York University EECS 3101 September 17, 2015

Homework Assignment #2
Due: September 24, 2015 at 4:00 p.m.

1. In this question we will consider functions $f : \mathbb{R} \to \mathbb{R}$. A value $x \in \mathbb{R}$ is called a *fixed point* of f if $f(x) = x$. A function f is called strictly decreasing if, for all $x, y \in \mathbb{R}$, $x < y \Rightarrow f(x) > f(y)$.

In solving this problem, you may use the following result from calculus.

Bolzano's Theorem Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function, and let a and b be real numbers with $a < b$, $g(a) < 0$ and $g(b) > 0$. Then, there exists a real number c such that $a < c < b$ and $g(c) = 0$.

For the remainder of this assignment, let $f : \mathbb{R} \to \mathbb{R}$ be any continuous, strictly decreasing function f that has $f(0) > 0$.

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous strictly decreasing function and let a and b be real numbers such that $a < b$, $f(a) \geq a$ and $f(b) < b$. Prove that there exists a fixed point c of f such that $a \leq c < b$.

(b) Show that f has a fixed point x such that $0 \leq x < f(0)$.

(c) Provide pseudocode for an algorithm that outputs an integer n such that there is a fixed point of f in the interval $[n, n+1) = \{x \in \mathbb{R} : n \leq x < n+1\}$. The number of times your algorithm calls the function f should be $O(\log |f(0)|)$.

Your pseudocode should include pre- and post-conditions. If you use a loop, you should also include a loop invariant in the pseudocode that you can use for part (c), below.

(d) Prove that the algorithm given in part (b) is correct.

Note: you do not have to prove that your algorithm calls f $O(\log |f(0)|)$ times, but it should be true in order for your algorithm to get full marks.

Example: The graph below shows the strictly decreasing function $f(x) = \frac{(7-x)^3}{30}$. It has a fixed point between 2 and 3, so if your algorithm is run with this function f, it should output 2.