CSE4421Z: Introduction to Robotics
Final Exam

Instructor: Dr. Burton Ma

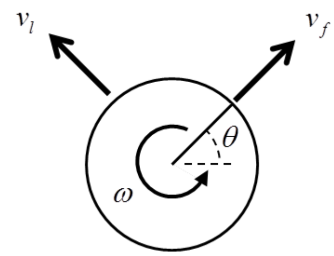
Tue 16 April 2013

Name: ____

Student Number: ____

Instructions

1. You have 2 hours to complete the exam.
2. Write your answers clearly and succintly in the space provided on the question sheets; use the back of the page and the extra blank page if you need additional space for your answer.
3. Textbook, notes, and a calculator are permitted.
4. This exam has 1 cover page and 15 pages for questions and answers.



1. (25 points) An omnibot is able to control its forward velocity v_f, its sideways velocity v_l, and its rotational velocity ω about its center point $[x\ y]^T$; the robot can control its velocities independently of one another so that it can simultaneously move in any direction and change its heading θ. The state of the omnibot at time t is given by the position of its center point and its heading; i.e.,

$$x_t = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}$$

Provide a velocity motion model for the omnibot; i.e., provide a procedure for calculating $p(x_t|u_t, x_{t-1})$. (Note: Table 5.1 is a velocity motion model for a differential drive robot.)

Solution: For the velocity model, you need to solve the inverse kinematics problem: find the control values $\hat{v}_f$, $\hat{v}_l$, and $\hat{\omega}$ required to move from x_{t-1} to x_t.

Let the motion model be a translation in the direction of v_f, followed by a translation in the direction of v_l, followed by an in-place rotation.

Let $x_t = \begin{bmatrix} x' \\ y' \\ \theta' \end{bmatrix}$, $x_{t-1} = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}$, and $u_t = \begin{bmatrix} v_f \\ v_l \\ \omega \end{bmatrix}$.

1. $\hat{v}_f = a \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ and $\hat{v}_l = b \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$ where a and b are the solutions to the following set of linear equations:

$$\begin{aligned} (x' - x)/\Delta t &= a\cos\theta - b\sin\theta \\ (y' - y)/\Delta t &= a\sin\theta + b\cos\theta \end{aligned}$$

   (there is no need to solve for a and b; a picture would have been sufficient; assuming $\Delta t = 1$ would have been acceptable)

2. $\hat{\omega} = (\theta' - \theta)/\Delta t$

3. return prob$(v_f - \hat{v}_f, \varepsilon_{v_f}) \cdot$ prob$(v_l - \hat{v}_l, \varepsilon_{v_l}) \cdot$ prob$(\omega - \hat{\omega}, \varepsilon_\omega)$ where the ε are suitable noise variances

Use this page for Question 1 if necessary.

2. For this question, consider the odometry motion model for a differential drive robot described in Section 5.4 of the textbook. Also, consider the bicycle model from Assignment 3 (Exercise 5.8.4 of the textbook) which has a steerable front wheel; assume that the bicycle has a motor that supplies power to the *front* wheel.

   (a) (5 points) The odometry motion model decomposes the motion into 3 steps: an in-place rotation, followed by a translation, followed by an in-place rotation. Is this a reasonable motion model for a bicycle? Briefly explain why or why not; one to three sentences should suffice. Consider reading part (b) before writing down your answer to this part.

   (b) (5 points) Assume that you are determined to use the odometry motion model from part (a); describe what the rider of the bicycle would have to do to duplicate the motion prescribed by the motion model (i.e., how would the rider have to turn the front wheel and how far would the bicycle have to travel for each of the three steps?).

(c) (15 points) Propose an alternate odometry motion model where the control inputs are described by the control vector

$$u_t = \begin{bmatrix} \alpha_t \\ d_t \end{bmatrix}$$

where α_t is the angle of the front wheel at time t, and d_t is the distance travelled by the rear wheel; all of the control inputs are measured by sensors in the bicycle (e.g., rotary encoders in the steering wheel and rear wheel). You can assume that α_t is constant over each time interval, and that each time interval is 1 unit of time in duration (i.e., $\Delta t = 1$).

