# CSE4421Z: Introduction to Robotics
# Final Exam

Instructor: Dr. Burton Ma

Thu 12 April 2012

Name: ______________________________

Student Number: ______________________________

Instructions

1. You have 2 hours to complete the exam.
2. Write your answers clearly and succintly in the space provided on the question sheets; use the back of the page and the extra blank page if you need additional space for your answer.
3. Textbook, notes, and a calculator are permitted.
4. This exam has 1 cover page, 12 pages for questions and answers, and one copy of Section 5.4 of the textbook.

1. Consider the RR arm shown in Figure 1.

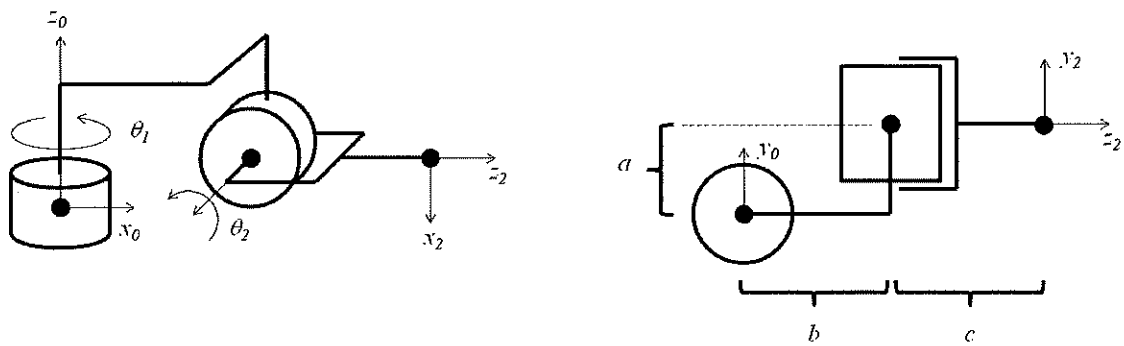



Figure 1: Left: Front view of arm. Right: Top-down view of arm. In this figure, all joint angles are shown at 0°.

Joints 1 and 2 are connected by a link with a 90° bend; the bend allows both joints to rotate through 360° without colliding with each other. The axis of joint 2 is always in the plane $z_0 = 0$. You may assume that the link dimensions $a$, $b$, and $c$ are always greater than zero, and that $b > c$.

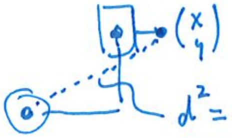
(a) (22 points) Given the location of frame $\{2\}$ expressed in frame $\{0\}$, $o_2^0 = (x,\ y,\ z)^T$, find the values of $\theta_1$ and $\theta_2$.

(b) (3 points) A high quality solution to part (a) would check that the point $o_2^0 = (x,\ y,\ z)^T$ is actually reachable by the robot; how would you check if $o_2^0$ is reachable? Provide a mathematical expression if possible, although a well written description could also receive full marks.

Use this page for Question 1 if necessary.

Two cases for $\theta_2$.

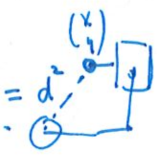
$-90 \le \theta_2 \le 90$ :




$d^2 = x^2 + y^2 \ge a^2 + b^2$

$\theta_2 = \operatorname{asin}\left(\frac{z}{c}\right)$

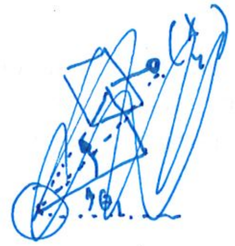
$90 \le \theta_2 \le 270$ :

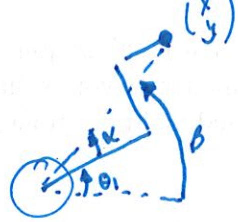



$a^2 + b^2 \ge x^2 + y^2 = d^2$

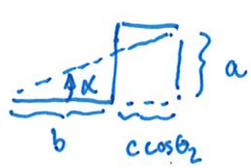
$\theta_2 = 180 - \operatorname{asin}\left(\frac{z}{c}\right)$

Once you have $\theta_2$ you can solve for $\theta_1$ :






$\beta = \operatorname{atan2}(y, x)$




$\alpha = \operatorname{atan2}(a, b + c\cos\theta_2)$

$\theta_1 = \beta - \alpha$

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$(x, y, z)^T$ lies on the surface of revolution given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R_{z,\theta_1}} \underbrace{\begin{bmatrix} c\cos\theta_2 + b \\ \cancel{c\sin\theta_2}\ a \\ c\sin\theta_2 \end{bmatrix}}_{\text{circle}}$$

You can check for reachability by solving the inverse kinematics and substituting $\theta_1, \theta_2$ into the equation on the left to see if it equals $o_2^0$

2. (25 points) Consider the RR arm shown in Figure 2; this arm is similar to the arm from Question 1 except that the link has a bend of $-45°$.

