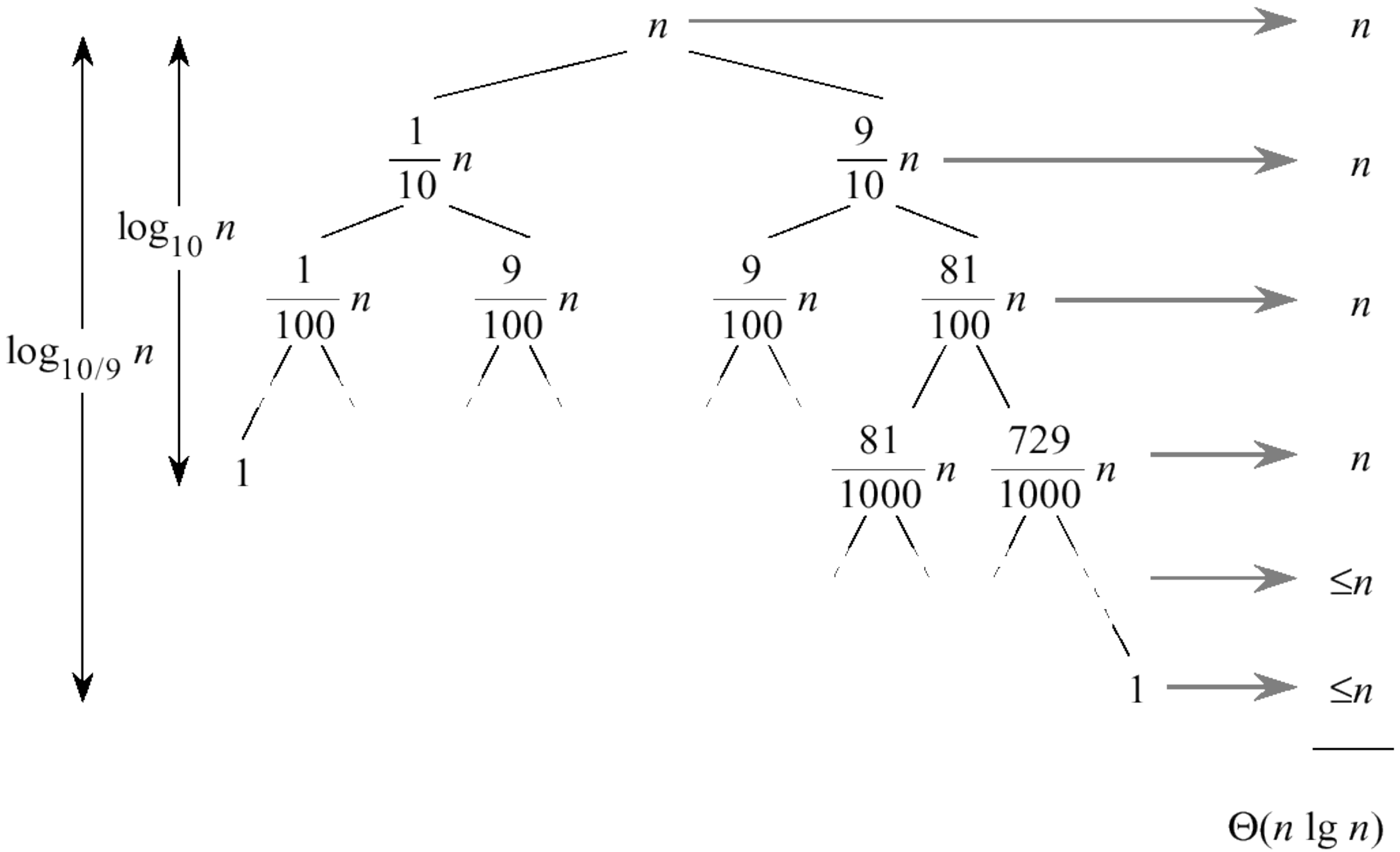
Worst Case (3)

- When does the worst case appear?
  - input is sorted
  - input reverse sorted
- Same recurrence for the worst case of insertion sort
- However, sorted input yields the best case for insertion sort!