For your answer, you should first briefly describe your motion model (what is the assumed motion of the bicycle). Second, you should provide an algorithm to compute $p(x_t|u_t, x_{t-1})$. (Note: Table 5.5 of the textbook describes an odometry motion model for an differential drive robot). Provide all necessary equations if you can, but a solution that describes the individual steps with supporting figures can also receive substantial marks.

Use this page for Question 2 if necessary.

3. Suppose that we have a mobile robot operating in a planar environment. The state of the robot is its x-y-location and its global heading direction θ in radians. Suppose that the initial estimate of the state is

$$\mathbf{x}_0 = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(a) (2 points) Suppose that initial estimate of the state is perfect. Assume that the robot moves flawlessly without any noise in its control inputs. What is the expected location of the robot after it moves $d = 1$ units forward?

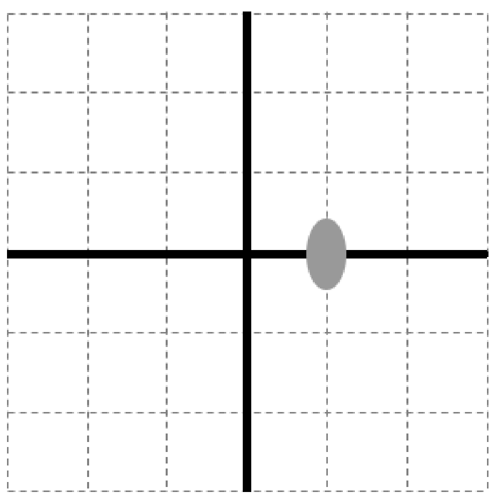
Solution:

$$\mu_1 = x_0 + \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

(b) (3 points) Suppose that initial estimate of the state is uncertain; we know x, y, and θ with high certainty which is reflected in the covariance matrix of the initial estimate:

$$\Sigma_0 = \begin{bmatrix} 0.01 & 0 & 0 \\ 0 & 0.01 & 0 \\ 0 & 0 & 0.01 \end{bmatrix}$$

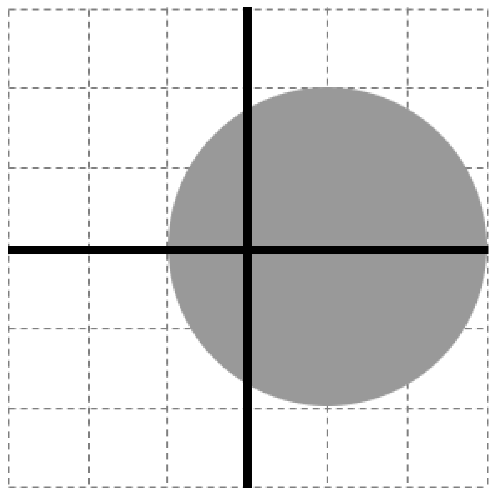
Sketch your best model of the posterior density of the robot position after the robot moves $d = 1$ unit forward. Note that $\sqrt{0.01 \text{ rad}^2} \approx 5.7°$. Label the axes with appropriate units.



(c) (3 points) Suppose that initial estimate of the state is uncertain; we know x and y with low certainty, and θ with high certainty which is reflected in the covariance matrix of the initial estimate:

$$\Sigma_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.01 \end{bmatrix}$$

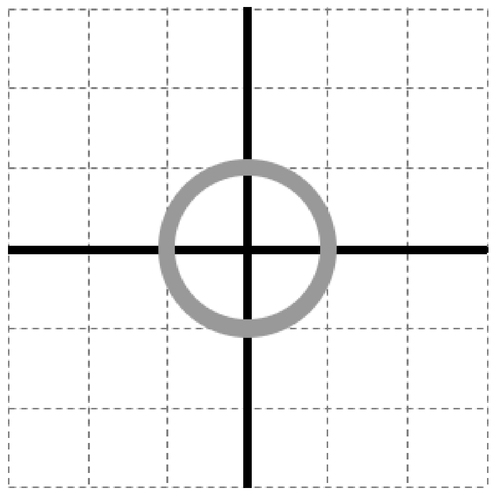
Sketch your best model of the posterior density of the robot position after the robot moves $d = 1$ unit forward. Label the axes with appropriate units.



