Signals and Systems I Have Known and Loved

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Part 1

Introduction

CHAPTER 1

Introduction to Signals and Systems

1.1. What are Signals and Systems?

1.1.1. Definitions. We start by describing systems. Abstractly, a *system* is a transformation S that maps an input function of time to an output function of time. The system input function is usually written $x(t)$. The system output function, representing the transformation of the input to the output of the system, is usually written $y(t)$. Using S, we can write

$$y(t) = S(x(t)), \tag{1.1}$$

so that the system S transforms the input $x(t)$ into the output $y(t)$.

Abstractly, a *signal* is any input or output of a system, described as a function of time. In (1.1), the input $x(t)$ and the output $y(t)$ are both signals. We will refer to these as the *input signal* and *output signal*, respectively.

EXAMPLE 1.1. *Suppose the system S is an integrator:*

$$S(x(t)) = \int_{-\infty}^{t} x(u)du. \tag{1.2}$$

(In the above equation, u is the dummy variable of integration.) For instance, with input signal $\cos(t)$*, we have*

$$S(\cos(t)) = \int_{-\infty}^{t} \cos(u)du \tag{1.3}$$

$$= \sin(t). \tag{1.4}$$

EXAMPLE 1.2. *Suppose the system S multiplies the input by t:*

$$S(x(t)) = tx(t). \tag{1.5}$$

For instance, with input signal e^t, we have

$$S(e^t) = te^t. \tag{1.6}$$

EXAMPLE 1.3. *Suppose the system S returns the square of the input:*

$$S(x(t)) = x(t)^2. \tag{1.7}$$

For instance, with input signal $\sin(t)$*, we have*

$$S(\sin(t)) = \sin^2(t). \tag{1.8}$$

1.1.2. Linear systems. A *linear system* has the following property:

DEFINITION 1.1. *For a system S, if*

$$S(\alpha x_1(t) + \beta x_2(t)) = \alpha S(x_1(t)) + \beta S(x_2(t)), \tag{1.9}$$

for any input signals $x_1(t)$ and $x_2(t)$ and for any constants α and β, then S is a ***linear system****.*

From (1.9), a linear system preserves *addition* and *scalar multiplication*:

- *Addition:* Let $\alpha = \beta = 1$. From (1.9), if two inputs are added, the output is the addition of the respective outputs, i.e.,

$$S(x_1(t) + x_2(t)) = S(x_1(t)) + S(x_2(t)). \tag{1.10}$$

- *Scalar multiplication:* Let $\beta = 0$. From (1.9), if the input is multiplied by a scalar, then the output is multiplied by the same scalar, i.e.,

$$S(\alpha x_1(t)) = \alpha S(x_1(t)). \tag{1.11}$$

EXAMPLE 1.4. *Consider the integrator from Example 1.1. Using (1.9), we can show that this system is linear:*

$$S(\alpha x_1(t) + \beta x_2(t)) = \int_{-\infty}^{t} \alpha x_1(u) + \beta x_2(u) du \tag{1.12}$$

$$= \alpha \int_{-\infty}^{t} x_1(u) du + \beta \int_{-\infty}^{t} x_2(u) du \tag{1.13}$$

$$= \alpha S(x_1(t)) + \beta S(x_2(t)). \tag{1.14}$$

EXAMPLE 1.5. *Consider the system from Example 1.3. Using (1.9), we can show that this system is not linear:*

$$S(\alpha x_1(t) + \beta x_2(t)) = (\alpha x_1(t) + \beta x_2(t))^2 \tag{1.15}$$

$$= \alpha^2 x_1(t)^2 + \beta^2 x_2(t)^2 + \alpha\beta x_1(t)x_2(t) \tag{1.16}$$

$$= \alpha^2 S(x_1(t)) + \beta^2 S(x_2(t)) + \alpha\beta x_1(t)x_2(t) \tag{1.17}$$

$$\neq \alpha S(x_1(t)) + \beta S(x_2(t)). \tag{1.18}$$

We leave it as an exercise to show that the system from Example 1.2 is linear.

1.1.3. Time-invariant systems. A time-invariant system is a system for which the response is the same, regardless of when the input arrives. Put differently, a delay in the input results in the same delay in the output, and no other change. More formally:

DEFINITION 1.2. *For a system S, let $y(t) = S(x(t))$. If*

$$S(x(t-d)) = y(t-d) \tag{1.19}$$

*for any d and any $x(t)$, then S is a **time-invariant system**.*

EXAMPLE 1.6. *Consider the integrator from Example 1.1. This system is time-invariant:*

$$S(x(t-d)) = \int_{-\infty}^{t} x(u-d)du. \tag{1.20}$$

Using the change of variables $v = u - d$,

$$\int_{-\infty}^{t} x(u-d)du = \int_{-\infty}^{t-d} \alpha x(v)dv \tag{1.21}$$

$$= y(t-d). \tag{1.22}$$

EXAMPLE 1.7. *Consider the system from Example 1.2. This system is not time-invariant:*

$$S(x(t-d)) = tx(t-d). \tag{1.23}$$

However,

$$y(t-d) = (t-d)x(t-d) \tag{1.24}$$

$$\neq tx(t-d) = S(x(t-d)). \tag{1.25}$$

We leave as an exercise to show that the system from Example 1.3 is time-invariant.

This course deals largely with the systems that are both linear and time-invariant. We refer to these as **linear time-invariant (LTI) systems**.

1.2. Properties of signals

1.2.1. Continuous-time vs. discrete-time. Signals are functions of time, normally written $x(t)$. If the domain of $x(t)$ is the set of real numbers $\mathbb{R}$, then the signal is *continuous-time*. If the domain of $x(t)$ is the set of integers $\mathbb{Z}$, then the signal is *discrete-time*.

To avoid confusion, from now on we will write continuous-time and discrete-time signals with distinct notation:

- A continuous-time signal is written $x(t)$, where t is a real number.
- A discrete-time signal is written $x[k]$, where k is an integer.

1.2.2. Periodic vs. non-periodic.

Definition 1.3. *If $f(t)$ has the property*

$$f(t) = f(t - T) \tag{1.26}$$

*for some $T > 0$, then $f(t)$ is **periodic** with period T, or frequency $1/T$.*

It is easy to show (see the exercises) that if $f(t)$ is periodic with period T, then it is also periodic with period $2T$, $3T$, and so on. If T is the *smallest* value of T for which $f(t)$ is periodic, then T is called the **fundamental period**. (Normally, we will simply say the period to mean the fundamental period, unless it is ambiguous.)

Similarly, a discrete-time signal is periodic with period T if $f[k] = f[k - T]$, but in this case, T must be an integer.

Since T must be restricted to the integers in discrete time, there is a slightly odd result: *sinusoidal functions are not always periodic in discrete time.* For example, consider the continuous-time function

$$f(t) = \sin(t). \tag{1.27}$$

Then $f(t)$ is periodic with period 2π, since

$$\sin(t) = \sin(t - 2\pi). \tag{1.28}$$

However, in discrete time,

$$f[k] = \sin[k] \tag{1.29}$$

is not periodic: $\sin[k-2\pi]$ is undefined, since $k-2\pi$ is not an integer, and discrete-time functions are only defined with integer arguments.

On the other hand, consider

$$f[k] = \sin\left[\frac{3\pi k}{16}\right]. \tag{1.30}$$

Is this periodic? It is if we can find integer T such that

$$\sin\left[\frac{3\pi k}{16}\right] = \sin\left[\frac{3\pi(k+T)}{16}\right]. \tag{1.31}$$

For this to be true, we need $3\pi T/16$ to be either equal to 2π, or an integer multiple of 2π. That is, we need to find integers T and j such that

$$\frac{3\pi T}{16} = j2\pi. \tag{1.32}$$

This is equivalent to

$$3T = 32j, \tag{1.33}$$

which is satisfied for integers with $T = 32$ and $j = 3$.

In general, a discrete-time sinusoid $\sin[\alpha k]$ is periodic if $\alpha/2\pi$ is a *rational number*.[1] Let $a/b = \alpha/2\pi$, with integers a and b. Then the period is equal to b. Let GCD(a, b) represent the greatest common divisor between a and b; if $\mathrm{GCD}(a,b) > 1$, then the fundamental period is $b/\mathrm{GCD}(a,b)$.

1.3. Energy and power of signals

We first consider continuous-time signals. The power dissipated by a resistor is $P = VI$, where V is the voltage across the resistor, and I is the current through the resistor. Suppose the signal $x(t)$ is applied to the circuit in Figure X: $x(t)$, measured in volts, is applied to a 1 Ω resistor. If the current through the resistor is given by $I_{x(t)}(t)$, then the instantaneous power dissipated by the resistor is given by

$$P(t) = x(t)I_{x(t)}(t). \tag{1.34}$$

[1] A rational number is any number that can be written a/b, where a and b are both integers.