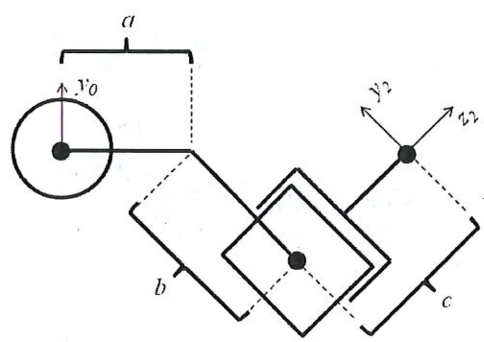



Figure 2: Top-down view of arm.

The axis of joint 2 is always in the plane $z_0 = 0$. You may assume that the link dimensions $a$, $b$, and $c$ are always greater than zero.

Solve for the forward kinematics of the arm; that is, given joint angles $\theta_1$ and $\theta_2$, find the orientation and position of frame $\{2\}$ as a $4 \times 4$ homogeneous matrix. If your solution involves a sequence of matrix multiplications then you do not need to perform the actual multiplications. Use of the Denavit-Hartenberg convention is acceptable, but not required.

This was a midterm question

3. For this question, consider the odometry motion model for a differential drive robot described in Section 5.4 of the textbook and Day 18 of the lecture slides. Also, consider the bicycle model from Assignment 3 (Exercise 5.8.4 of the textbook and slide 12 from Day 12 of the lecture slides) which has a steerable front wheel; assume that the bicycle has a motor that supplies power to the *front* wheel.

   (a) (5 points) The odometry motion model decomposes the motion into 3 steps: an in-place rotation, followed by a translation, followed by an in-place rotation. Is this a reasonable motion model for a bicycle? Briefly explain why or why not; one to three sentences should suffice. Consider reading part (b) before writing down your answer to this part.

   **Solution:** No, pivoting the bicycle frame about the front wheel (the in-place rotation) is unusual for normal location of a bicycle.

   (b) (5 points) Assume that you are determined to use the odometry motion model from part (a); describe what the rider of the bicycle would have to do to duplicate the motion prescribed by the motion model (i.e., how would the rider have to turn the front wheel and how far would the bicycle have to travel for each of the three steps?).

   **Solution:**

   1. pivot the bicycle about the front wheel so that the frame of the bicycle is pointing in the desired direction
   2. point the front wheel straight ahead ($\alpha = 0$) and move forward by the desired amount
   3. pivot the bicycle about the front wheel so that the frame of the bicycle is pointing in the final direction

(c) (15 points) Propose an alternate odometry motion model where the control inputs are described by the control vector

$$u_t = \begin{bmatrix} \alpha_{t-1} \\ \alpha_t \\ d_t \end{bmatrix}$$

where $\alpha_{t-1}$ is the angle of the front wheel at time $t-1$, $\alpha_t$ is the angle of the front wheel at time $t$, and $d_t$ is the distance travelled by the front wheel; all of the control inputs are measured by sensors in the bicycle (e.g., rotary encoders in the steering wheel and front wheel). You can assume that the time steps between odometry measurements is small so that $|\alpha_t - \alpha_{t-1}|$ is small.

For your answer, you should first briefly describe your motion model (what is the assumed motion of the bicycle, and how is the motion decomposed into separate steps if necessary). Second, you should provide an algorithm to compute the probability density of the motion (similar to the algorithm given in Table 5.5 of the textbook).

**Solution:**

1. Move with smooth rolling motion with the front wheel at angle $\alpha_{t-1}$ over a distance $d_t$
2. Turn the front wheel to $\alpha_t$
3. Allow for a small rotation to achieve the final orientation $\theta'$. This step is needed because moving with smooth rolling motion with the front wheel at angle $\alpha_{t-1}$ over a distance $d_t$ will almost certainly not result in the orientation $\theta'$.

