Rough notes on binary representations from EECS3101 tutorial, Sep 10, 2015.

Definition 1. A natural number n is *even* if there is a natural number k such that $n = 2k$.

Definition 2. A natural number n is *odd* if there is a natural number k such that $n = 2k + 1$.

Proposition 3. very natural number is either even or odd.

Exercise: prove by induction.

Proposition 4. No natural number is both even and odd.

Exercise: prove by contradiction.

Definition 5. In the binary representation of numbers, the string $b_\ell b_{\ell-1} \ldots b_2 b_1 b_0$ (where each $b_i \in \{0, 1\}$) the number $\sum_{i=0}^{\ell} b_i \cdot 2^i$.

Proposition 6. There is a binary representation for every natural number.

The proposition can be stated more formally: For all $n \in \mathbb{N}$, there exist $\ell \in \mathbb{N}$ and $b_\ell, \ldots, b_0 \in \{0, 1\}$ such that $n = \sum_{i=0}^{\ell} b_i \cdot 2^i$.

Proof. We use induction on n.

Base case ($n = 0$): Take $\ell = 1$ and $b_0 = 0$. Then $\sum_{i=0}^{\ell} b_i \cdot 2^i = b_0 \cdot 2^0 = 0 = n$.

Inductive step: Let $n \geq 1$. Assume that for $0 \leq k < n$, there is a binary representation of k. Our goal is to prove there is a binary representation of n. We will do this by explicitly constructing the binary representation of n.

By Proposition 3 n is either even or odd. So $n = 2k + j$ for some $k \in \mathbb{N}$ and $j \in \{0, 1\}$.

Then, $k = \frac{n-j}{2} \leq \frac{n}{2} < n$ (since $n \geq 1$).

So by the induction hypothesis, there exist $\ell \in \mathbb{N}$ and $b_\ell, \ldots, b_0 \in \{0, 1\}$ such that $k = \sum_{i=0}^{\ell} b_i \cdot 2^i$

Let $b'_0 = j$. For $1 \leq i \leq \ell + 1$ let $b'_i = b_{i-1}$. We check that $b'_{\ell+1} \ldots b'_0$ is a binary representation of n:

$$
\begin{aligned}
\sum_{i=0}^{\ell+1} b'_i \cdot 2^i &= (\sum_{i=1}^{\ell+1} b'_i \cdot 2^i) + b'_0 \\
&= (\sum_{i=1}^{\ell+1} b_{i-1} \cdot 2^i) + j \\
&= 2(\sum_{i=1}^{\ell+1} b_{i-1} \cdot 2^{i-1}) + j \\
&= 2(\sum_{i=0}^{\ell} b_i \cdot 2^i) + j \\
&= 2k + j \\
&= n
\end{aligned}
$$

□

Proposition 7. Each positive natural number has at most one binary representation that starts with 1.

Proof. We use strong induction on n.

Base case ($n = 1$): We argue that the only binary representation of n is 1. The single bit 0 is clearly not a representation of 1. No string of length $\ell \geq 2$ that starts with 1 can represent the number 1, since any such string will represent a sum whose leading term is $2^{\ell-1} = 2$, so the sum will be at least 2.

Inductive step: Let $n \geq 2$. Assume that for all k with $1 \leq k < n$, k has at most one binary representation that starts with 1.

Let $b_\ell \dots b_0$ and $a_m \dots a_0$ are two representations of n that both start with 1. Our goal is to show they are the same binary strings (i.e., $\ell = m$ and for all i, $b_i = a_i$).

Case 1: n is even. Then there is a k such that $n = 2k$.

To derive a contradiction, assume $b_0 = 1$. Then $n = \sum_{i=0}^{\ell} b_i \cdot 2^i = 2(\sum_{i=1}^{\ell} b_i \cdot 2^{i-1}) + 1$, which is odd. This contradicts the fact that n is even (since a number cannot be both odd and even). Thus $b_0 = 0$.

Similarly, it can be shown that $a_0 = 0$.

Now, let $k = \frac{n}{2} = \sum_{i=1}^{\ell} b_i \cdot 2^{i-1}$. This is a natural number, and $1 \leq k < n$, so by the induction hypothesis, there is only one binary representation of k that starts with a 1.

By definition, $b_\ell ... b_1$ represents $\sum_{i=0}^{\ell-1} b_{i+1} \cdot 2^i = \sum_{i=1}^{\ell} b_i \cdot 2^{i-1} = \frac{1}{2} \sum_{i=0}^{\ell} b_i \cdot 2^i = \frac{n}{2} = k$.

So, $b_\ell ... b_1$ represents k (and starts with a 1). Similarly, $a_m ... a_1$ represents k (and starts with a 1).

Since there is only one binary representation of k that starts with a 1, we must have $m = \ell$ and $a_i = b_i$ for $1 \leq i < \ell$. Also, we have $b_0 = a_0 = 0$, so the two representations $b_\ell \dots b_0$ and $a_m \dots a_0$ of n are identical binary strings.

Case 2: n is odd: can be done similarly. □

Putting the two propositions above together, we see that every positive natural number has a unique binary representation that starts with 1.

Exercise: Prove that if n is represented by a binary string of length ℓ that starts with 1, then $2^{\ell-1} \leq n < 2^\ell$.

This means that the (unique) binary string that represents the positive integer n and starts with 1 has length $\lfloor \log_2 n \rfloor + 1$.