By Ohm's law,

$$I_{x(t)}(t) = \frac{x(t)}{R} \tag{1.35}$$

$$= x(t), \tag{1.36}$$

since $R = 1\ \Omega$. Thus,

$$P(t) = x(t)^2. \tag{1.37}$$

Then (1.37) gives the **instantaneous power** of the signal $x(t)$, assuming $x(t)$ is real-valued. More generally, if $x(t)$ is complex-valued, then the instantaneous power is given by

$$P(t) = |x(t)|^2. \tag{1.38}$$

For real-valued signals, (1.37) and (1.38) are identical.

Power is the rate of energy dissipated over time. Thus, if the power P is constant, then energy is the product of power and time: $E = PT$. However, the instantaneous power $P(t)$ from (1.38) may be time-varying. The **energy** of the signal $x(t)$ on the interval $[T_1, T_2]$ is given by the integral of instantaneous power over time:

$$E^{[T_1,T_2]} = \int_{T_1}^{T_2} P(t)dt \tag{1.39}$$

$$= \int_{T_1}^{T_2} |x(t)|^2 dt. \tag{1.40}$$

The **average power** of the signal $x(t)$ over the interval $[T_1, T_2]$ is then given by

$$P_{\text{avg}}^{[T_1,T_2]} = \frac{E^{[T_1,T_2]}}{T_2 - T_1} \tag{1.41}$$

$$= \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} |x(t)|^2 dt. \tag{1.42}$$

The **total energy** of the signal $x(t)$, written E, is the energy in the signal over all time: $E = E^{(-\infty,\infty)}$. We will state this as a limit:

$$E = \lim_{T \to \infty} E^{[-T,T]} \tag{1.43}$$

$$= \lim_{T \to \infty} \int_{-T}^{T} |x(t)|^2 dt. \tag{1.44}$$

The **total average power** of the signal $x(t)$ is defined similarly:

$$P_{\text{avg}} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt. \tag{1.45}$$

Discrete time energy and power are defined similarly. The respective quantities for discrete-time signals are given by

$$P[k] = |x[k]|^2 \tag{1.46}$$

$$E^{[T_1,T_2]} = \sum_{k=T_1}^{T_2} |x[k]|^2 \tag{1.47}$$

$$P_{\text{avg}}^{[T_1,T_2]} = \frac{1}{T_2 - T_1 + 1} \sum_{k=T_1}^{T_2} |x[k]|^2 \tag{1.48}$$

$$E = \lim_{T\to\infty} \sum_{k=-T}^{T} |x[k]|^2 \tag{1.49}$$

$$P_{\text{avg}} = \lim_{T\to\infty} \frac{1}{2T} \sum_{k=-T}^{T} |x[k]|^2 \tag{1.50}$$

A signal is a *power signal* if $0 < P_{\text{avg}} < \infty$. A signal is an *energy signal* if $0 < E < \infty$. In the problems, you will show that a power signal has $E = \infty$, and an energy signal has $P_{\text{avg}} = 0$.

1.4. Special signals and signal forms

1.4.1. Time shifting and time scaling. Let $x(t)$ represent a signal. We can transform the signal in the following ways:

- *Time shifting.* We can delay $x(t)$, i.e. shift it later in time, as follows: if $\tau > 0$ is the desired delay, then the delayed signal is $x(t - \tau)$. Then the value of the signal that occurs at $t = 0$ in the original signal occurs at $t = \tau$ in the delayed signal. Similarly, we can advance $x(t)$, i.e. shift it ahead in time, by writing $x(t + \tau)$.
- *Time scaling.* We can expand $x(t)$ in time as follows: if α is the time scaling factor, where t seconds in the original signal becomes t/α seconds in the new signal, then the scaled signal is $x(t/\alpha)$. For example, $\alpha = 2$ dilates (i.e., slows down) the signal in time by a factor of 2. If $\alpha < 1$, then the signal is contracted in time (i.e., it speeds up). If $\alpha < 0$, then the

signal is time-reversed (where $\alpha = -1$ corresponds to exact time reversal, without scaling).

In both of the above cases, the transformation is a full *change of variables*, which will be illustrated in the examples throughout the remainder of the section.

1.4.2. Even and odd signals.

DEFINITION 1.4. *If a signal $x(t)$ has the property*

$$x(t) = x(-t), \tag{1.51}$$

*then $x(t)$ is an **even** signal.*

DEFINITION 1.5. *If a signal $x(t)$ has the property*

$$x(t) = -x(-t), \tag{1.52}$$

*then $x(t)$ is an **odd** signal.*

If $x(t)$ is even, then

$$\int_{-t}^{t} x(u)du = \int_{-t}^{0} x(u)du + \int_{0}^{t} x(u)du \tag{1.53}$$

$$= \int_{0}^{t} x(-u)du + \int_{0}^{t} x(u)du \tag{1.54}$$

$$= \int_{0}^{t} x(u)du + \int_{0}^{t} x(u)du \tag{1.55}$$

$$= 2\int_{0}^{t} x(u)du. \tag{1.56}$$

If $x(t)$ is odd, then

$$\int_{-t}^{t} x(u)du = \int_{-t}^{0} x(u)du + \int_{0}^{t} x(u)du \tag{1.57}$$

$$= \int_{0}^{t} x(-u)du + \int_{0}^{t} x(u)du \tag{1.58}$$

$$= \int_{0}^{t} -x(u)du + \int_{0}^{t} x(u)du \tag{1.59}$$

$$= 0. \tag{1.60}$$

1.4.3. Step and rectangle signals. The unit step function is given by

$$U(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0. \end{cases} \tag{1.61}$$

The rectangle function is given by

$$\mathrm{rect}(t) = \left\{ \begin{array}{ll} 1, & |t| < 1/2 \\ 0, & |t| \geq 1/2. \end{array} \right\} \tag{1.62}$$

Note that $\mathrm{rect}(t) = U(t + 1/2) - U(t - 1/2)$. These functions can also be defined in discrete time:

$$U[k] = \begin{cases} 0, & k < 0 \\ 1, & k \geq 0 \end{cases} \tag{1.63}$$

and

$$\mathrm{rect}[k] = \left\{ \begin{array}{ll} 1, & |k| < 1/2 \\ 0, & |k| \geq 1/2 \end{array} \right\} \tag{1.64}$$

Example 1.8. *Suppose we want a step function that starts not at time $t = 0$, but at time $t = 5$. Thus, we want to delay the start of the signal to $t = 5$, and we can write $U(t - 5)$.*

Example 1.9. *Suppose we want to both reverse a step function (i.e., make it start as 1, then go to 0), and delay the time of the transition to time $t = 5$. We can first reverse the function by writing $U(-t)$. To delay, we make a change of variables, substituting $t-5$ for t. Thus, the final signal is $U(-(t-5)) = U(-t+5)$.*

Example 1.10. *Suppose we want a rectangular signal centred around $t = -3$, i.e., advancing the signal by 3 seconds; moreover, we want the width of the signal to be 2 seconds. We can first scale the signal:* $\mathrm{rect}(t/2)$*, and then change t for $t + 3$: the final signal is*

$$\mathrm{rect}\left(\frac{t+3}{2}\right) = \mathrm{rect}\left(\frac{t}{2} + \frac{3}{2}\right). \tag{1.65}$$

Note the *change of variables* in the above examples; for instance, in Example 1.9, one may be tempted to write $U(-t-5)$ to delay the reversed step $U(-t)$, but this is not correct.

Using the rectangle function, we can create a periodic *train* of rectangle functions, also called a *square wave function*, as follows. We will use the original rectangle function, specified above. For a period T, the periodic square wave will have one rectangle centred around 0, one centred around T, one centred around $2T$, and so on (without forgetting about the negative time axis either: $-T$, $-2T$, and so on). So the square wave function $S(t)$ will look like an infinite series of delayed and advanced rectangle functions:

$$S(t)$$

$$= \dots \operatorname{rect}(t+2T) + \operatorname{rect}(t+T) + \operatorname{rect}(t) + \operatorname{rect}(t-T) + \operatorname{rect}(t-2T)\dots \tag{1.66}$$

$$= \sum_{j=-\infty}^{\infty} \operatorname{rect}(t - jT). \tag{1.67}$$

In the exercises, you will show that $S(t)$ is periodic with period T.

The *duty cycle* of a square wave is the percent of time in one period that it spends "on" (i.e., equal to 1). If $T = 2$, then the duty cycle is 50%.

1.4.4. Delta function and signals. The *Dirac delta function*, $\delta(t)$, is usually defined in terms of its integral:

$$\int_{-\infty}^{t} \delta(u)du = U(t). \tag{1.68}$$

Noting that the integral gives the area under the curve, we conclude that:

- For all $t < 0$, the area under $\delta(t)$ is constant at zero. Thus, $\delta(t) = 0$ for all $t < 0$.
- For all $t > 0$, the area under $\delta(t)$ is constant at 1. Thus, $\delta(t) = 0$ for all $t > 0$.
- Precisely at $t = 0$, the area under $\delta(t)$ is 1. Thus, the amplitude at this point is infinite: an infinitely tall function with an infinitely short base, the area under which is exactly 1.