$x_t = [x',\ y',\ \theta']^T$
$x_{t-1} = [x,\ y,\ \theta]^T$
$u_t = [\alpha_{t-1},\ \alpha_t,\ d_t]^T$
Algorithm odometry-model$(x_t, u_t, x_{t-1})$

1. Compute $\hat{\alpha}_{t-1}$ and $\hat{d}_t$ required to move from $x_{t-1}$ to $x_t$ using smooth rolling motion. This can be done using the equations from the velocity motion model discussed in class.
2. Compute $\hat{\theta}'$, the bearing of the bicycle after moving with smooth rolling motion.
3. $\delta\alpha_{t-1} = \alpha_{t-1} - \hat{\alpha}_{t-1}$
4. $\delta\alpha_t = \alpha_t - \hat{\alpha}_{t-1}$
5. $\delta d_t = d_t - \hat{d}_t$
6. $\delta\theta = \theta' - \hat{\theta}'$
7. return prob$(\delta\alpha_{t-1}, \varepsilon_{\alpha_{t-1}})$ $\cdot$ prob$(\delta\alpha_t, \varepsilon_{\alpha_t})$ $\cdot$ prob$(\delta d_t, \varepsilon_{d_t})$ $\cdot$ prob$(\delta\theta, \varepsilon_\theta)$ where the $\varepsilon$ are variances of the various noises.

4. Consider a differential drive robot moving in a planar environment, as shown in Figure 3.

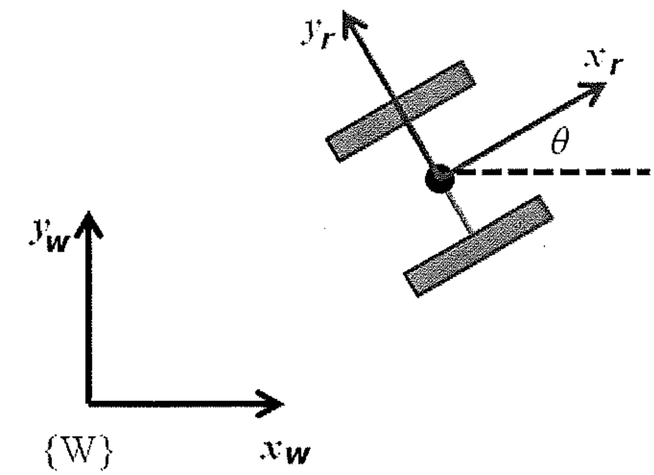



Figure 3: Differential drive moving in a planar enviroment with a world coordinate frame $\{W\}$. The position of the robot at time $t$ in the world frame is $(x,\ y)^T$ and the heading of the robot at time $t$ is $\theta$. The robot has its own internal reference frame with the $x$ axis of the frame pointing in the heading direction.

A map of the environment (in the world frame) is shown in Figure 4; there are three point landmarks in the environment.

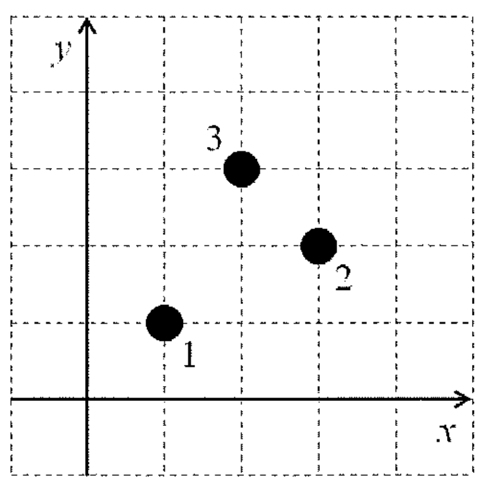
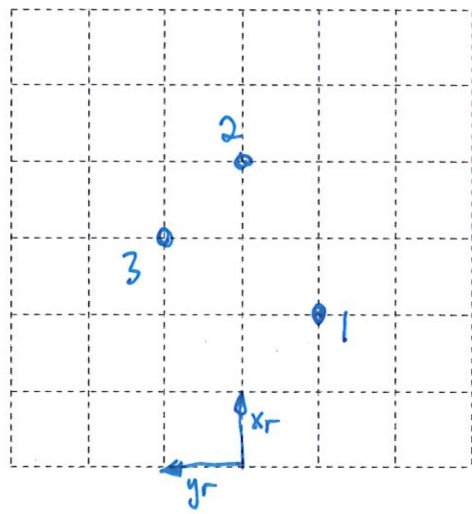


