Solution to Homework Assignment #4

1. Yes. Consider any $n \geq e$. Then, $\sqrt{\log n} \geq \sqrt{\log e} = 1$ (since log and $\sqrt{\cdot}$ are both increasing functions. So,

$$\begin{aligned} \sqrt{\log n} &\leq \sqrt{\log n} \cdot \sqrt{\log n} \\ &= \log n \\ &= 2 \cdot \frac{1}{2} \log n \\ &= 2 \cdot \log \sqrt{n}. \end{aligned}$$

(In the definition of big-O notation, we are using $n_0 = e$ and $c = 2$.)

2. No. Consider any c, n_0. We must show there is an $n \geq n_0$ such that $n! > c \cdot 2^n$.
For $n \geq 3$,

$$\begin{aligned} n! &= \prod_{i=1}^{n} i \\ &= 2 \cdot \prod_{i=3}^{n} i \\ &\geq 2 \cdot \prod_{i=3}^{n} 3 \\ &= 2 \cdot 3^{n-2}. \end{aligned}$$

Now,

$$\begin{aligned} 2 \cdot 3^{n-2} > c \cdot 2^n &\Leftrightarrow 3^{n-1} > 3c \cdot 2^{n-1} \\ &\Leftrightarrow (1.5)^{n-1} > 3c \\ &\Leftrightarrow n > \frac{\log(3c)}{\log(1.5)} + 1. \end{aligned}$$

So, let $n = \max\{n_0, 3, \frac{\log(3c)}{\log(1.5)} + 2\}$. Then, $n \geq n_0$ and, by the reasoning above, $n! > c \cdot 2^n$.

3.

(a) It's easy to write a programme to compute this value. It is $298/32 = 9.3125$.

(b) We will use $n_0 = 1$. So, we want to prove that, for some constant c, $T(n) \leq cn$ for all $n \geq 1$. Since T is defined using a recurrence, induction is a natural way to prove such a claim. Before we do the proof, let's do some rough work to figure out which value of c will make the induction step work.

The induction step will look something like this:

$$\begin{aligned} T(n) &\leq T(\lfloor n/2 \rfloor) + T(\lceil n/6 \rceil + 1) + 3n \\ &\leq c\lfloor n/2 \rfloor + c(\lceil n/6 \rceil + 1) + 3n \text{ (we'll use the induction hypothesis here)} \\ &\leq c(n/2) + c(n/6+2) + 3n \\ &= (2/3)cn + 2c + 3n \end{aligned}$$

Now we need to prove that $2c + 3n \leq cn/3$, which is equivalent to $n(c-9) \geq 6c$. If $c \geq 10$ and $n \geq 6c$, then we're done.

So let's try using $c = 10$. (You might also have guessed this value would work after doing part (a).) Note that this means the induction step will only work when $n \geq 60$. So we'll need a lot of base cases.

Now we're ready to do the actual proof.

Claim: For all $n \geq 1$, $T(n) \leq 10n$.

Base Case: $1 \leq n \leq 59$. By part (a), $T(n) \leq 10n$.

Inductive step: Let $n \geq 60$. Assume that $T(k) \leq 10k$ for $1 \leq k < n$. Note that $1 \leq \lfloor n/2 \rfloor \leq n/2 < n$ and $1 \leq \lceil n/6 \rceil + 1 \leq n/6 + 2 < n$ (the last inequality follows from the fact that $n > 12/5$). So, by the inductive hypothesis, $T(\lfloor n/2 \rfloor) \leq 10\lfloor n/2 \rfloor$ and $T(\lceil n/6 \rceil + 1) \leq 10(\lceil n/6 \rceil + 1)$.

Also, since $n \geq 60$, $20 + 3n \leq 10n/3$. So, we have

$$\begin{aligned} T(n) &\leq T(\lfloor n/2 \rfloor) + T(\lceil n/6 \rceil + 1) + 3n \\ &\leq 10\lfloor n/2 \rfloor + 10(\lceil n/6 \rceil + 1) + 3n \\ &\leq 10(n/2) + 10(n/6+2) + 3n \\ &= 20n/3 + 20 + 3n \\ &\leq 20n/3 + 10n/3 \\ &= 10n \end{aligned}$$