Also, note that $\delta(t) = \frac{d}{dt}U(t)$.

The Dirac delta function has the following useful integral property:

$$\int_{-\infty}^{\infty} x(t)\delta(t)dt = x(0)\int_{-\infty}^{\infty} x(t)dt \tag{1.69}$$

$$= x(0), \tag{1.70}$$

which follows since $\delta(t) = 0$ everywhere except at $t = 0$.

The equivalent function in discrete time is called the *Kronecker delta function*:

$$\delta[k] = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases} . \tag{1.71}$$

Either delta function is also called the *impulse function*. If a system input $x(t) = \delta(t)$, then the system output $y(t)$ is called the *impulse response*.

1.4.5. Real and complex sinusoidal signals. A sinusoidal signal $x(t)$ with frequency f and phase θ is written

$$x(t) = \sin(2\pi f t + \theta). \tag{1.72}$$

(Note that the period $T = 1/f$.) The equivalent *angular frequency* ω is given by

$$\omega = 2\pi f, \tag{1.73}$$

so (1.72) can be written

$$x(t) = \sin(\omega t + \theta). \tag{1.74}$$

The sinusoids sin and cos are equivalent up to a phase shift:

$$\sin(\omega t) = \cos\left(\omega t - \frac{\pi}{2}\right). \tag{1.75}$$

Furthermore, $\sin(\omega t)$ is an odd function, while $\cos(\omega t)$ is an even function.

The function $e^{j\omega t}$ is a complex-valued sinusoid. By Euler's formula,

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t). \tag{1.76}$$

With some manipulation, we can write

$$\cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2} \tag{1.77}$$

$$\sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j} \tag{1.78}$$

1.5. Complex-valued signals

Let $x(t)$ represent a complex-valued signal. The real part of $x(t)$ will be written $x_R(t) = \Re(x(t))$, and the imaginary part will be written $x_I(t) = \Im(x(t))$. Then

$$x(t) = \Re(x(t)) + j\Im(x(t)) \tag{1.79}$$

$$= x_R(t) + jx_I(t). \tag{1.80}$$

DEFINITION 1.6. *Let $x(t) = x_R(t) + jx_I(t)$ represent a complex-valued signal. The **complex conjugate** $x(t)$, written $x^*(t)$, is*

$$x^*(t) = x_R(t) - jx_I(t). \tag{1.81}$$

In general, the complex conjugate is formed by substituting j with $-j$.

DEFINITION 1.7. *The **magnitude** of a complex signal, $|x(t)|$, is given by*

$$|x(t)| = \sqrt{x_R(t)^2 + x_I(t)^2}. \tag{1.82}$$

This leads to the following result.

THEOREM 1.1. *Let $x(t)$ be a complex-valued signal. Then*

$$|x(t)| = \sqrt{x(t)x^*(t)}. \tag{1.83}$$

Proof:

$$x(t)x^*(t) = (x_R(t) + jx_I(t))(x_R(t) - jx_I(t))] \tag{1.84}$$

$$= x_R(t)^2 + jx_I(t)x_R(t) - jx_R(t)x_I(t) - (j^2)x_I(t)^2 \tag{1.85}$$

$$= x_R(t)^2 + x_I(t)^2 \tag{1.86}$$

$$= |x(t)|^2. \tag{1.87}$$

The theorem follows by taking the square root of both sides. ■

EXAMPLE 1.11. *The complex conjugate of* $e^{j\omega t}$ *is* $e^{-j\omega t}$. *The magnitude of* $e^{j\omega t}$, $|e^{j\omega t}|$, *is*

$$|e^{j\omega t}| = \sqrt{e^{j\omega t}(e^{j\omega t})^*} \tag{1.88}$$

$$= \sqrt{e^{j\omega t}e^{-j\omega t}} \tag{1.89}$$

$$= \sqrt{1} \tag{1.90}$$

$$= 1. \tag{1.91}$$

1.6. Problems

(1) Show that the system from Example 1.2 is linear.

(2) Show that the system from Example 1.3 is time-invariant.

(3) If $f(t)$ is periodic with period T, then:

  (a) Show that $f(t-\tau)$ is periodic for any τ.

  (b) Using the above result, show that $f(t)$ is periodic with period kT, for any integer k.

(4) Let $f(t)$ and $g(t)$ be two periodic signals, with fundamental period T_f and T_g, respectively. If the least common multiple[2] of T_f and T_g, $\mathrm{LCM}(T_f, T_g)$, is finite, show that $f(t)+g(t)$ is periodic, with period equal to $\mathrm{LCM}(T_f, T_g)$.

(5) Show that a power signal has $E = \infty$.

(6) Show that an energy signal has $P_{\text{avg}} = 0$.

(7) If $f(t)$ and $g(t)$ are signals:

  (a) If $f(t)$ and $g(t)$ are both even, show that $f(t) + g(t)$ is even.

  (b) If $f(t)$ and $g(t)$ are both odd, show that $f(t) + g(t)$ is odd.

  (c) If $f(t)$ and $g(t)$ are both even or both odd, show that $f(t)g(t)$ is even.

(8) Show that the square wave function $S(t)$ is periodic with period T, and show that $S(t)$ is a power signal.

(9) Let $x(t)$ be any signal. Show that

$$\sum_{j=-\infty}^{\infty} x(t - jT) \tag{1.92}$$

is a periodic signal with period T.

[2]Least common multiple of a and b is the smallest positive number c such that $ja = kb = c$, for any integers j and k. Note that a, b, and c do *not* need to be integers.

CHAPTER 2

Systems Using Ordinary Differential and Difference Equations

2.1. Properties and solutions of ordinary differential equations

Ordinary differential equations (ODEs) can be written in the form

$$c_0 y(t) + \sum_{i=1}^{n} c_i \frac{d^i}{dt^i} y(t) = x(t). \tag{2.1}$$

with an input signal $x(t)$ and an output signal $y(t)$. In this course, we will only deal with systems up to second order, i.e.,

$$c_0 y(t) + c_1 \frac{d}{dt} y(t) + c_2 \frac{d^2}{dt^2} y(t) = x(t). \tag{2.2}$$

For shorthand, we will write

$$\frac{d^i}{dt^i} y(t) = y^{(i)}(t), \tag{2.3}$$

with $y^{(0)}(t) = y(t)$, so (2.2) becomes

$$c_0 y(t) + c_1 y^{(1)}(t) + c_2 y^{(2)}(t) = x(t). \tag{2.4}$$

Normally, for systems described as ODEs, we provide the input signal $x(t)$ and ask what is the resulting output signal $y(t)$. Thus, $y(t)$ is called the *solution* for input $x(t)$.

2.1.1. ODEs are LTI systems.

We first show that ODEs are linear. Suppose we have two inputs, $x_a(t)$ and $x_b(t)$, with corresponding outputs $y_a(t)$ and $y_b(t)$, respectively. Then it is true that

$$c_0 y_a(t) + c_1 y_a^{(1)}(t) + c_2 y_a^{(2)}(t) = x_a(t), \tag{2.5}$$

$$c_0 y_b(t) + c_1 y_b^{(1)}(t) + c_2 y_b^{(2)}(t) = x_b(t). \tag{2.6}$$

What happens if we provide the input $\alpha x_a(t) + \beta x_b(t)$? We can show that the resulting solution is $\alpha y_a(t) + \beta y_b(t)$:

$$\begin{aligned} &\alpha x_a(t) + \beta x_b(t) \\ &= \alpha c_0 y_a(t) + \alpha c_1 y_a^{(1)}(t) + \alpha c_2 y_a^{(2)}(t) + \beta c_0 y_b(t) + \beta c_1 y_b^{(1)}(t) + \beta c_2 y_b^{(2)}(t) \qquad (2.7) \\ &= c_0\Big(\alpha y_a(t) + \beta y_b(t)\Big) + c_1\Big(\alpha y_a^{(1)}(t) + \beta y_b^{(1)}(t)\Big) + c_2\Big(\alpha y_a^{(2)}(t) + \beta y_b^{(2)}(t)\Big) \qquad (2.8) \\ &= c_0\Big(\alpha y_a(t) + \beta y_b(t)\Big) + c_1 \frac{d}{dt}\Big(\alpha y_a(t) + \beta y_b(t)\Big) + c_2 \frac{d^2}{dt^2}\Big(\alpha y_a(t) + \beta y_b(t)\Big), \qquad (2.9) \end{aligned}$$

where the last line follows since the differentiation operator d^i/dt^i is linear. Therefore, $\alpha y_a(t) + \beta y_b(t)$ is a solution for $\alpha x_a(t) + \beta x_b(t)$, which implies that the system is linear.