Analysis of Quicksort

- Suppose the split is 1/10 : 9/10

$$T(n) = T(n/10) + T(9n/10) + \Theta(n) = \Theta(n \log n)!$$



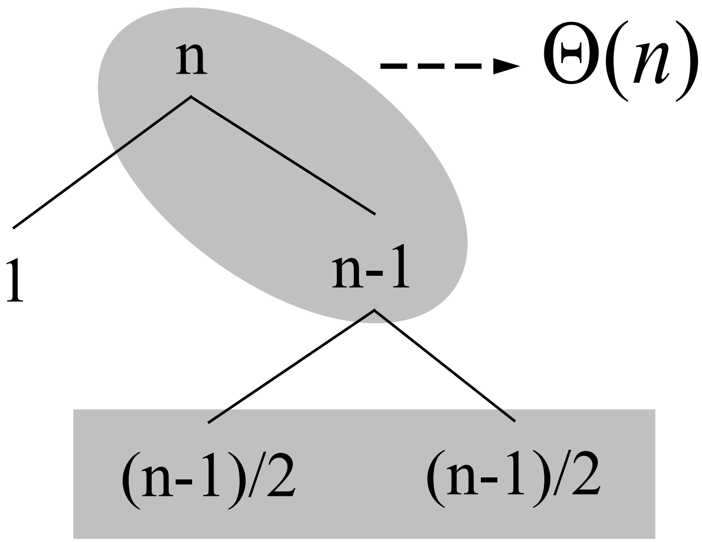
An Average Case Scenario

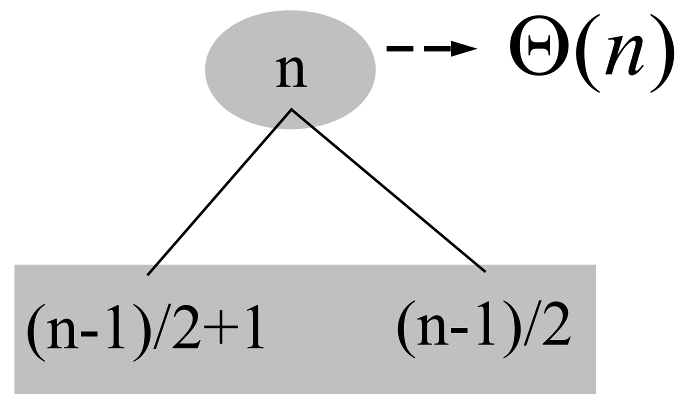
- Suppose, we alternate lucky and unlucky cases to get an average behavior

$$L(n) = 2U(n/2) + \Theta(n) \quad \text{lucky}$$
$$U(n) = L(n-1) + \Theta(n) \quad \text{unlucky}$$

we consequently get

$$L(n) = 2(L(n/2-1) + \Theta(n/2)) + \Theta(n)$$
$$= \quad 2L(n/2-1) + \Theta(n)$$
$$= \quad \Theta(n \log n)$$





An Average Case Scenario (2)

- How can we make sure that we are usually lucky?
  - Partition around the ”middle” (n/2th) element?
  - Partition around a random element (works well in practice)
- Randomized algorithm
  - running time is independent of the input ordering
  - no specific input triggers worst-case behavior
  - the worst-case is only determined by the output of the random-number generator

Randomized Quicksort

- Assume all elements are distinct
- Partition around a random element
- Randomization is a general tool to improve algorithms with bad worst-case but good average-case complexity

Next: Lower bounds

Q: Can we beat the Ω(n log n) lower bound for sorting?

A: In general no, but in some special cases YES!

Ch 7: Sorting in linear time

Let's prove the Ω(n log n) lower bound.

Lower bounds

- What are we counting?

  Running time? Memory? Number of times a specific operation is used?
- What (if any) are the assumptions?
- Is the model general enough?

Here we are interested in lower bounds for the WORST CASE. So we will prove (directly or indirectly):

for any algorithm for a given problem, for each $n>0$, there exists an input that make the algorithm take

$\Omega(f(n))$ time. Then f(n) is a lower bound on the worst case running time.