(d) (3 points) Suppose that initial estimate of the state is uncertain; we know x and y with high certainty, and θ is more or less unknown which is reflected in the covariance matrix of the initial estimate:

$$\Sigma_0 = \begin{bmatrix} 0.01 & 0 & 0 \\ 0 & 0.01 & 0 \\ 0 & 0 & 100 \end{bmatrix}$$

Sketch your best model of the posterior density of the robot position after the robot moves $d = 1$ unit forward. Label the axes with appropriate units.



(e) (7 points) If we assume that the robot moves flawlessly without any noise in its control inputs, then an appropriate plant model of the robot for use in an extended Kalman filter is

$$\mathbf{x}_t = \begin{bmatrix} x_{t-1} + \cos\theta_{t-1} \\ y_{t-1} + \sin\theta_{t-1} \\ \theta_{t-1} \end{bmatrix} = g(u_t, \mathbf{x}_{t-1})$$

The Jacobian of g is given by

$$G_t = \begin{bmatrix} 1 & 0 & -\sin\theta_{t-1} \\ 0 & 1 & \cos\theta_{t-1} \\ 0 & 0 & 1 \end{bmatrix}$$

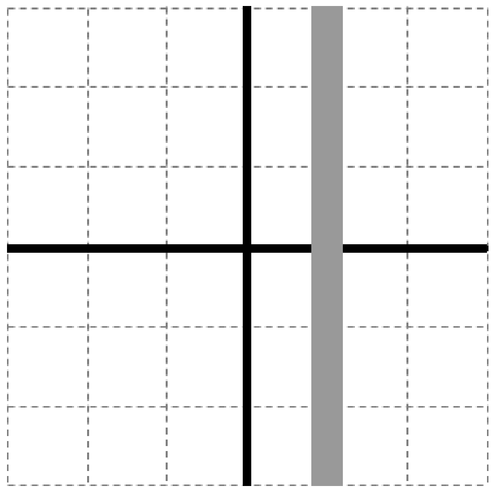
What is the estimated state and covariance computed during the prediction phase of an extended Kalman filter for the situation described in part (d)?

Solution:

$$\mu_1 = \begin{bmatrix} 0 + \cos\theta \\ 0 + \sin\theta \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{\Sigma}_1 = G_t \Sigma_0 G_t^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.01 & 0 & 0 \\ 0 & 0.01 & 0 \\ 0 & 0 & 100 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0.01 & 0 & 0 \\ 0 & 100.01 & 100 \\ 0 & 100 & 100 \end{bmatrix}$$

(f) (7 points) Sketch the uncertainty ellipse of the estimated state and covariance from part (e); label the axes with appropriate units. Discuss the difference between your solution from part (d) and the EKF prediction from part (e).



Solution: The linearization of the plant model fails in this case. The x coordinate of the robot is estimated with high precision and the high uncertainty in the bearing direction appears only in the y coordinate of the robot. It is not surprising that the linearization fails: The ring shaped distribution from (d) cannot be represented faithfully as a Gaussian distribution, and the uncertainty in the bearing direction is very large.

4. Consider a differential drive robot moving in a planar environment, as shown in Figure 3.

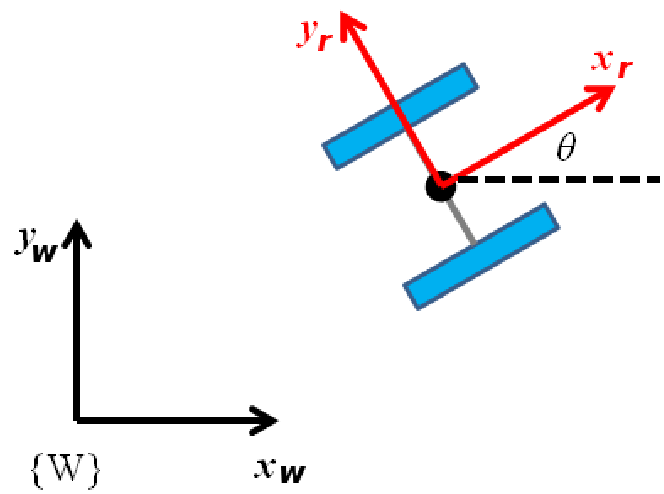


Figure 1: Differential drive moving in a planar enviroment with a world coordinate frame $\{W\}$. The position of the robot at time t in the world frame is $(x,\ y)^T$ and the heading of the robot at time t is θ. The robot has its own internal reference frame with the x axis of the frame pointing in the heading direction.