To show that ODEs are time-invariant, first note that the differentiation operator is time invariant: if $z(t) = d/dt\, y(t)$, then

$$z(t - \tau) = \frac{d}{dt} y(t - \tau). \qquad (2.10)$$

Thus, if $y(t)$ is a solution for $x(t)$, then

$$\begin{aligned} &c_0 y(t-\tau) + c_1 y^{(1)}(t-\tau) + c_2 y^{(2)}(t-\tau) \\ &= c_0 y(t-\tau) + c_1 \frac{d}{dt} y(t-\tau) + c_2 \frac{d^2}{dt^2} y(t-\tau) \qquad (2.11) \\ &= x(t-\tau). \qquad (2.12) \end{aligned}$$

2.1.2. Solutions to ODEs. Up to second order, a solution to an ODE is of the form

$$y(t) = re^{st}, \qquad (2.13)$$

where r and s are (possibly complex valued) constants. Note that

$$y^{(1)}(t) = sre^{st}, \qquad (2.14)$$

and

$$y^{(2)}(t) = s^2 re^{st}. \qquad (2.15)$$

We will first consider the *transient* solution, where $x(t) = 0$. Substituting into (2.4), we have

$$0 = c_0 re^{st} + c_1 sre^{st} + c_2 s^2 re^{st} \tag{2.16}$$

$$= re^{st}(c_0 + c_1 s + c_2 s^2). \tag{2.17}$$

Equation (2.17) is satisfied with equality if:

(1) $r = 0$; or

(2) $c_0 + c_1 s + c_2 s^2 = 0$.

If $r = 0$, then $y(t) = 0$, which is a trivial solution; we ignore this case. The equation $c_0 + c_1 s + c_2 s^2$ has two solutions in s, s_a and s_b, from the quadratic equation:

$$s_a = \frac{-c_1 + \sqrt{c_1^2 - 4c_2 c_0}}{2c_2} \tag{2.18}$$

$$s_b = \frac{-c_1 - \sqrt{c_1^2 - 4c_2 c_0}}{2c_2} \tag{2.19}$$

Thus the ODE has two solutions:

$$y_a(t) = r_a e^{s_a t} \tag{2.20}$$

$$y_b(t) = r_b e^{s_b t}, \tag{2.21}$$

but the ODE is linear, so we can combine the two solutions:

$$y(t) = r_a e^{s_a t} + r_b e^{s_b t}. \tag{2.22}$$

The solution in (2.22) is a general solution, but the constants r_a and r_b can be obtained by specifying initial conditions.

Example 2.1.

We now consider the *steady-state* solution, in which $x(t) \neq 0$. For the purposes of this course, it will be sufficient to consider $x(t)$ as a complex sinusoid with amplitude a:

$$x(t) = ae^{j\omega t}. \tag{2.23}$$

Thus, we have

$$ae^{j\omega t} = c_0 re^{st} + c_1 sre^{st} + c_2 s^2 re^{st} \tag{2.24}$$

$$= re^{st}(c_0 + c_1 s + c_2 s^2). \tag{2.25}$$

Substituting $s = j\omega$, we have

$$ae^{j\omega t} = r(c_0 + c_1 j\omega + c_2(j\omega)^2)e^{j\omega t} \tag{2.26}$$

so

$$r = \frac{a}{c_0 + c_1 j\omega + c_2(j\omega)^2} \tag{2.27}$$

and the steady state solution becomes

$$y(t) = \frac{a}{c_0 + c_1 j\omega - c_2\omega^2} e^{j\omega t}. \tag{2.28}$$

Again, since the system is linear, the overall solution is formed by the sum of the transient and steady-state solutions. However, we will be most interested in the steady-state solution.

EXAMPLE 2.2.

2.2. Difference equations

A *difference equation* is the discrete-time counterpart to an ODE. A difference equation can be written

$$d_0 y[k] + d_1 y[k-1] + \ldots + d_n y[k-n] = x[k], \tag{2.29}$$

or more succinctly as

$$\sum_{i=0}^{n} d_i y[k-i] = x[k]. \tag{2.30}$$

Difference equations share many similarities with ODEs. For example, difference equations are LTI, which we leave for the problems.

Consider the transient solution of a difference equation. For convenience, we will consider a first-order equation: setting $x[k] = 0$, we must solve

$$d_0 y[k] + d_1 y[k-1] = 0. \tag{2.31}$$

The solution is of the form $y[k] = r\alpha^k$. Substituting into (2.31),

$$d_0 r\alpha^k + d_1 r\alpha^{k-1} = 0 \tag{2.32}$$

which can be rearranged

$$r\alpha^{k-1}\Big(d_0\alpha + d_1\Big) = 0. \tag{2.33}$$

Equation (2.35) is satisfied if $r = 0$ (which is trivial), or if $d_0\alpha + d_1 = 0$, which implies

$$\alpha = -\frac{d_1}{d_0}. \tag{2.34}$$

As in the ODE case, the value of r can be found by specifying an initial condition. Second (and higher) order difference equations can be solved in a similar manner, which you will show in the problems.

The steady-state solution is found in a similar manner to the ODE case. Let $x[k] = e^{j\omega k}$. For the first-order difference equation, following (2.35) we can set

$$r\alpha^{k-1}\Big(d_0\alpha + d_1\Big) = e^{j\omega k}. \tag{2.35}$$

Setting $\alpha = e^{j\omega}$,

$$e^{j\omega k} = re^{j\omega(k-1)}\Big(d_0 e^{j\omega} + d_1\Big) \tag{2.36}$$

$$= re^{j\omega k}\frac{d_0 e^{j\omega} + d_1}{e^{j\omega}}. \tag{2.37}$$

which leads to $r = e^{j\omega}/(d_0 e^{j\omega} + d_1)$, and

$$y[k] = \left(d_0 + d_1 e^{-j\omega}\right) e^{j\omega k}. \tag{2.38}$$

Consider the form of (2.38): it closely follows the form of (2.31). In fact, the steady-state solution for $x[k] = e^{j\omega k}$ in the general difference equation (2.30) is

$$y[k] = e^{j\omega k}\sum_{i=0}^{n} d_i e^{-j\omega i}. \tag{2.39}$$

Example 2.3.

2.3. Problems

(1) Consider the general first-order ODE

$$c_0 y(t) + c_1 y^{(1)}(t) = x(t). \tag{2.40}$$

Give the general transient solution for this ODE, assuming that $y(0) = a$, and $x(t) = 0$.

(2) Show that difference equations of the form of (2.30) are LTI.

(3) Consider the general second-order difference equation

$$d_0 y[k] + d_1 y[k-1] + d_2 y[k-2] = x[k]. \tag{2.41}$$

Give the general transient solution for this difference equation, assuming that $y[0] = a$, $y[1] = b$, and $x[k] = 0$.

Part 2

Continuous-time signals and systems

CHAPTER 3

Periodic Signals and the Fourier Series

3.1. A bit about vector spaces

Let $v[k]$ and $w[k]$, $k = 0, 1, \dots, n-1$ be vectors. Suppose we want to find the *projection* of $w[k]$ on $v[k]$: that is, the value of α such that the error ϵ, given by

$$\epsilon = \Big| w[k] - \alpha v[k] \Big|, \tag{3.1}$$

is minimized. It turns out this projection is given by

$$\frac{w[k] \cdot v[k]}{|v[k]|} v[k], \tag{3.2}$$

where $w[k] \cdot v[k]$ is the *dot product*, given by

$$w[k] \cdot v[k] = \sum_{i=0}^{n-1} w[i]v[i]. \tag{3.3}$$

If the vectors are complex, then dot product is defined

$$w[k] \cdot v[k] = \sum_{i=0}^{n-1} w[i]v[i]^*, \tag{3.4}$$

recalling that * represents complex conjugation. The magnitude of a vector, $|w[k]|$, can be expressed in terms of the dot product:

$$|w[k]|^2 = \sum_{i=0}^{n-1} w[k]^2 \tag{3.5}$$

$$= w[k] \cdot w[k]. \tag{3.6}$$

Example 3.1.

If there are two or more vectors $v_1[k], v_2[k], \dots, v_m[k]$, those vectors form a *vector space*, and the vectors $v_i[k]$ are *basis vectors*. We can ask how the vector $w[k]$ is projected into the vector space. That is, can we find constants α_i that minimizes the error ϵ, which is now

$$\epsilon = \left| w[k] - \sum_{i=1}^{m} \alpha_i v_i[k] \right| ? \tag{3.7}$$

For now we will restrict ourselves to *orthonormal basis vectors*, that is, for any pair $v_i[k]$ and $v_j[k]$,

- For any i, $|v_i[k]| = 1$; and
- For any pair i and j, if $i \neq j$, then $v_i[k] \cdot v_j[k] = 0$.