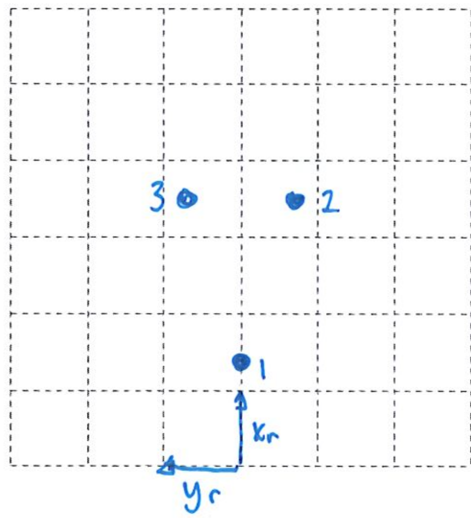

Figure 4: Map of the environment with three point landmarks.

Suppose that the robot can measure the location of all of the landmarks, with the coordinates of each landmark expressed in the internal reference frame of the robot (not the world frame).

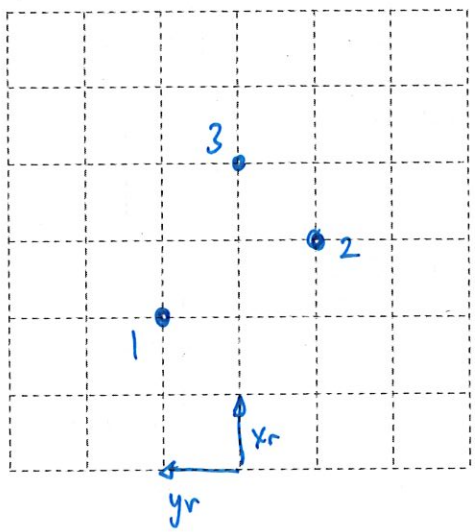
(a) (5 points) Suppose that the robot pose at time $t$ is $(-1,\ 2,\ 0°)^T$. Draw the (noise free) landmark locations in the robot's internal reference frame; make sure to include the $x$ and $y$ axes of the robot's reference frame.




(b) (5 points) Suppose that the robot pose at time $t$ is $(0,\ 0,\ 45°)^T$. Draw the (noise free) landmark locations in the robot's internal reference frame; make sure to include the $x$ and $y$ axes of the robot's reference frame.



(c) (5 points) Suppose that the robot pose at time $t$ is $(2,\ -1,\ 90°)^T$. Draw the (noise free) landmark locations in the robot's internal reference frame; make sure to include the $x$ and $y$ axes of the robot's reference frame.



(d) (10 points) Let the landmark locations in the world be

$$\begin{aligned} M_1 &= (M_{1,x},\ M_{1,y})^T \\ M_2 &= (M_{2,x},\ M_{2,y})^T \\ M_3 &= (M_{3,x},\ M_{3,y})^T \end{aligned}$$

Let the state of the robot at time $t$ be

$$x_t = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}$$

Let the measurement at time $t$ be

$$z_t = \begin{bmatrix} m_{1,x} \\ m_{1,y} \\ m_{2,x} \\ m_{2,y} \\ m_{3,x} \\ m_{3,y} \end{bmatrix}$$

where $(m_{i,x},\ m_{i,y})^T$ is the noisy measured location of landmark $i$ in the robot's internal reference frame. Give a measurement model (suitable for use in an EKF or UKF) that relates the measurements $z_t$ to the state of the robot $x_t$.

The $m_i$ are rotated and translated versions of the $M_i$

The pose of the robot with state $\begin{bmatrix} x \\ y \\ \theta \end{bmatrix}$ is given by

$$T^w_r = \begin{bmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{bmatrix}$$

The $m_i$ are measured in frame $\{r\}$; therefore

$$T^w_r \underbrace{\begin{bmatrix} m_i \\ 1 \end{bmatrix}}_{\text{marker } i \text{ in frame } \{r\}} = \underbrace{\begin{bmatrix} M_i \\ 1 \end{bmatrix}}_{\text{marker } i \text{ in frame } \{w\}}$$

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$$\begin{bmatrix} m_i \\ 1 \end{bmatrix} = (T_r^w)^{-1} \begin{bmatrix} M_i \\ 1 \end{bmatrix}$$

compute $m_1, m_2, m_3$ using and stack the values to get $z_t$