A map of the environment (in the world frame) is shown in Figure 4; there are three point landmarks in the environment.

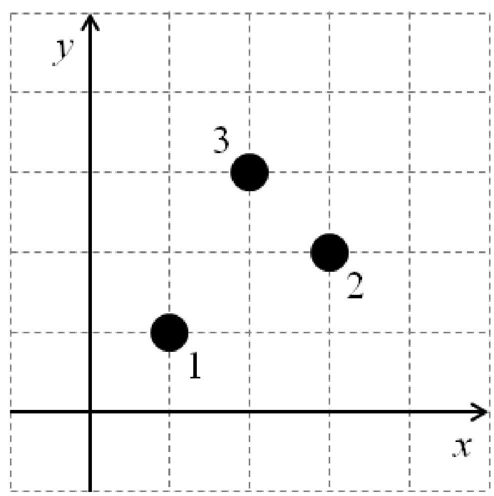
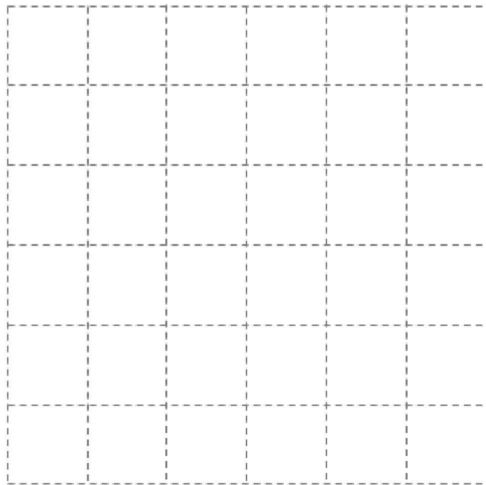


Figure 2: Map of the environment with three point landmarks.

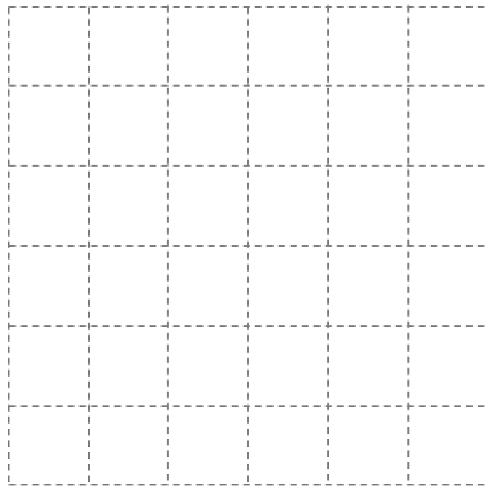
Suppose that the robot can measure the location of all of the landmarks, with the coordinates of each landmark expressed in the internal reference frame of the robot (not the world frame).

(a) (3 points) Suppose that the robot pose at time t is $(-1,\ 2,\ 0°)^T$. Draw the (noise free) landmark locations in the robot's internal reference frame; make sure to include the x and y axes of the robot's reference frame.



(b) (3 points) Suppose that the robot pose at time t is $(0,\ 0,\ 45°)^T$. Draw the (noise free) landmark locations in the robot's internal reference frame; make sure to include the x and y axes of the robot's reference frame.

(c) (3 points) Suppose that the robot pose at time t is $(2,\ -1,\ 90°)^T$. Draw the (noise free) landmark locations in the robot's internal reference frame; make sure to include the x and y axes of the robot's reference frame.



(d) (16 points) Let the landmark locations in the world be

$$\begin{aligned} M_1 &= (M_{1,x},\ M_{1,y})^T \\ M_2 &= (M_{2,x},\ M_{2,y})^T \\ M_3 &= (M_{3,x},\ M_{3,y})^T \end{aligned}$$

Let the state of the robot at time t be

$$x_t = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}$$

Let the measurement at time t be

$$z_t = \begin{bmatrix} m_{1,x} \\ m_{1,y} \\ m_{2,x} \\ m_{2,y} \\ m_{3,x} \\ m_{3,y} \end{bmatrix}$$

where $(m_{i,x},\ m_{i,y})^T$ is the noisy measured location of landmark i in the robot's internal reference frame. Give a measurement model (suitable for use in an EKF or UKF) that relates the measurements z_t to the state of the robot x_t.

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