That is, the vectors are unit-length and orthogonal to each other. If a basis is orthonormal, then from (3.2), the the projection onto each basis vector is

$$\alpha_i = w[k] \cdot v_i[k], \tag{3.8}$$

and the minimum-error projection is

$$\sum_{i=1}^{m} (w[k] \cdot v_i[k]) v_i[k]. \tag{3.9}$$

EXAMPLE 3.2. *An example of an orthonormal basis is the set of Cartesian basis vectors, for example with $n = 3$,*

$$\begin{aligned} v_1 &= [1, 0, 0] \\ v_2 &= [0, 1, 0] \\ v_3 &= [0, 0, 1]. \end{aligned} \tag{3.10}$$

Another example is

$$\begin{aligned} v_1 &= \left[\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right] \\ v_2 &= [0, 1, 0] \\ v_3 &= \left[\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}} \right]. \end{aligned} \tag{3.11}$$

A set of basis vectors $v_i[k]$ is said to span the space if *any* vector in the space can be expressed exactly as a linear combination of those vectors, as in (3.9). From linear algebra, we know that any orthonormal basis with n vectors will span an n-dimensional vector space.

3.2. Sinusoids as an orthonormal basis in discrete time

For $k = 0, 1, \ldots, n-1$, consider the pair of vectors

$$v_s[k] = \sqrt{\frac{2}{n}} \sin\left(\frac{2\pi k}{n}\right) \tag{3.12}$$

$$v_c[k] = \sqrt{\frac{2}{n}} \cos\left(\frac{2\pi k}{n}\right) \tag{3.13}$$

These signals are a single period of the discrete-time sinusoid with period $1/n$. In the exercises, you show that

$$|v_s[k]| = |v_c[k]| = 1. \tag{3.14}$$

and that

$$v_s[k] \cdot v_c[k] = 0. \tag{3.15}$$

3.3. The Fourier series in discrete time

For any discrete-time periodic signal, we can formulate the discrete-time Fourier series as follows.

Let n represent the period; for convenience, we will assume that n is even, but similar analysis may be used if n is odd. We start with the discrete-time functions

$$w_{s,j}[k] = \sin\left(\frac{2\pi jk}{n}\right), j = 0, 1, 2, \ldots, n/2 \tag{3.16}$$

$$w_{c,j}[k] = \cos\left(\frac{2\pi jk}{n}\right), j = 0, 1, 2, \ldots, n/2. \tag{3.17}$$

However, if $j = 0$, $w_{s,0}[k] = 0$, which is trivial (meanwhile, $w_{c,0}[k] = 1$). Further, if $j = n/2$, $w_{c,n/2}[k] = 0$ (meanwhile, $w_{s,n/2}[k] = \sin(k\pi)$, which oscillates between +1 and -1).

3.4. Signal space in continuous time

Consider the continuous-time version of a discrete-time signal.

Following the above discussion, suppose we redefine dot product as follows: for signals $x(t)$ and $y(t)$, defined on the interval $[0, T]$, let

$$x(t) \cdot y(t) = \int_0^T x(t)y(t)dt. \tag{3.18}$$

Using this definition, the magnitude of a signal —x(t)— becomes

$$|x(t)| = \sqrt{x(t) \cdot x(t)} \tag{3.19}$$

$$= \sqrt{\int_0^T x(t)^2 dt}. \tag{3.20}$$

Note that $|x(t)|^2$ is the total energy in the signal $x(t)$.

Example 3.3.

Using the continuous-time dot product from (3.18), we can define an orthonormal basis in a similar way to the discrete case. A collection of signals $v_1(t), v_2(t), \ldots, v_n(t)$ forms an *orthonormal basis* if:

- For any i, $|v_i(t)| = 1$; and
- For any pair i and j, if $i \neq j$, then $v_i(t) \cdot v_j(t) = 0$.

Moreover, a signal $x(t)$ can be projected onto the basis as follows. To obtain the coordinate of any basis vector in an orthonormal basis, e.g. $v_i(t)$, we find $x(t) \cdot v_i(t)$:

$$\hat{x}(t) = \sum_{i=1}^{n} v_i(t)(v_i(t) \cdot x(t)) \tag{3.21}$$

$$= \sum_{i=1}^{n} v_i(t) \int_0^T v_i(t)x(t). \tag{3.22}$$

If $\hat{x}(t) = x(t)$, then $x(t)$ is in the space spanned by the basis signals $v_i(t)$.

Example 3.4.

Similarly to the discrete-time case, there exists an orthonormal basis consisting of sinusoidal signals. Consider

$$v_0(t) = \sqrt{\frac{1}{T}} \tag{3.23}$$

$$v_{c,k}(t) = \sqrt{\frac{2}{T}} \cos\left(\frac{2\pi kt}{T}\right), k = 1, 2, \ldots \tag{3.24}$$

$$v_{s,k}(t) = \sqrt{\frac{2}{T}} \sin\left(\frac{2\pi kt}{T}\right), k = 1, 2, \ldots. \tag{3.25}$$

It can be shown that this set of signals forms an orthonormal basis in continuous-time signal space.

3.5. The Fourier Series in Continuous Time

3.5.1. Trigonometric Fourier series. The equations in (3.23)-(3.25) form an orthonormal basis. Therefore, using the dot product, we can easily find the projection of any periodic signal $x(t)$ onto this basis. We now show how to construct the Fourier series with respect to this basis.

Consider a periodic signal $x(t)$ with period T. From (3.23)-(3.25), we can obtain the projection of $x(t)$ onto the basis using the dot product. Calling the coefficients a'_0, a'_k, and b'_k corresponding to v_0, $v_{s,k}$, and $v_{c,k}$, respectively:

$$a'_0 = v_0(t) \cdot x(t) \tag{3.26}$$

$$= \int_0^T \sqrt{\frac{1}{T}} x(t) dt \tag{3.27}$$

$$a'_k = v_{c,k} \cdot x(t) \tag{3.28}$$

$$= \int_0^T \sqrt{\frac{2}{T}} \cos\left(\frac{2\pi kt}{T}\right) x(t) dt, k = 1, 2, \ldots \tag{3.29}$$

$$b'_k = v_{c,k} \cdot x(t) \tag{3.30}$$

$$= \int_0^T \sqrt{\frac{2}{T}} \sin\left(\frac{2\pi kt}{T}\right) x(t) dt, k = 1, 2, \ldots \tag{3.31}$$

To reassemble the function $x(t)$ from the basis functions, we take a linear combination of the basis functions with the coefficients obtained in the above equations:

$$x(t) = a'_0 v_0^t + \sum_{k=1}^{\infty} a'_{c,k} v_{c,k}(t) + \sum_{k=1}^{\infty} b'_{c,k} v_{s,k}(t) \tag{3.32}$$

$$= a'_0 \sqrt{\frac{1}{T}} + \sum_{k=1}^{\infty} a'_{c,k} \sqrt{\frac{2}{T}} \cos\left(\frac{2\pi kt}{T}\right) + \sum_{k=1}^{\infty} b'_{c,k} \sqrt{\frac{2}{T}} \sin\left(\frac{2\pi kt}{T}\right) \tag{3.33}$$

In (3.33), the radical terms $\sqrt{1/T}$ and $\sqrt{2/T}$ can be eliminated by collecting them together with a'_0, a'_k, and b'_k, by letting

$$a_0 = a'_0 \sqrt{\frac{1}{T}} \tag{3.34}$$

$$a_k = a'_k \sqrt{\frac{2}{T}} \tag{3.35}$$

$$b_k = b'_k \sqrt{\frac{2}{T}}, \tag{3.36}$$

which also eliminates the radicals from the calculation of the coefficients. Using this substitution, the trigonometric Fourier series is defined as follows.

Definition 3.1. *For (almost) any periodic signal $x(t)$, with period T, the signal can be represented as*

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos\left(\frac{2\pi k t}{T}\right) + \sum_{k=1}^{\infty} b_k \sin\left(\frac{2\pi k t}{T}\right), \tag{3.37}$$

where

$$a_0 = \frac{1}{T}\int_0^T x(t)dt \tag{3.38}$$

$$a_k = \frac{2}{T}\int_0^T x(t)\cos\left(\frac{2\pi k t}{T}\right), k = 1, 2, \ldots \tag{3.39}$$

$$b_k = \frac{2}{T}\int_0^T x(t)\sin\left(\frac{2\pi k t}{T}\right), k = 1, 2, \ldots. \tag{3.40}$$

The form given in (3.37) is called the ***trigonometric Fourier series****.*

The coefficients (3.38)-(3.40) differ from those obtained using the orthonormal basis method by a constant factor: $\sqrt{1/T}$ for a_0 and $\sqrt{2/T}$ for a_k and b_k. This constant multiplication is taken into account in the Fourier series form in (3.37), resulting in an equivalent representation to the orthonormal projection.

We will sometimes substitute $f = 1/T$ as the fundamental frequency of $x(t)$, or $\omega = 2\pi f = 2\pi/T$ as the fundamental angular frequency of $x(t)$.

The integrals in (3.38)-(3.40) can be taken over *any complete period* of $x(t)$ other than $[0, T]$ (e.g., $[-T/2, T/2]$, $[T, 2T]$). In some cases, a more convenient period may be found to make the calculation easier.

3.5.2. Exponential Fourier series.

Recall Euler's formula (1.76), in which $\sin(\omega t)$ and $\cos(\omega t)$ are expressed in terms of complex exponentials $e^{j\omega t}$. Using this formula, there exists a form of the Fourier series, specified in terms of $e^{j\omega t}$. For convenience, let $\exp(x) = e^x$.

Starting with the trigonometric Fourier series defined in (3.37), we substitute Euler's formula:

(3.41)
$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos\left(\frac{2\pi kt}{T}\right) + \sum_{k=1}^{\infty} b_k \sin\left(\frac{2\pi kt}{T}\right)$$

(3.42)
$$= a_0 + \sum_{k=1}^{\infty} a_k \frac{\exp\left(j\frac{2\pi kt}{T}\right) + \exp\left(-j\frac{2\pi kt}{T}\right)}{2} + \sum_{k=1}^{\infty} b_k \frac{\exp\left(j\frac{2\pi kt}{T}\right) - \exp\left(-j\frac{2\pi kt}{T}\right)}{2j}$$

(3.43)
$$= a_0 + \sum_{k=1}^{\infty} \frac{a_k - jb_k}{2} \exp\left(j\frac{2\pi kt}{T}\right) + \sum_{k=1}^{\infty} \frac{a_k + jb_k}{2} \exp\left(-j\frac{2\pi kt}{T}\right).$$

To simplify, define a coefficient d_k as follows:

$$d_k = \begin{cases} \frac{a_k - jb_k}{2}, & k \geq 1 \\ a_0, & k = 0 \\ \frac{a_{-k} + jb_{-k}}{2}, & k \leq -1 \end{cases} \tag{3.44}$$

Then the Fourier series becomes

$$x(t) = d_0 + \sum_{k=1}^{\infty} d_k \exp\left(j\frac{2\pi kt}{T}\right) + \sum_{k=-\infty}^{-1} d_k \exp\left(j\frac{2\pi kt}{T}\right) \tag{3.45}$$

$$= \sum_{k=-\infty}^{\infty} d_k \exp\left(j\frac{2\pi kt}{T}\right). \tag{3.46}$$

The coefficients d_k can be obtained starting with the trigonometric Fourier series. For example, for $k \geq 1$:

$$d_k = \frac{a_k - jb_k}{2} \tag{3.47}$$

$$= \frac{1}{T}\int_0^T x(t)\cos\left(\frac{2\pi kt}{T}\right) dt - \frac{j}{T}\int_0^T x(t)\sin\left(\frac{2\pi kt}{T}\right) dt \tag{3.48}$$

$$= \frac{1}{T}\int_0^T x(t) \frac{\exp\left(j\frac{2\pi kt}{T}\right) + \exp\left(-j\frac{2\pi kt}{T}\right)}{2} dt$$

$$- \frac{j}{T}\int_0^T x(t) \sum_{k=1}^{\infty} b_k \frac{\exp\left(j\frac{2\pi kt}{T}\right) - \exp\left(-j\frac{2\pi kt}{T}\right)}{2j} dt \tag{3.49}$$

$$= \frac{1}{T}\int_0^T x(t)\exp\left(-j\frac{2\pi kt}{T}\right) dt. \tag{3.50}$$

(The reader can verify that the right hand side of the calculation is the same for $k < 1$.) Thus, we have the following definition.

DEFINITION 3.2. *For (almost) any periodic signal $x(t)$, with period T, the signal can be represented as*

$$x(t) = \sum_{k=-\infty}^{\infty} d_k \exp\left(j\frac{2\pi kt}{T}\right), \tag{3.51}$$

where

$$d_k = \frac{1}{T}\int_0^T x(t) \exp\left(-j\frac{2\pi kt}{T}\right). \tag{3.52}$$

The form given in (3.51) is called the ***exponential Fourier series****.*

Once again, we can take the integral over any one complete period.

3.6. Convergence of Fourier series

We now give a sufficient condition on the convergence of the Fourier series.

First, for a function $x(t)$ that is discontinuous at t_0, let the *magnitude* $M(t_0)$ of the discontinuity be

$$M(t_0) = |\lim_{t\to t_0^+} x(t) - \lim t \to t_0^- x(t)|, \tag{3.53}$$

where superscript $^+$ represents the limit counting down from values greater than t_0, and the superscript $^-$ represents the limit counting up from values less than t_0.

The *Dirichlet conditions* give sufficient conditions on the convergence of the Fourier series, either in the trigonometric or exponential form.

DEFINITION 3.3. *Let $x(t)$ be a periodic signal with period T. The* ***Dirichlet conditions*** *on $x(t)$ are given as follows:*

(1) *The signal $x(t)$ is absolutely integrable, i.e.,*

$$\int_0^T |x(t)|dt < \infty. \tag{3.54}$$

(The integral may be taken over any complete period.)

(2) *In any one period of $x(t)$, the number of maxima and minima is finite.*

(3) *In any one period of $x(t)$, the number of discontinuities is finite, and all discontinuities have finite magnitude.*

If $x(t)$ satisfies the Dirichlet conditions, then its Fourier series converges.

If a signal $x(t)$ achieves a maximum and remains constant at that maximum, that counts as one maximum for the purposes of the Dirichlet conditions.

An example of a signal that does not satisfy the second Dirichlet condition is $\sin(1/t)$: as $t \to 0$, there are an infinite number of distinct maxima.

An example of a signal that does not satisfy the third Dirichlet condition is $\tan(t)$: the period is π, and there is at most one discontinuity per period (at $t = k\pi$, for integer k), but that discontinuity is infinite in magnitude.

3.7. The Fourier series and systems

Consider a 2nd-order ODE system described by

$$x(t) = c_0 y(t) + c_1 y^{(1)}(t) + c_2 y^{(2)}(t). \tag{3.55}$$

In Chapter 2 we described the *steady-state* solution to this system: if the input is a complex sinusoid $ae^{j\omega t}$, then the response $y(t)$ is given by

$$y(t) = \frac{a}{c_0 + c_1 j\omega - c_2\omega^2} e^{j\omega t}. \tag{3.56}$$

Now suppose the input is periodic with period T, and described by an exponential Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} d_k \exp\left(j\frac{2\pi kt}{T}\right) \tag{3.57}$$

If $\omega = 2\pi/T$, then

$$x(t) = \sum_{k=-\infty}^{\infty} d_k e^{jk\omega t}. \tag{3.58}$$

Systems described by ordinary differential equations are LTI. Therefore,

$$y(t) = \sum_{k=-\infty}^{\infty} \frac{d_k}{c_0 + c_1 jk\omega - c_2 k^2\omega^2} e^{jk\omega t}. \tag{3.59}$$

3.8. Problems

(1) For the signals $x_s[k]$ and $x_c[k]$ in (3.12), show that $|x_s[k]| = |x_c[k]| = 1$ and $x_s[k] \cdot x_c[k] = 0$.

(2) Consider the periodic signal $x(t)$, with period $T = 1$, defined over one period by

$$x(t) = \begin{cases} 1, & 0 \leq t < 0.5; \\ -1, & 0.5 \leq t < 1. \end{cases} \tag{3.60}$$

Sketch several periods of $x(t)$, and find the trigonometric Fourier series of this signal.

(3) Consider the periodic signal $x(t)$ with period $T = 2$, defined over one period by

$$x(t) = t - 1, \;\; 0 \leq t < 2. \tag{3.61}$$

Sketch several periods of $x(t)$, and find the exponential Fourier series of this signal. You may use the indefinite integral: $\int te^{\alpha t}dt = \frac{1}{\alpha^2}(e^{\alpha t}(\alpha t - 1))$.

(4) Show that, for any odd signal $x(t)$, the coefficients a_k from (3.39) are equal to zero.

(5) Show that, for any even signal $x(t)$, the coefficients b_k from (3.40) are equal to zero.

(6) Show that the signals in (3.60) and (3.61) satisfy the Dirichlet conditions.

CHAPTER 4

Non-Periodic Signals and the Fourier Transform

4.1. From periodic to non-periodic

In the previous chapter we introduced the exponential Fourier series, given by

$$x(t) = \sum_{k=-\infty}^{\infty} d_k \exp\left(j\frac{2\pi kt}{T}\right), \tag{4.1}$$

where

$$d_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\exp\left(-j\frac{2\pi kt}{T}\right). \tag{4.2}$$

Letting

$$\omega_k = \frac{2\pi k}{T}, \tag{4.3}$$

these equations become

$$x(t) = \sum_{\omega_k=-\infty}^{\infty} d_k \exp\left(j\omega_k t\right) \tag{4.4}$$

$$d_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\exp\left(-j\omega_k t\right). \tag{4.5}$$

(Note that these are changed slightly from their definition in Chapter 3.)

How do we generalize the Fourier series to non-periodic signals? Note that for a periodic signal, we only need to consider the signal over one period, e.g., on the interval $[-T/2, T/2]$, since all periods are the same. However, for a non-periodic signal, we must consider the signal over all time: on the interval $(-\infty, \infty)$. Thus, one way to think of a non-periodic signal is to see it as a "periodic" signal as $T \to \infty$.

How does this affect the Fourier series (4.4)-(4.5)? First, the difference between adjacent values of ω_k from (4.3) is given by

$$\Delta\omega_k = \omega_k - \omega_{k-1} \tag{4.6}$$

$$= \frac{2\pi k}{T} - \frac{2\pi(k-1)}{T} \tag{4.7}$$

$$= \frac{2\pi}{T}. \tag{4.8}$$

As $T \to 0$, $\Delta\omega_k$ becomes infinitesimal. This means ω_k becomes a continuous variable, which we will write from now on as ω, so the coefficients d_k from (4.5) become $d(\omega)$, a function of the continuous ω. Now, the coefficient calculation becomes

$$d(\omega) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t) \exp(-j\omega t) \tag{4.9}$$

$$= \lim_{T\to\infty} \frac{\Delta\omega}{2\pi} \int_{-T/2}^{T/2} x(t) \exp(-j\omega t). \tag{4.10}$$

Finally, $x(t)$ from (4.4) becomes

$$x(t) = \lim_{T\to\infty} \sum_{\omega_k=-\infty}^{\infty} d(\omega_k) \exp(j\omega_k t) \tag{4.11}$$

$$= \lim_{T\to\infty} \sum_{\omega_k=-\infty}^{\infty} \frac{\Delta\omega_k}{2\pi} \left(\int_{-T/2}^{T/2} x(t) \exp(-j\omega_k t)\, dt \right) e^{j\omega t}. \tag{4.12}$$

As $T \to \infty$, the sum over infinitely many, infinitely small steps (multiplied by the step size $\Delta\omega_k$) becomes an integral, and we can write

$$x(t) = \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \right) e^{j\omega t} d\omega. \tag{4.13}$$

Equation (4.13) gives us the Fourier transform and inverse Fourier transform: the inner integral under the parentheses generalizes the Fourier series coefficients, while the outer integral recovers the original signal $x(t)$. Thus:

DEFINITION 4.1. *The **Fourier transform of the signal** $x(t)$ is defined as*

$$\mathcal{F}_x(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt, \tag{4.14}$$

*and the **inverse Fourier transform of** $\mathcal{F}_x(j\omega)$ is defined as*

$$x(t) = \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} \mathcal{F}_x(j\omega) e^{j\omega t} d\omega. \tag{4.15}$$

EXAMPLE 4.1.

4.2. Properties of the Fourier transform

4.2.1. Useful properties. We give several useful properties of the Fourier transform.

- linear
- Fourier transform of a delta function
- Inverse Fourier transform of a delta function
- Fourier transform of sin/cos
- time delay
- modulation
- Fourier transform of derivatives
- Fourier transform vs. Laplace transform

A helpful hint: A signal $x(t)$ and its Fourier transform $\mathcal{F}_x(j\omega)$ form a unique pair. In some cases, it is easier to work with a signal in its Fourier transform form than in its original form. Working with the Fourier transform is called the *frequency domain*, while the original form of the signal is called the *time domain*.

4.2.2. Convolution and the Fourier transform. The convolution of two signals $x(t)$ and $y(t)$, written $x(t) \star y(t)$, is given by

$$x(t) \star y(t) = \int_{\tau=-\infty}^{\infty} x(\tau)y(t-\tau)d\tau. \tag{4.16}$$

We wish to show that

$$X(j\omega)Y(j\omega) = \mathcal{F}\left[x(t) \star y(t)\right]. \tag{4.17}$$

To do so, starting on the left side of (4.17),

$$X(j\omega)Y(j\omega) = \mathcal{F}[x(t)]\mathcal{F}[y(t)] \tag{4.18}$$

$$= \int_{\tau=-\infty}^{\infty} x(\tau)e^{-j\omega\tau}d\tau \int_{u=-\infty}^{\infty} y(u)e^{-j\omega u}du \tag{4.19}$$

using different names for the variable of integration for each Fourier transform on the right: τ and u, respectively. Rearranging the integral,

$$= \int_{\tau=-\infty}^{\infty}\int_{u=-\infty}^{\infty} x(\tau)y(u)e^{-j\omega\tau}e^{-j\omega u}dud\tau \tag{4.20}$$

$$= \int_{\tau=-\infty}^{\infty}\int_{u=-\infty}^{\infty} x(\tau)y(u)e^{-j\omega(\tau+u)}dud\tau \tag{4.21}$$

Now make the change of variables: $t = \tau + u$: we have $u = t - \tau$, and $d\tau = dt$. Continuing,

$$= \int_{\tau=-\infty}^{\infty} \int_{t=-\infty}^{\infty} x(\tau) y(t-\tau) e^{-j\omega t} d\tau dt \tag{4.22}$$

Rearranging,

$$= \int_{t=-\infty}^{\infty} \int_{\tau=-\infty}^{\infty} x(\tau) y(t-\tau) d\tau e^{-j\omega t} dt \tag{4.23}$$

$$= \int_{t=-\infty}^{\infty} \left(x(t) \star y(t) \right) e^{-j\omega t} dt \tag{4.24}$$

$$= \mathcal{F}\left[x(t) \star y(t) \right]. \tag{4.25}$$

A similar property applies when multiplying in the time domain: one then convolves in the frequency domain. The analysis is similar: we now have

$$X(j\omega) \star Y(j\omega) = \frac{1}{2\pi} \mathcal{F}[x(t)y(t)]. \tag{4.26}$$

We leave it as an exercise to show this case.

4.2.3. Obtaining the Fourier transform of a periodic signal. In Chapter 3, we discussed the Fourier series. Although the Fourier transform generalizes the concept of the Fourier series to nonperiodic signals, it is still possible to calculate the Fourier transform of a periodic signal.

It is simplest to start with the trigonometric Fourier series. To recall Chapter 3, a periodic signal $x(t)$ with period T has a Fourier series representation

$$x(t) = \sum_{\ell=-\infty}^{\infty} d_\ell \exp\left(j \frac{2\pi\ell}{T} t \right). \tag{4.27}$$

where the constants d_ℓ are calculated as in (3.52). Taking the Fourier transform of this expression,

$$X(j\omega) = \mathcal{F}\left[x(t) \right] \tag{4.28}$$

$$= \mathcal{F}\left[\sum_{\ell=-\infty}^{\infty} d_\ell \exp\left(j \frac{2\pi\ell}{T} t \right) \right] \tag{4.29}$$

$$= \sum_{\ell=-\infty}^{\infty} d_\ell \mathcal{F}\left[\exp\left(j \frac{2\pi\ell}{T} t \right) \right] \tag{4.30}$$

It can be shown that[1]

$$\mathcal{F}\left[\exp\left(j\frac{2\pi\ell}{T}t\right)\right] = 2\pi\delta\left(\omega - \frac{2\pi\ell}{T}\right). \tag{4.31}$$

Thus,

$$X(j\omega) = \sum_{\ell=-\infty}^{\infty} 2\pi d_\ell \delta\left(\omega - \frac{2\pi\ell}{T}\right). \tag{4.32}$$

That is, the Fourier transform of a continuous-time periodic signal yields a series of Dirac delta functions in the frequency domain.

Example 4.2.

4.3. Problems

(1) Give the Fourier transform of

$$x(t) = \begin{cases} 1+t, & -1 \le t \le 0 \\ 1-t, & 0 \le t \le 1 \\ 0, & \text{otherwise} \end{cases} \tag{4.33}$$

(2) Give the inverse Fourier transform of

$$F_x(j\omega) = \begin{cases} 1, & |\omega| < 1 \\ 0, & \text{otherwise} \end{cases} \tag{4.34}$$

(3) Consider a system with input $x(t)$ and output $y(t)$ where

$$y(t) = \int_{-\infty}^{t} x(t)dt. \tag{4.35}$$

a. Give the transfer function $T(j\omega)$ of this system in the frequency domain.
b. Sketch the magnitude $|T(j\omega)|$ as a function of ω.
c. Based on your sketch, what is the response of $T(j\omega)$ to high frequencies?

(4) Considering the system in question 3, what is the time-domain impulse response $h(t)$ of the system?

[1] To show this property yourself, start in the Fourier domain with the right hand side, and show that the inverse Fourier transform is $e^{-j2\pi\ell t/T}$.

(5) Let

$$x(t) = \begin{cases} 1, & 0 \le t \le 2 \\ 0, & \text{otherwise} \end{cases} \tag{4.36}$$

and

$$h(t) = \begin{cases} t, & 0 \le t \le 1 \\ 0, & \text{otherwise} \end{cases} \tag{4.37}$$

Find and sketch the convolution $x(t) * h(t)$.

(6) Show that

$$X(j\omega) \star Y(j\omega) = \frac{1}{2\pi}\mathcal{F}[x(t)y(t)]. \tag{4.38}$$

Part 3

Discrete-time signals and systems

CHAPTER 5

Sampling and Quantization

5.1. Introduction

Sampling is the process by which a continuous-time signal is transformed into a discrete-time signal. Under some circumstances (that we describe in this chapter), it is possible to exactly reconstruct the continuous-time signal from the discrete-time version.

5.2. Mathematical sampling

Let $x(t)$ be a signal, and suppose we want to sample it at a sampling time of T_s (or a sampling rate of $1/T_s$). That is, we want to create a discrete-time signal $x[k]$ where

$$x[k] = x(xT_s). \tag{5.1}$$

We will also use $\omega_s = 2\pi/T_s$ as the angular sampling frequency.

Mathematically, sampling may be thought of as multiplying $x(t)$ by a series of Dirac delta functions. Let

$$s(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_s) \tag{5.2}$$

represent the sampling function. Multiplying this by the signal $x(t)$,

$$x(t)s(t) = \sum_{k=-\infty}^{\infty} x(t)\delta(t - kT_s). \tag{5.3}$$

On the right side of this equation we have $x(t)\delta(t-kT_s)$. However, the Dirac delta function is zero everywhere except at $t = kT_s$; thus,

$$x(t)\delta(t - kT_s) = x(kT_s)\delta(t - kT_s) \tag{5.4}$$

$$= x[k]\delta(t - kT_s). \tag{5.5}$$

Substituting back into (5.3),

$$x(t)s(t) = \sum_{k=-\infty}^{\infty} x[k]\delta(t - kT_s). \tag{5.6}$$

We now consider some properties of $s(t)$, which will be useful to the derivation of the sampling theorem.

Example 5.1.

We now obtain the Fourier transform of $s(t)$. We start by noting that $s(t)$ is periodic, with period T_s; it therefore has a Fourier series, and as we saw in Section 4.2.3, this Fourier series can be converted to a Fourier transform.[1] The exponential Fourier series of $s(t)$ is written (using ℓ as the index of summation, to distinguish it from k in (5.2))

$$s(t) = \sum_{\ell=-\infty}^{\infty} d_\ell \exp\left(j\frac{2\pi\ell}{T_s}t\right) \tag{5.7}$$

$$= \sum_{\ell=-\infty}^{\infty} d_\ell e^{j\omega_s \ell t}, \tag{5.8}$$

recalling that $\omega_s = 2\pi/T_s$. The series coefficients are given by

$$d_\ell = \frac{1}{T}\int_{-T_s/2}^{T_s/2} s(t)e^{-j\omega_s\ell t}dt \tag{5.9}$$

$$= \frac{1}{T}\int_{-T_s/2}^{T_s/2} \sum_{k=-\infty}^{\infty} \delta(t-kT_s)e^{-j\omega_s\ell t}dt \tag{5.10}$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty}\int_{-T_s/2}^{T_s/2} \delta(t-kT_s)e^{-j\omega_s\ell t}dt \tag{5.11}$$

$$= \frac{1}{T}\int_{-T_s/2}^{T_s/2} \delta(t)e^{-j\omega_s\ell t}dt \tag{5.12}$$

$$= \frac{1}{T}e^{-j\omega_s\ell 0} \tag{5.13}$$

$$= \frac{1}{T} \tag{5.14}$$

where (5.12) follows since every delta function of the form $\delta(t - kT_x) = 0$ over the range $[-T_s/2, T_s/2]$, except $\delta(t)$, at $k = 0$; and where (5.13) follows from the properties of integrals of $\delta(t)$.

[1] Strictly, $s(t)$ violates the Dirichlet conditions, since the function $\delta(t)$ has a discontinuity with infinite amplitude; however, the Dirichlet conditions are sufficient, not necessary.

Then

$$\mathcal{F}[s(t)] = \mathcal{F}\left[\frac{1}{T}\sum_{\ell=-\infty}^{\infty} e^{j\omega_s \ell t}\right] \tag{5.15}$$

$$= \frac{1}{T}\sum_{\ell=-\infty}^{\infty} \mathcal{F}\left[e^{j\omega_s \ell t}\right] \tag{5.16}$$

$$= \frac{2\pi}{T}\sum_{\ell=-\infty}^{\infty} \delta(\omega - \ell\omega_s), \tag{5.17}$$

where (5.15) follows from (5.8) and (5.14); (5.16) follows from the linearity of the Fourier transform; and (5.17) follows from (4.31).

5.3. Nyquist's sampling theorem

Define the bandwidth of a signal ω_0 as the maximum frequency for which the Fourier transform is nonzero; this can be a positive or negative frequency. That is,

$$\omega_0 = \max_{\omega > 0} \left\{ |F(j\omega)| > 0, |F(-j\omega)| > 0 \right\}. \tag{5.18}$$

Due to the symmetry of the Fourier transform, for real-valued signals, the bandwidth for positive and negative frequencies is the same.

Nyquist's sampling theorem is stated as follows:

Theorem 5.1 (Nyquist's sampling theorem). *Let T_s represent a sampling interval, and let $\omega_s = 2\pi/T_s$ represent the corresponding angular sampling frequency. Suppose the signal $x(t)$, with bandwidth ω_0, is sampled to form $x[k]$, where $x[k] = x(kT_s)$ for all integers k. If*

$$\omega_s > 2\omega_0. \tag{5.19}$$

then $x(t)$ can be reconstructed exactly from $x[k]$.

We prove the sampling theorem by examining the Fourier transform of $x(t)s(t)$.

5.4. Signal reconstruction

5.5. Quantization

5.6. Problems

(1) Suppose the signal $x(t)$ is given by

$$x(t) = \cos(2\pi t) + \sin(4\pi t) - \cos(6\pi t). \tag{5.20}$$

a. What is the bandwidth of $x(t)$?

b. What is the minimum sampling frequency that would permit accurate reconstruction of this signal?

c. Sketch the Fourier transform of $x(t)$.

d. Suppose the sampling time is $T_s = 0.1$ s. Sketch the Fourier transform of $x(t)s(t)$, where $s(t)$ is the ideal (impulse train) sampling function.

(2) Suppose $x(t) = \sin(2\pi t)$, and zero-order hold is used to reconstruct the signal, where the sampling time $T_s = 1/8$ s, and the hold time $\tau = 1/16$ s.

a. Sketch the sampled signal using zero-order hold.

b. Briefly explain how this signal is reconstructed from its samples.

(3) Repeat question 2 for pulse-train sampling.

(4) For a sampling time of T_s and any ϵ, show that the signals $\cos((\pi+\epsilon)t/T_s)$ and $\cos((\pi-\epsilon)t/T_s)$ are indistinguishable after sampling.

(5) Suppose a signal $x(t)$ has a bandwidth of 2 MHz, and suppose that -1 V $\leq x(t) \leq 1$ V. The signal is sampled at the minimum required sampling frequency, and quantized using PCM. A maximum quantization error of 0.1 V is desired. Find:

a. The number of required PCM levels; and

b. The bit rate required to represent the signal using the number of levels from part a.

CHAPTER 6

Discrete-time Fourier series and transforms

6.1. Introduction

6.2. Problems

(1) Let $x[k]$ be a periodic signal with period $K = 6$, and let

$$x[k] = k \tag{6.1}$$

on the range $k = 0, 1, \ldots, 5$.

a. Sketch several periods of $x[k]$.

b. Find the discrete-time Fourier series for $x[k]$.

(2) Show that the discrete-time Fourier transform is periodic in Ω with period equal to 2π.

(3) Let $x[k]$ be a non-periodic signal, defined as follows.

$$x[k] = \begin{cases} -2, & k = -2 \\ 1, & k = -1 \\ 2, & k = 0 \\ -1, & k = 1 \\ -2, & k = 2 \\ 0 & \text{otherwise} \end{cases} \tag{6.2}$$

a. Sketch $x[k]$.

b. Find the discrete-time Fourier transform of $x[k]$.

(4) Find the Z transform of $x[k]$ in question 3.

(5) Let

$$h[k] = \begin{cases} -3, & k = 0 \\ 2, & k = 1 \\ -1, & k = 2 \\ 0 & \text{otherwise} \end{cases} \tag{6.3}$$

Find the Z transform of $h[k]$, and find the convolution $x[k] \star h[k]$, where $x[k]$ is given in question 3. (Perform the convolution in the Z transform domain.) Sketch the result of the convolution.

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